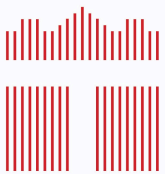
Slido: #ADA2021

CSIE 2136 Algorithm Design and Analysis, Fall 2021



National Taiwan University
國立臺灣大學

NP Completeness - I

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Annoucement

- HW4
  - Due Time (hand-written): 2022/01/04 14:20
  - Due Time (programming): 2022/01/11 14:20
  - Don’t get spooked by the length; many of them are installation guides and tool/library manuals

- Final exam: 2022/01/06
  - Location TBD
  - Closed book
  - Scope: all topics taught in ADA, focusing on those after the midterm

To solve, or not to solve?

Design an algorithm to solve the computational problem efficiently

Prove that the problem is too hard to solve (no easier than solving NP-complete problems!)

We want to avoid…

"I can't find an efficient algorithm, I guess I'm just too dumb."

"I can't find an efficient algorithm, because no such algorithm is possible!"

Origin: Michael Garey and David S. Johnson "Computers and Intractability: A Guide to the Theory of NP-Completeness" (1979)
Cartoon: https://www.ac.tuwien.ac.at/people/szeider/cartoon/

A better way

"I can't find an efficient algorithm, but neither can all these famous people."

Origin: Michael Garey and David S. Johnson "Computers and Intractability: A Guide to the Theory of NP-Completeness" (1979)
Cartoon: https://www.ac.tuwien.ac.at/people/szeider/cartoon/

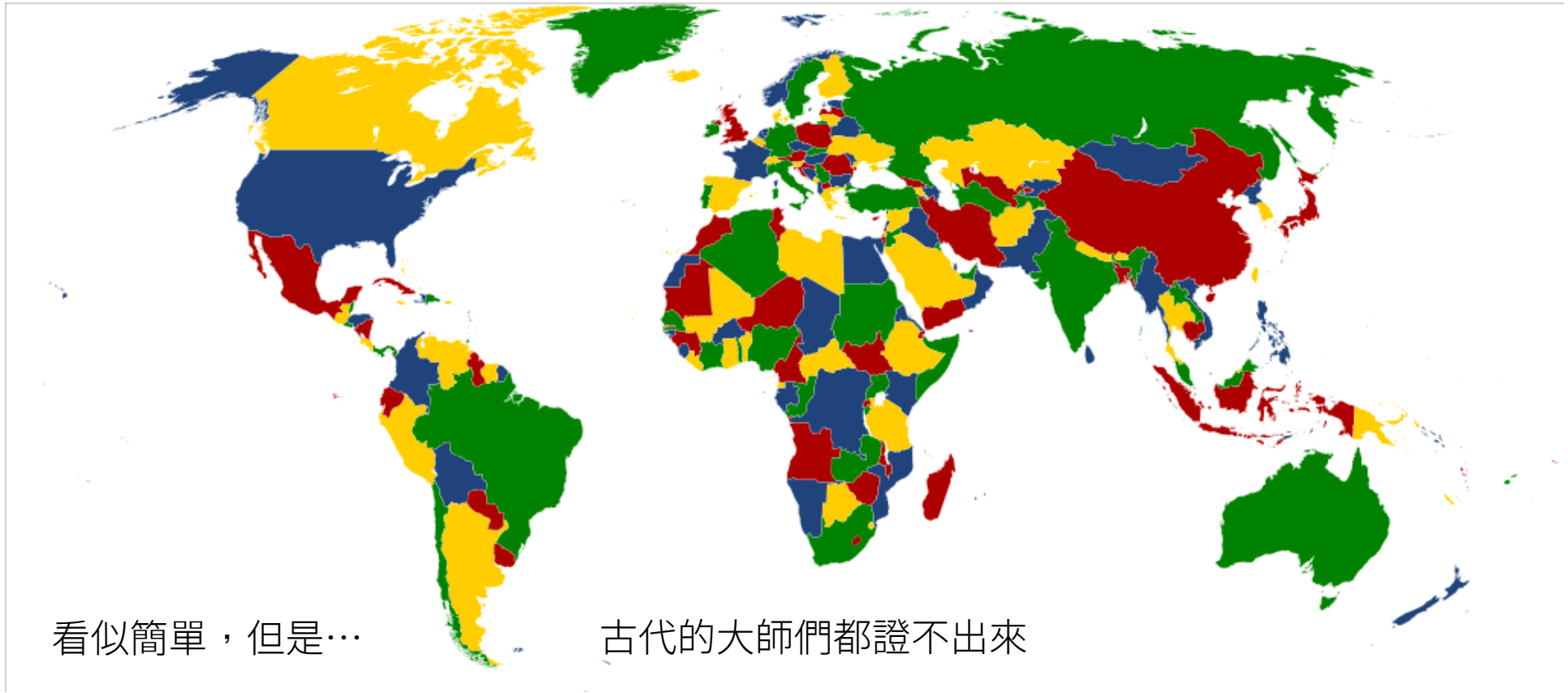
2-week Agenda

- NP-Completeness Overview
  - Warm up: graph coloring
  - Complexity classes
  - Decision problems
  - Reduction
  - Appendix: P-time solving vs. verification
- Proving NP-Completeness
  - Formula satisfiability
  - 3-CNF satisfiability
  - Clique
  - Vertex cover
  - Independent set
  - Traveling salesman
  - Hamiltonian cycle

Warm up: Graph Coloring

四色問題 (Four Color Problem)

