Problem 5

> Refs & people discussed with:
>
> https://en.wikipedia.org/wiki/Trinomial_triangle
> b09902100

(1) Asymptotic Notations

(a)

$$
\begin{aligned}
& \ln n! = \sum_{i=1}^{n} \ln i \le \sum_{i=1}^{n} \ln n = n \ln n = \ln n^n \\
\Rightarrow\ & \ln n! = O(\ln n^n)
\end{aligned}
$$

(b)

$$
\begin{aligned}
& n^{\ln c} = (e^{\ln n})^{\ln c} = e^{\ln n \cdot \ln c} = c^{\ln n} \\
\Rightarrow\ & n^{\ln c} = \Theta(c^{\ln n})
\end{aligned}
$$

(c)

Assume that $\sqrt{n} = O(n^{\sin n})$ is true, then $\exists\ n_0,\ c > 0$ such that $\forall\ n > n_0,\ \sqrt{n} \le c \cdot n^{\sin n}$.

Let $n_1 = \lceil \frac{c^2+n_0}{\pi} \rceil \pi$, we have:

$$
\begin{aligned}
& n_1 \ge \frac{c^2+n_0}{\pi}\pi = c^2 + n_0 > n_0 \\
& \sqrt{n_1} \ge \sqrt{\frac{c^2+n_0}{\pi}\pi} = \sqrt{c^2+n_0} > c \\
& \sin n_1 = \sin\left(\lceil \frac{c^2+n_0}{\pi} \rceil \pi\right) = 0 \\
& c \cdot {n_1}^{\sin n_1} = c \cdot {n_1}^{0} = c \\
\Rightarrow\ & \sqrt{n_1} > c = c \cdot {n_1}^{\sin n_1}
\end{aligned}
$$

This conflicts with the assumption, therefore the assumption is false, $\sqrt{n} \ne O(n^{\sin n})$

(d)

Let $f(x) = x - (\ln x)^3$, we have:

$$
\begin{aligned}
f(x) &= x - (\ln x)^3 \\
f'(x) &= 1 - \frac{3(\ln x)^2}{x} = \frac{x - 3(\ln x)^2}{x} \\
f''(x) &= -\frac{3(2\ln x - (\ln x)^2)}{x^2} = \frac{3\ln x(\ln x - 2)}{x^2}
\end{aligned}
$$

And:

$$
\begin{aligned}
f(e^6) &= e^6 - 6^3 = (e^2)^3 - 6^3 > 0 \\
f'(e^6) &= \frac{e^6 - 3\cdot 6^2}{e^6} > \frac{e^6 - 6^3}{e^6} > 0 \\
f''(x) &= \frac{3\ln x(\ln x - 2)}{x^2} > 0,\ \forall\, x > e^2 \\
\Rightarrow &\ \forall\, x > e^6,\ f(x) = x - (\ln x)^3 > 0 \\
\Rightarrow &\ \forall\, x > e^6,\ (\ln x)^3 < x
\end{aligned}
$$

Choose $n_0 = \lceil e^6 \rceil,\ c = 1$:

$$
\begin{aligned}
&\forall\, n > n_0,\ (\ln n)^3 < c\cdot n \\
&\Rightarrow (\ln n)^3 = o(n)
\end{aligned}
$$

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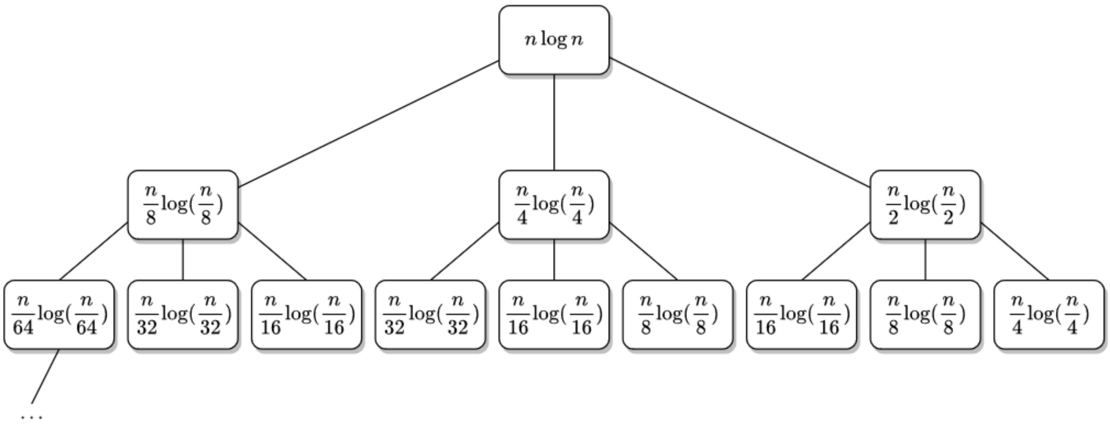
(2) Solve Recurrences