- 只能使用四種顏色，使地圖上每兩個鄰接區域的顏色都不一樣

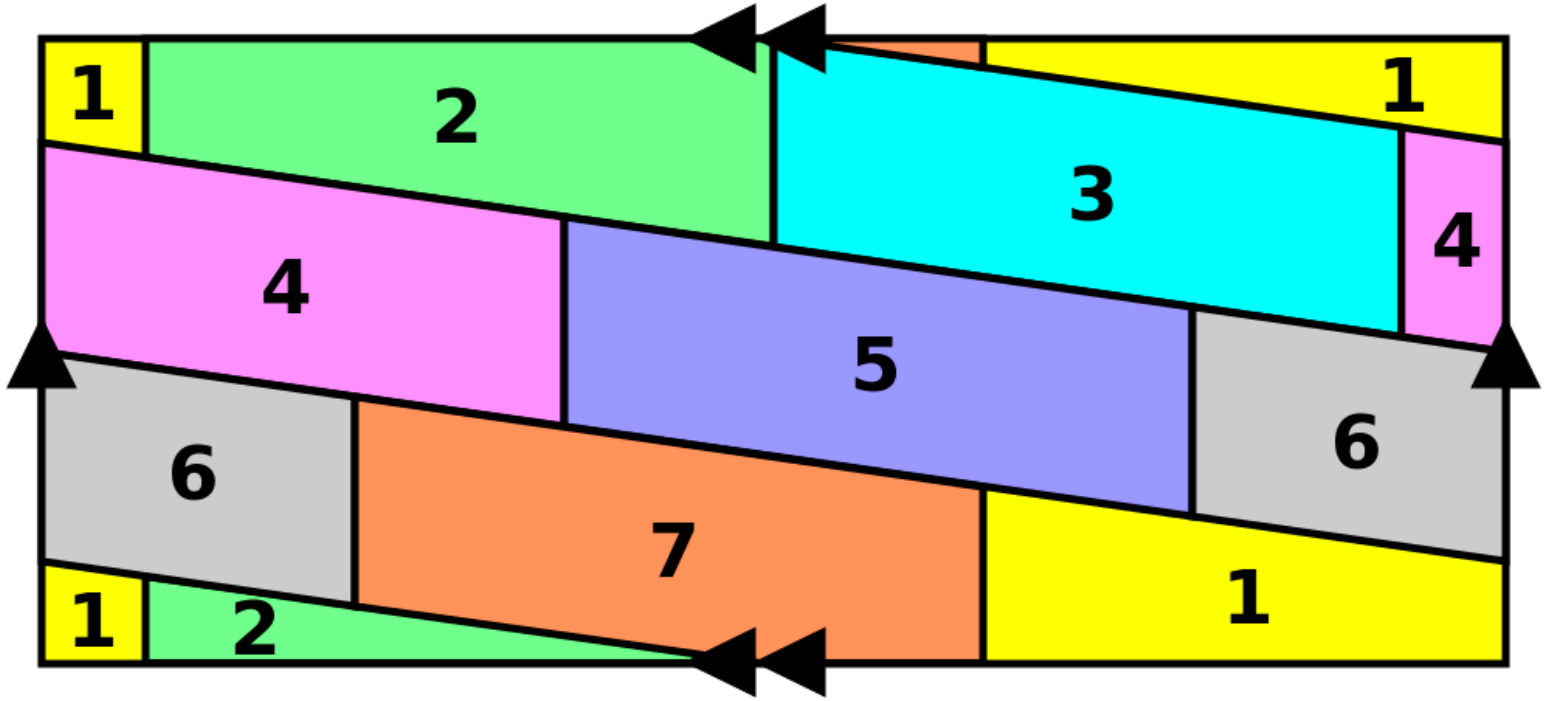


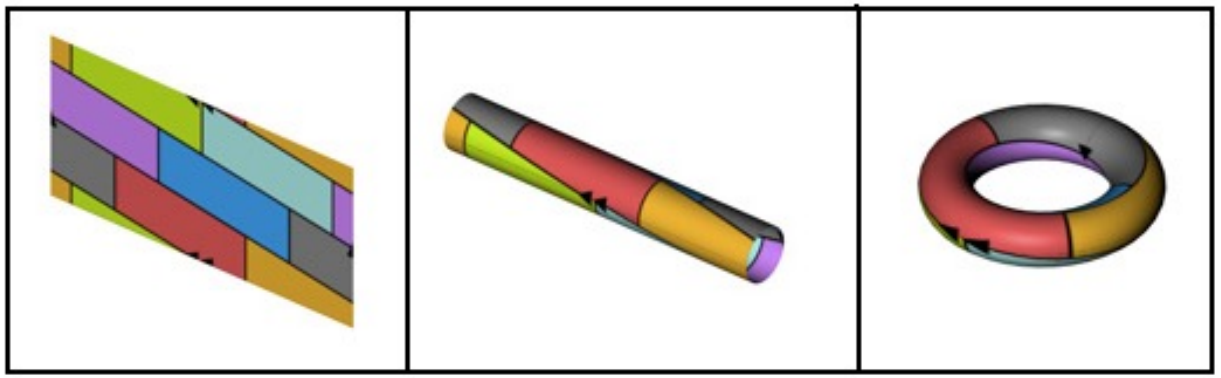
看似簡單，但是…　　古代的大師們都證不出來

一百多年後的四色定理

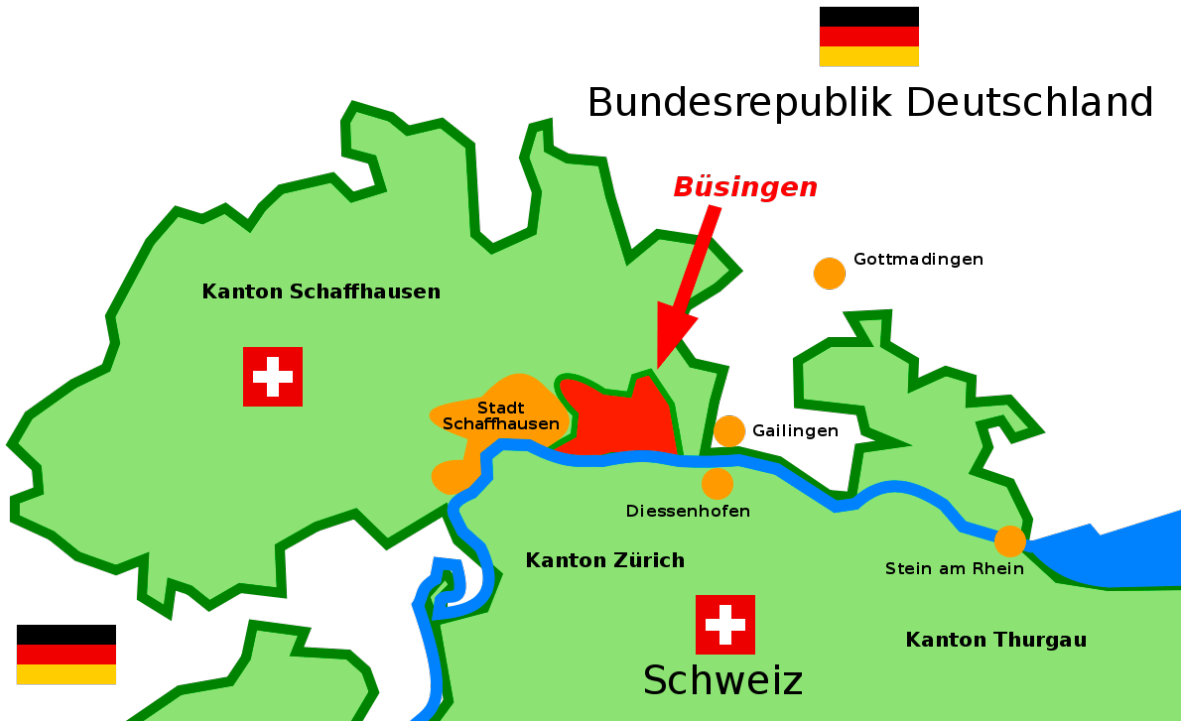
- Finally proven (with the help of computers) by Kenneth Appel and Wolfgang Haken in 1976
  - Their algorithm runs in $O(n^2)$ time
- First major theorem proved by a computer
- Open problems remain…
  - Linear time algorithms to find a solution
  - Concise, human-checkable, mathematical proofs

沒那麼簡單之一





沒那麼簡單之二



就像台東縣金峰鄉的這兩個村里，離金峰鄉的位置有段距離，並且自成一格的出現在別的鄉鎮：

Q: Can you color this map using only 2 colors? Using only 3 colors? Using only 4 colors?

2 colors: No

3 colors: Yes

4 colors: Yes

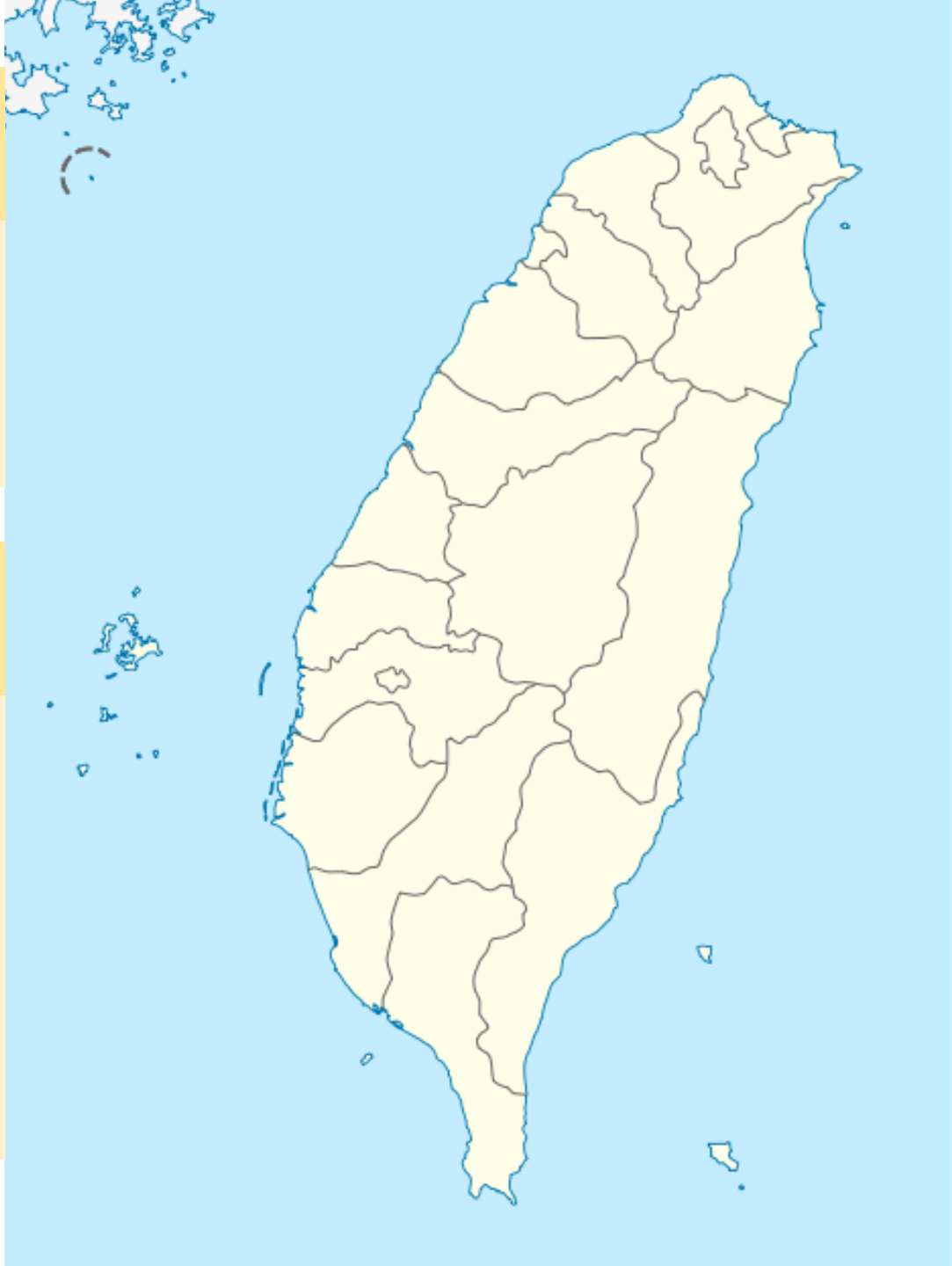
Q: Given a map, can you efficiently determine whether k colors is enough?

$k = 1$: trivial

$k = 2$: trivial

$k = 3$: the planar 3-colorability problem is NP-complete.

$k \geq 4$: always possible because of the four-color theorem



K 色問題 (k-colorability)

Given a graph G, can we color the vertices with k colors such that no adjacent vertices have the same color?

Q: Can we efficiently solve the k-colorability problem when $k = 1$? $k = 2$? $k \geq 3$?

$k = 1$: trivial

$k = 2$: trivial

$k \geq 3$: NP complete

Complexity Classes

What is a “hard” problem

- A problem is hard if it has no known efficient algorithms to solve it
  - Efficient = tractable = polynomial running time (in the size of input)
  - K-colorability, Hamiltonian, knapsack, ⋯

- How to classify problems by hardness?
  - Decide which complexity classes the problem belongs to via polynomial-time reduction
  - 已知問題 A 很難。若能證明問題 B 至少跟 A 一樣難，那麼問題 B 也很難。

Complexity classes

- A complexity class is defined as a set of problems of related resource-based complexity
  - Resource = time, memory, communication, ⋯
- This lecture focuses on decision problems and time resource

Complexity classes

- Class P: class of problems that can be solved in $O(n^k)$
  - $O(n^k)$ means polynomial in the input size (k is a constant)
  - Problems in P are considered tractable
- Class NP: class of problems that can be verified in $O(n^k)$
  - NP = Nondeterministic Polynomial

Q: Is P $\subseteq$ NP? Is NP $\subseteq$ P?