(a)

$$
\begin{aligned}
T(n) &= 2T(n-1) + 1 \\
&= 2(2T(n-2) + 1) + 1 = 2^2T(n-2) + 3 \\
&= 2^2(2T(n-3) + 1) + 3 = 2^3T(n-3) + 7 \\
&\dots \\
&= 2^kT(n-k) + (2^k - 1) \\
&= 2^{n-2}T(n-(n-2)) + 2^{n-2} - 1 \\
&= 2^{n-1} - 1 \\
\Rightarrow T(n) &= \Theta(2^n)
\end{aligned}
$$

(b)

The recursion tree looks like this:



Let R_k be the sum of k-th row of the recursion tree.

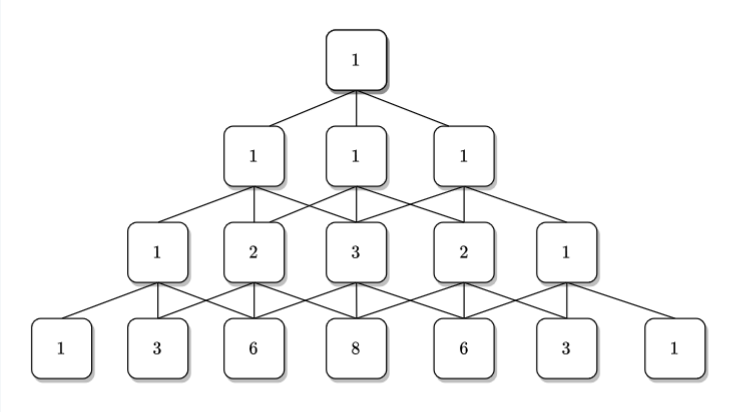
$$R_1 = n \log n$$

$$\begin{aligned}
R_2 &= \frac{n}{2}\log\left(\frac{n}{2}\right) + \frac{n}{4}\log\left(\frac{n}{4}\right) + \frac{n}{8}\log\left(\frac{n}{8}\right) \\
&= \frac{7}{8}n\log n - \frac{(2^2 \cdot 1 + 2 \cdot 2 + 1 \cdot 3)\log 2}{8}n \\
&= \frac{7}{8}n\log n - \frac{11\log 2}{8}n \\
&\le \frac{7}{8}n\log n
\end{aligned}$$

$$\begin{aligned}
R_3 &= \frac{n}{4}\log\left(\frac{n}{4}\right) + 2\cdot\frac{n}{8}\log\left(\frac{n}{8}\right) + 3\cdot\frac{n}{16}\log\left(\frac{n}{16}\right) + 2\cdot\frac{n}{32}\log\left(\frac{n}{32}\right) + \frac{n}{64}\log\left(\frac{n}{64}\right) \\
&= \left(\frac{7}{8}\right)^2 n\log n - \frac{154\log 2}{64}n \\
&\le \left(\frac{7}{8}\right)^2 n\log n
\end{aligned}$$

$$R_k \le \left(\frac{7}{8}\right)^{k-1} n\log n$$

> To prove the coefficient of $n\log n$, we combine the same terms in each row, and look only at the coefficient:



This triangle is called "trinomial triangle".
The i-th term in j-th row (both starts from 0) is the coefficient of x^i in $(1+x+x^2)^j$.