P $\subseteq$ NP: Yes, because a problem solvable in polynomial time is verifiable in polynomial time as well

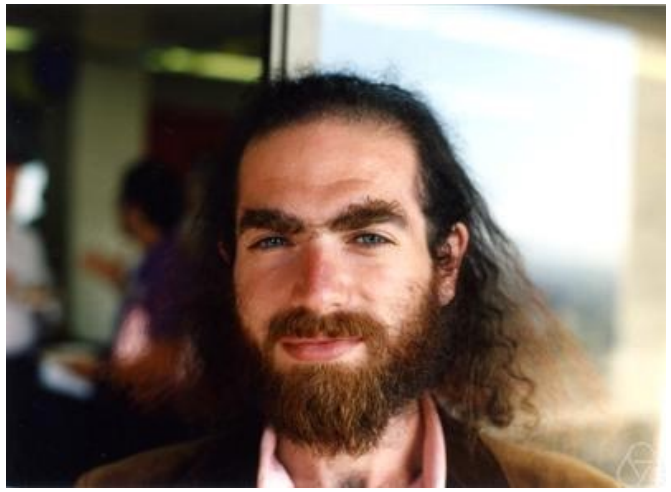
NP $\subseteq$ P: We don't know. If so, P = NP.

Complexity classes

A question worth $1,000,000: Is P = NP?

Millennium Prize

- Solve one and get $1,000,000!
  - P versus NP problem
  - Hodge conjecture
  - Riemann hypothesis
  - Yang-Mills existence and mass gap
  - Navier-Stokes existence and smoothness
  - Birch and Swinnerton-Dyer conjecture
  - Poincaré conjecture (solved by Grigori Perelman)



Grigori Perelman
Fields Medal (2006), declined
Millennium Prize (2010), declined

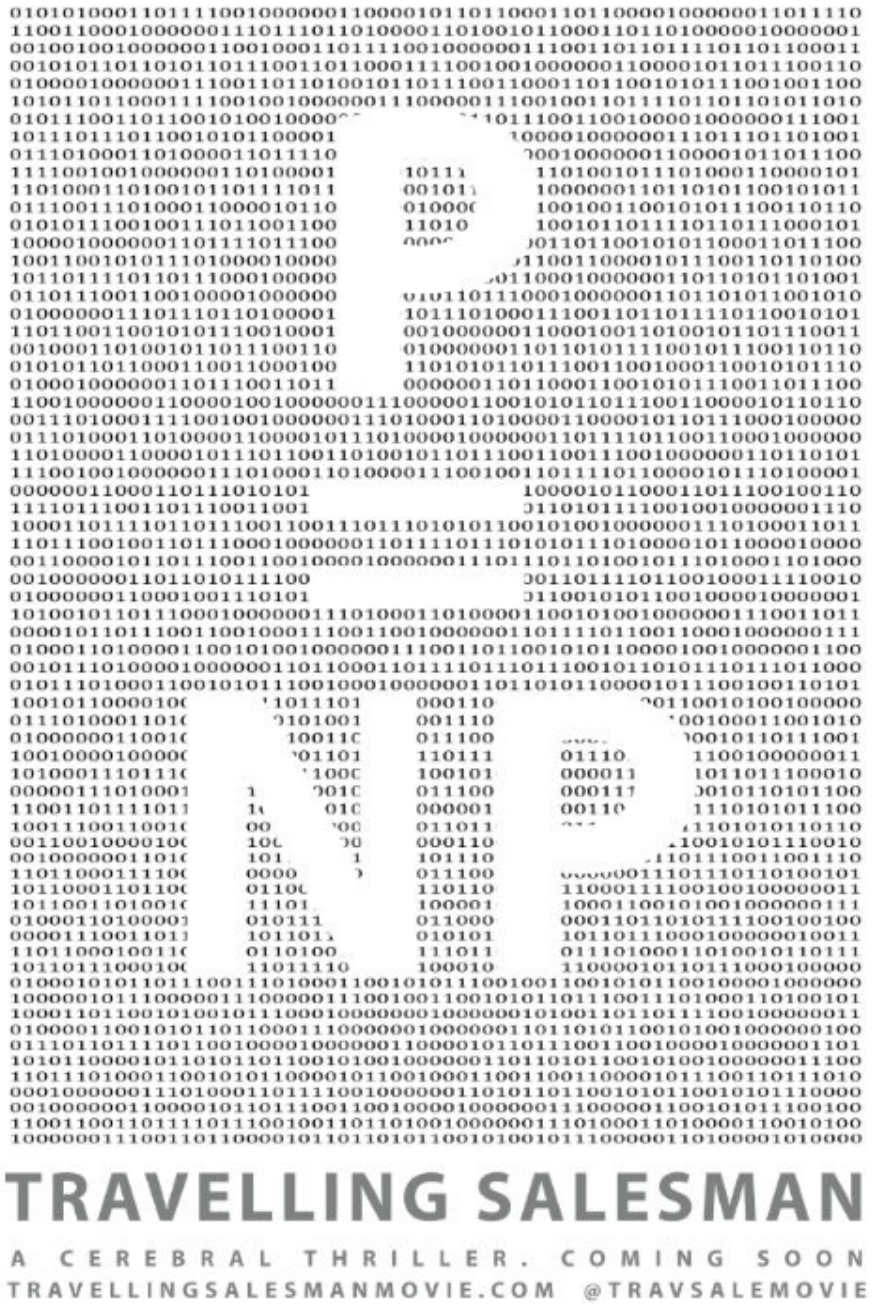
Implications of P=NP

Problems that are verifiable => solvable

Public-key cryptography will be broken

"If P = NP, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss..."
– Scott Aaronson, MIT

Widespread belief in P ≠ NP



“Travelling Salesman” (2012)

A movie about P = NP

Best Feature Film in Silicon Valley Film Festival 2012

http://www.travellingsalesmanmovie.com/

Complexity classes

Class P: class of problems that can be solved in $O(n^k)$

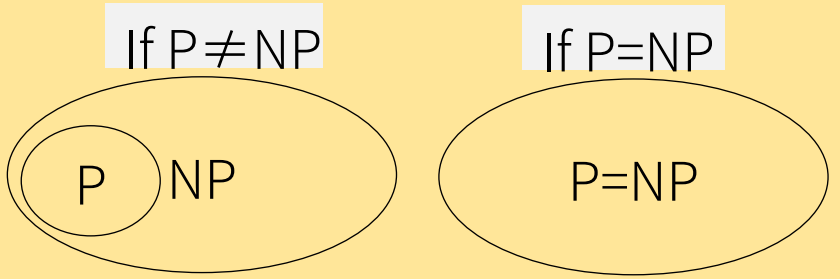
Class NP: class of problems that can be verified in $O(n^k)$

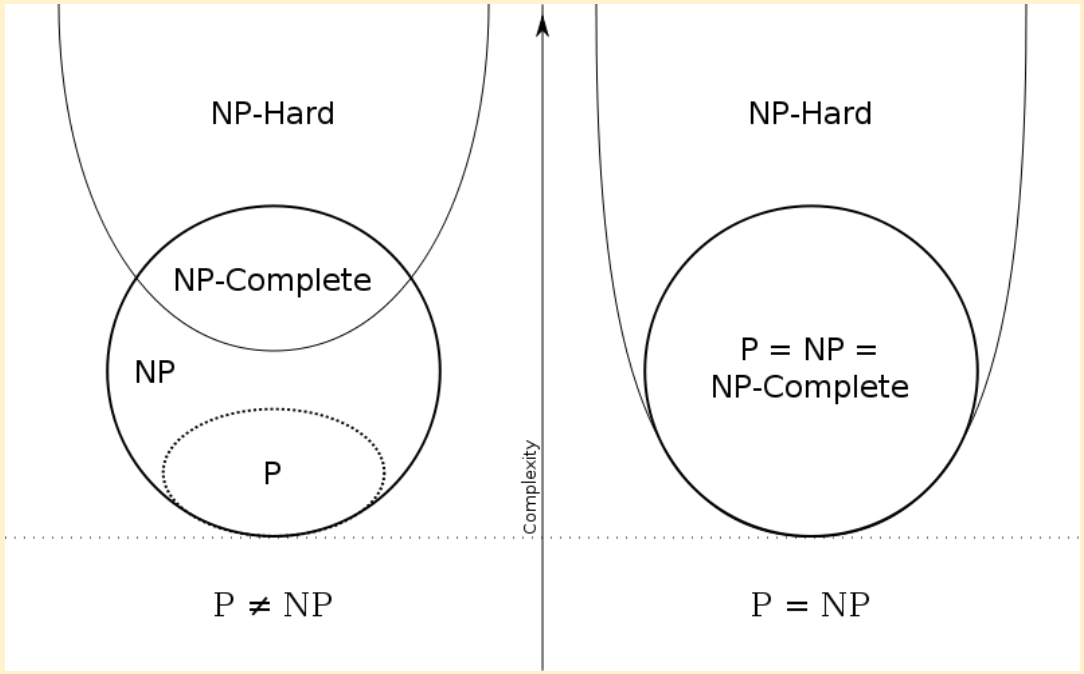
Class NP-hard: a class of problems that are "at least as hard as the hardest problems" in NP

- Hardness relationship can be determined via polynomial-time reduction

Class NP-complete (NPC): class of problems in both NP and NP-hard

Q: Draw the relationship diagrams among P, NP, NP-hard, and NP-complete, when (1) P≠NP, (2) P=NP