Using $f(i,j)$ to denote the i-th term in j-th row in trinomial triangle, we can rewrite the $n\log n$ term in k-th row (starts from 1) as:

$$\begin{aligned}\sum_{i=0}^{2(k-1)}(f(i,k)\frac{n}{2^{3(k-1)-i}}\log n) &= \sum_{i=0}^{2(k-1)}\frac{f(i,k)\cdot 2^i}{8^{k-1}}n\log n \\ &= \frac{\sum_{i=0}^{2(k-1)}f(i,k)\cdot 2^i}{8^{k-1}}n\log n \\ &= \frac{(1+2+2^2)^{k-1}}{8^{k-1}}n\log n \\ &= (\frac{7}{8})^{k-1}n\log n\end{aligned}$$

Then:

$$\begin{aligned}T(n) &\le \sum_{k=1}^{\infty}R_k \\ &\le \sum_{k=1}^{\infty}(\frac{7}{8})^{k-1}n\log n \\ &= \sum_{k=0}^{\infty}(\frac{7}{8})^{k}n\log n \\ &= \frac{1}{1-\frac{7}{8}}n\log n \\ \Rightarrow T(n) &= O(n\log n)\end{aligned}$$

And:

$$\begin{aligned}T(n) &\ge R_1 \\ &= n\log n \\ \Rightarrow T(n) &= \Omega(n\log n)\end{aligned}$$

Therefore, $T(n) = \Theta(n \log n)$.

(c)

Choose $n_1 = e,\ c_1 = 1,\ \epsilon = 0.5$

$$\forall\, n > n_1,\ n \log n \geq n \cdot 1 = c_1 \cdot n$$
$$\Rightarrow n \log n = \Omega(n^1) = \Omega(n^{(\log_4 2)+\epsilon})$$

Choose $n_2 = 2,\ c_2 = 2$

$$\forall\, n > n_2,\ 4 \cdot \frac{n}{2} \log \frac{n}{2} = 2n(\log n - \log 2) \leq 2n \log n = c_2 \cdot n \log n$$

By case 3 of master theorem, $T(n) = \Theta(n \log n)$

(d)

$$n = 2^m \Rightarrow T(2^m) = 2^{m/2} T(2^{m/2}) + 2^m$$
$$F(m) = T(2^m) \Rightarrow F(m) = 2^{m/2} F(\frac{m}{2}) + 2^m$$

Let $\lg x = \log_2 x$.

Claim: $F(m) \leq (2 \lg m) 2^m\ \forall\, m \geq 2$

For $m = 2$, $F(2) = T(4) = 2 \cdot T(2) + 4 = 6 \leq 2 \cdot 2^2 = 8$
If it's true for $m = k/2$:

$$\begin{aligned} F(k) &= 2^{k/2} F(\frac{k}{2}) + 2^k \\ &\leq 2^{k/2}(2(\lg k - \lg 2) 2^{k/2}) + 2^k \\ &= (2 \lg k) 2^k - 2^k \\ &< (2 \lg k) 2^k \end{aligned}$$

By induction, $F(m) \leq (2 \lg m) 2^m\ \forall\, m \geq 2 \Rightarrow F(m) = O((\log m) 2^m)$.

Claim: $F(m) \geq (\lg m) 2^m\ \forall\, m \geq 2$

For $m = 2$, $F(2) = T(4) = 2 \cdot T(2) + 4 = 6 \geq 2^2 = 4$
If it's true for $m = k/2$:

$$
\begin{aligned}
F(k) &= 2^{k/2}F(\frac{k}{2}) + 2^k \\
&\geq 2^{k/2}((\lg k - \lg 2)2^{k/2}) + 2^k \\
&= (\lg k)2^k
\end{aligned}
$$

By induction, $F(m) \geq (\lg m)2^m \;\forall\; m \geq 2 \Rightarrow F(m) = \Omega((\log m)2^m)$.

$\Rightarrow F(m) = \Theta((\log m)2^m) \Rightarrow T(2^m) = \Theta((\log m)2^m) \Rightarrow T(n) = \Theta(n \log\log n)$