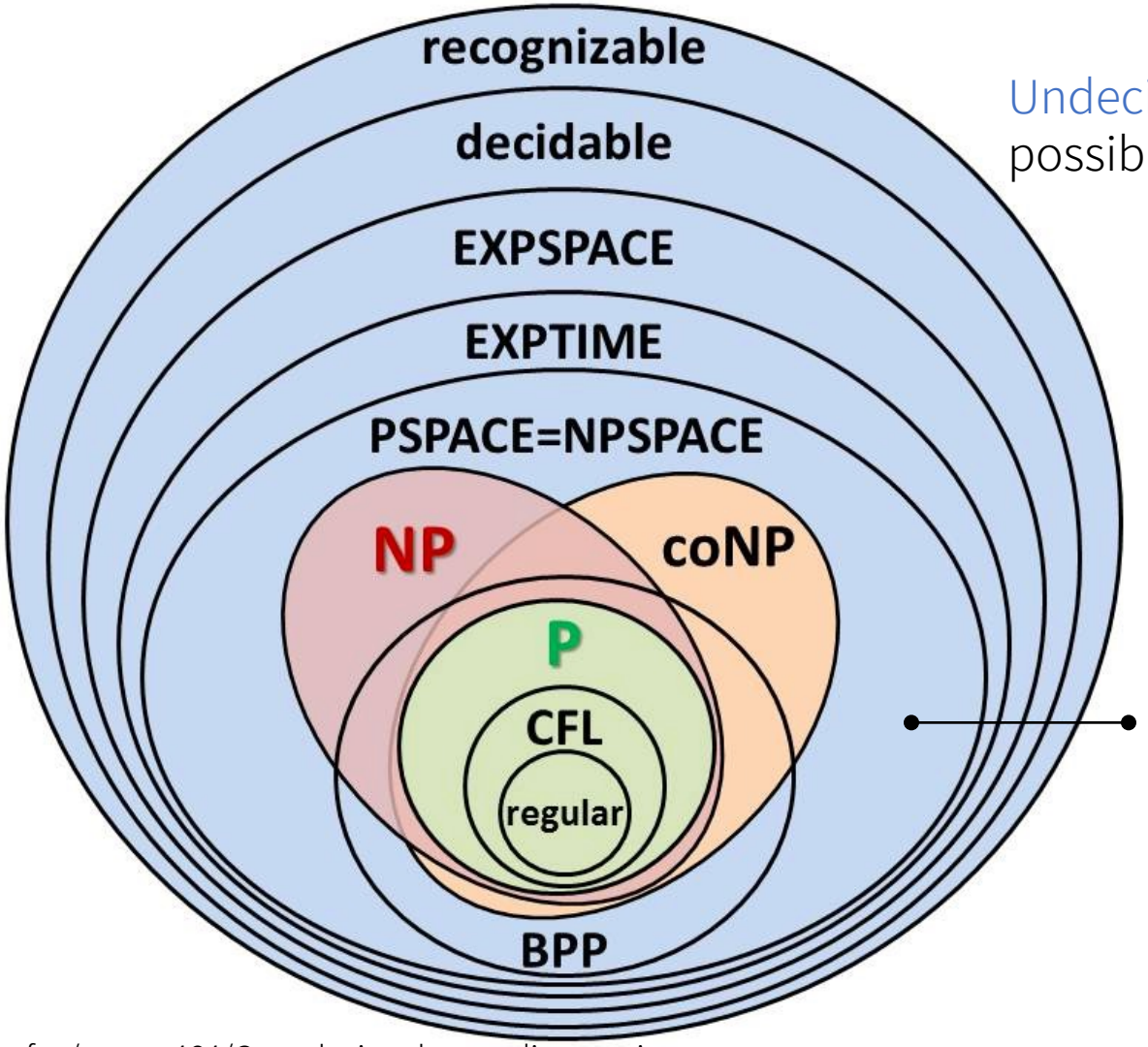


Source: http://en.wikipedia.org/wiki/File:P_np_np-complete_np-hard.svg

The importance of class NPC

- Solving one NPC problem can lead to solving all NP problems
  - NPC problems are the hardest in NP
  - This is why it's called NP-"complete"
- If P≠NP, we can show that a problem is "too hard to solve" by proving it is no easier than a NPC problem
  - NPC problems can be good reference points to judge whether an unknown problem is solvable in polynomial time
  - Then we can turn our focus to designing approximate algorithms or solving special cases

More complexity classes



Undecidability. No algorithm possible, such as Halting problem.

Example: chess and Go

http://www.cse.psu.edu/~sofya/cmpsc464/Complexity-classes-diagram.jpg

Halting Problem

Given a program and an input, determine whether the program will finish running, or continue to run forever.

Theorem: Halting problem is undecidable

Proof

- Let's show that the halting problem is undecidable by contradiction
- Suppose h can determine whether a program p halts on input x
  - $h(p, x)$ = return (p halts on input x) [Eq. 1]
- Define $g(p)$ = if $h(p, p)$ is 0 then return 0 else $HANG$
- => $g(g)$ = if $h(g, g)$ is 0 then return 0 else $HANG$ [Eq. 2]
- Consider whether g halts on g, both cases contradict the assumption:
  - g halts on g: then $h(g, g) = 1$ [Eq. 1], which would make $g(g)$ hang [Eq. 2]
  - g does not halt on g: then $h(g, g) = 0$ [Eq. 1], which would make $g(g)$ halt [Eq. 2]

Decision Problems

Decision problems

The answer to the problem is simply “yes” or “no” (or “1” or “0”)

- Examples
  - PRIME: Given a natural number x, is x a prime?
  - MST: Given a graph $G = (V, E)$ and a bound K, is there a spanning tree with a cost at most K?
  - KNAPSACK: Given a knapsack of capacity C, a set of objects with weights and values, and a target value V, is there a way to fill the knapsack with at least V value?
  - COLOR: Given a graph $G = (V, E)$ and a value k, can we color the vertices with k colors such that no adjacent vertices have the same color?

Optimization problems

Each feasible solution to the problem has an associated value, and we wish to find a feasible solution with the best value (i.e., maximum or minimum)

- Examples
  - MST-OPT: Given a graph $G = (V, E)$, find the minimum spanning tree of G
  - KNAPSACK-OPT: Given a knapsack of capacity C and a set of objects with weights and values, fill the knapsack so as to maximize the total value
  - COLOR-OPT: Given a graph $G = (V, E)$, find the minimum number of colors required to color G such that no adjacent vertices have the same color

Converting an optimization problem to a decision problem

- Every optimization problem has a decision version that is no harder than* the optimization problem.
- In fact, they have equal* computational difficulty.

* In the sense that they are in the same complexity class

P_{opt}: given a graph and (s, t), find the length of the shortest path between s and t

P_{dec}: given a graph, (s, t), and k, determine whether there is a path between s and t whose length $\leq k$

Q: Suppose algorithm A_{opt} solves problem P_{opt}, and algorithm A_{dec} solves problem P_{dec}. Can we use A_{opt} to solve P_{dec}? Can we use A_{dec} to solve P_{opt}?

Using A_{opt} to solve P_{dec}: check if the optimal value $\leq k$ (in the minimization problem)

Using A_{dec} to solve P_{opt}: apply binary search on the value range, logarithmic overhead

Converting an optimization problem to a decision problem

- Every optimization problem has a decision version that is no harder than* the optimization problem.
- In fact, they have equal* computational difficulty.

* In the sense that they are in the same complexity class; we will explain more later

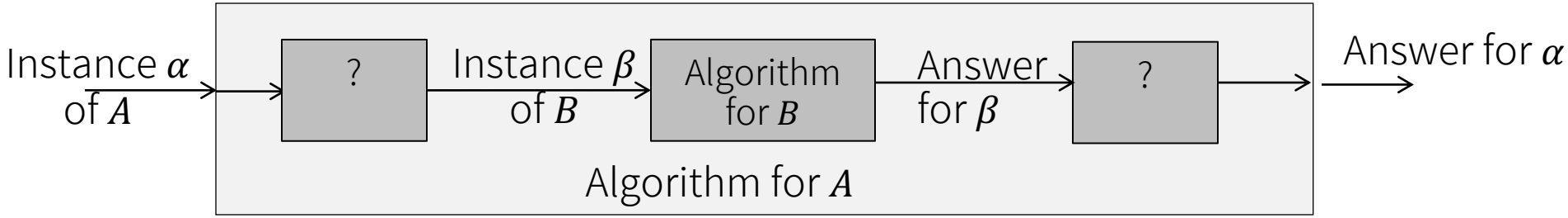
Q: To convert a maximization problem to a related decision problem, we can impose a _____ bound on the value to be optimized. Vice versa.

Lower

Reduction

Reduction: general concept

- **General concept**: Problem A reduces to problem B if we can use an algorithm that solves B to help solve A

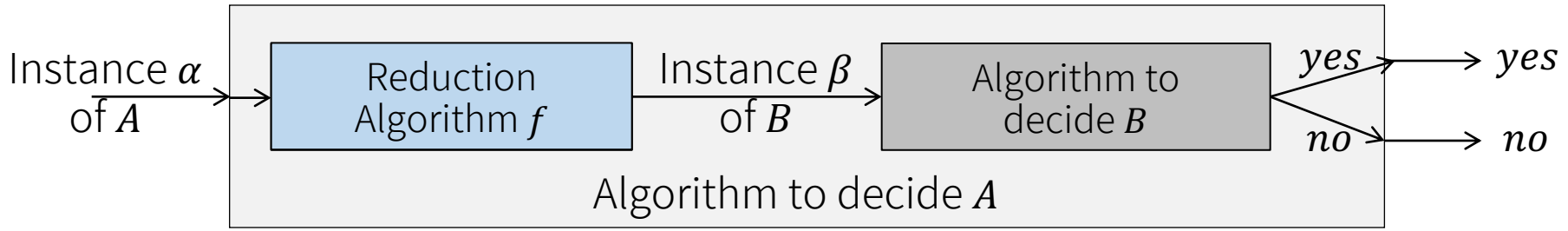


Complexity of the Algorithm for A?

Reduction of decision problems

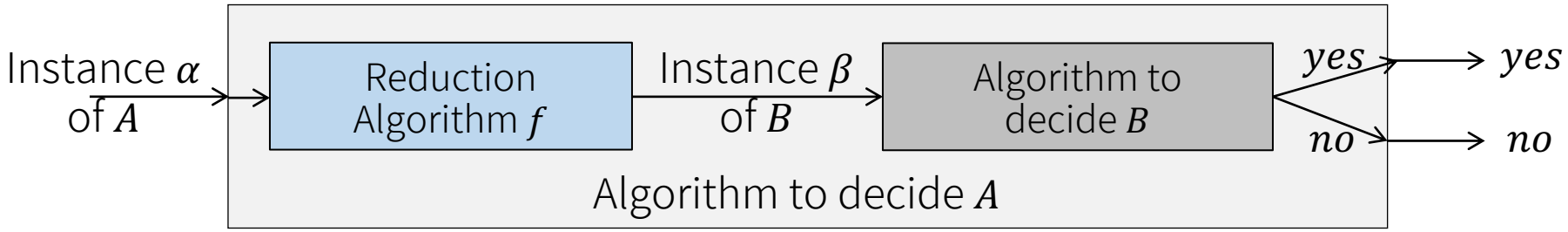
- We focus on decision problems here
- A reduction is an algorithm that transforms one problem instance to another
- For all α, $AlgA(\alpha) = 1$ iff $AlgB(f(\alpha)) = 1$
- $A \leq_p B$ means…
  - A can be reduced to B in polynomial time
  - A is no harder than B

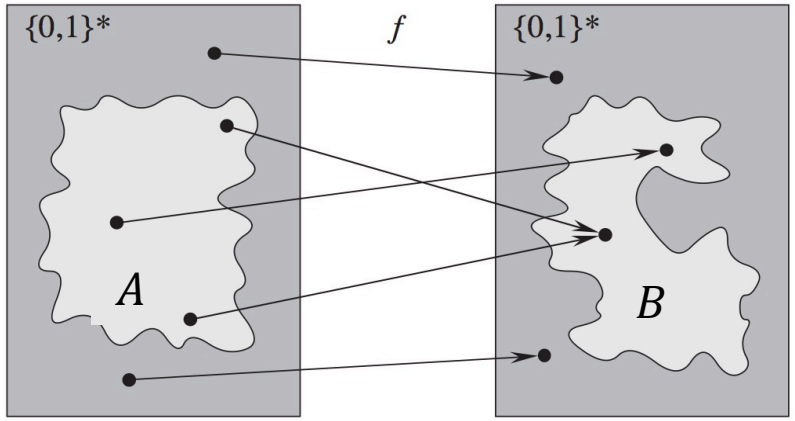
$AlgA(\alpha)$ outputs YES iff
$AlgB(f(\alpha))$ outputs YES



Polynomial-time reduction

- To prove that $A \leq_p B$, we need to construct a reduction algorithm f s.t.
  - The reduction algorithm f is in polynomial time
  - $AlgB(f(\alpha))$ outputs 1 if and only if $AlgA(\alpha)$ outputs 1





Q: Suppose
Problem A is “Can 2 divide x?”
Problem B is “Can y divide x?”
x and y are both nature numbers.
Answer the following questions:
(1) What are Problem A’s instances?
(2) What are Problem B’s instances?
(3) Show that $A \leq_p B$ by constructing a p-time reduction f

(1) $\alpha \in \{1, 2, 3, \dots\}$

(2) $\beta \in \{(1,1), (1,2), \dots, (2,1), (2,2), \dots\}$

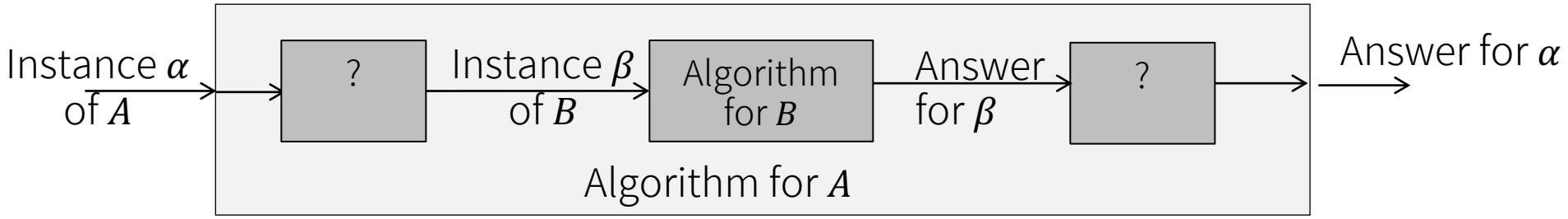
(3) $f(\alpha) = (\alpha, 2)$, this can be done in $O(1)$.

Also, trivially $AlgB(f(\alpha))$ outputs 1 if and only if $AlgA(\alpha)$.

Applications of reduction

1. **Designing algorithms**: if we can reduce A to B, then we can solve A using a given algorithm for B.
2. **Classifying problems**: if $A \leq_p B$ and $B \leq_p A$, then A and B have the same difficulty.
3. Proving limits: if $A \leq_p B$ and A is hard, then so is B
   - Use frequently in NP-completeness proofs.
   - 目的是證明 B 為 intractable，而非設計可用的演算法！

Applications of reduction: designing algorithms



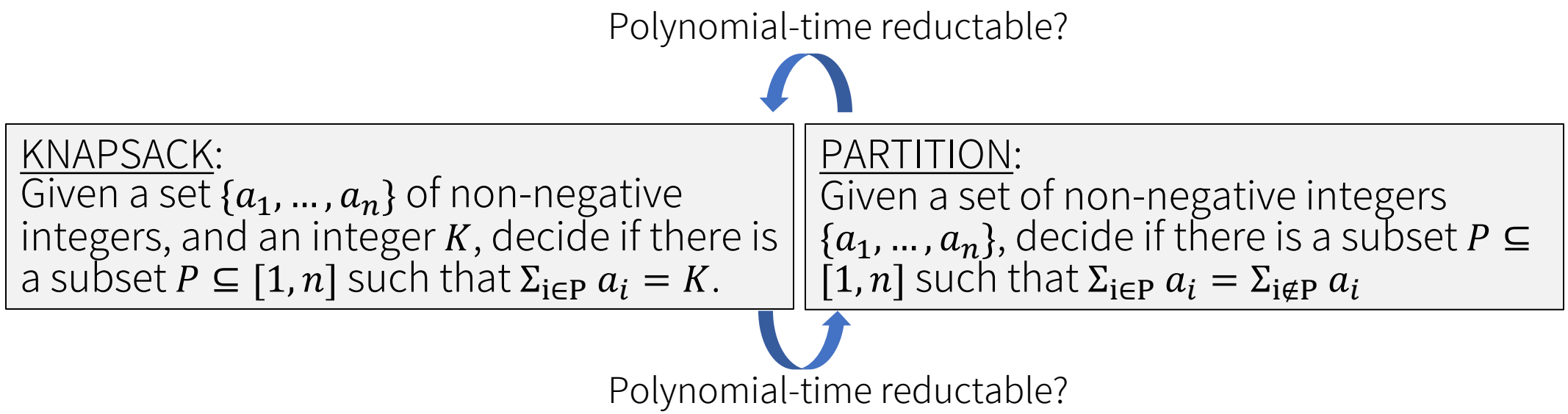
- Example #1:
  - A = all-pairs shortest path problem
  - B = single-source shortest path problem
- Example #2:
  - A = finding arbitrage opportunity
  - B = detecting negative cycles
- Any other examples?

Applications of reduction: classifying problems

Goal: show KNAPSACK and PARTITION have equal difficulty

Approach: show that

1. PARTITION $\leq_p$ KNAPSACK
2. KNAPSACK $\leq_p$ PARTITION

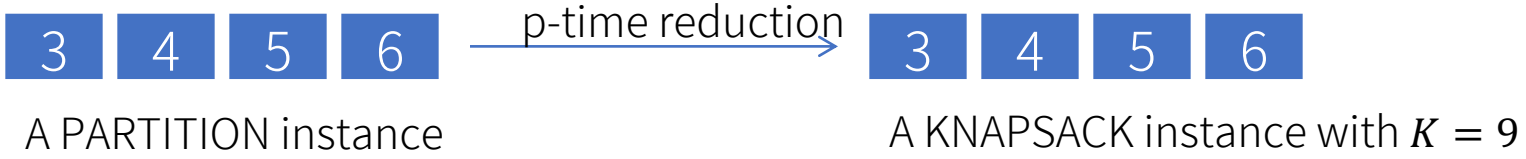


練習：證明 PARTITION $\leq_p$ KNAPSACK

KNAPSACK:
Given a set $\{a_1, \ldots, a_n\}$ of non-negative integers, and an integer K, decide if there is a subset $P \subseteq [1, n]$ such that $\Sigma_{i \in P}\, a_i = K$.

PARTITION:
Given a set of non-negative integers $\{a_1, \ldots, a_n\}$, decide if there is a subset $P \subseteq [1, n]$ such that $\Sigma_{i \in P}\, a_i = \Sigma_{i \notin P}\, a_i$

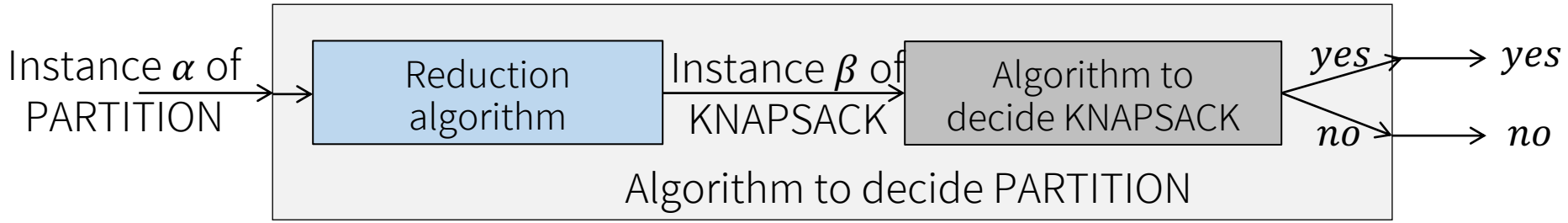
- If we can solve KNAPSACK, how can we use that to solve PARTITION?
- Polynomial-time reduction: Set $K = \frac{1}{2}\Sigma_{i \in 1,\ldots,n} a_i$



A PARTITION instance

A KNAPSACK instance with $K = 9$

練習：證明 PARTITION $\leq_p$ KNAPSACK



- If we can solve KNAPSACK, how can we use that to solve PARTITION?
- Polynomial-time reduction: Set $K = \frac{1}{2}\Sigma_{i\in 1,\ldots,n}a_i$
- Correctness proof: show that $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes if and only if $\mathbf{PARTITION}(\{a_1, \ldots, a_n\})$ returns yes

練習：證明 PARTITION $\leq_p$ KNAPSACK

- Polynomial-time reduction: Set $K = \frac{1}{2}\Sigma_{i\in 1,\ldots,n} a_i$
- Correctness proof: show that $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes if and only if $\mathbf{PARTITION}(\{a_1, \ldots, a_n\})$ returns yes

(1) If $\mathbf{PARTITION}(\{a_1, \ldots, a_n\})$ returns yes -> $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes

- If $\mathbf{PARTITION}(\{a_1, \ldots, a_n\})$ returns yes, then there is a subset $P \subseteq [1, n]$ such that $\Sigma_{\mathrm{i}\in\mathrm{P}}\, a_i = \Sigma_{\mathrm{i}\notin\mathrm{P}}\, a_i = K$.
- Thus, $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ will return yes

(2) If $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes -> $\mathbf{PARTITION}(\{a_1, \ldots, a_n\})$ returns yes

- If $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes, then there is a subset $P \subseteq [1, n]$ such that $\Sigma_{\mathrm{i}\in\mathrm{P}}\, a_i = K$.
- The rest (not in P) must sum up to K as well. Thus, $\mathbf{PARTITION}(\{a_1, \ldots, a_n\})$ will return yes

練習：證明 KNAPSACK $\leq_p$ PARTITION

KNAPSACK:
Given a set $\{a_1, \ldots, a_n\}$ of non-negative integers, and an integer K, decide if there is a subset $P \subseteq [1, n]$ such that $\Sigma_{i \in P}\, a_i = K$.

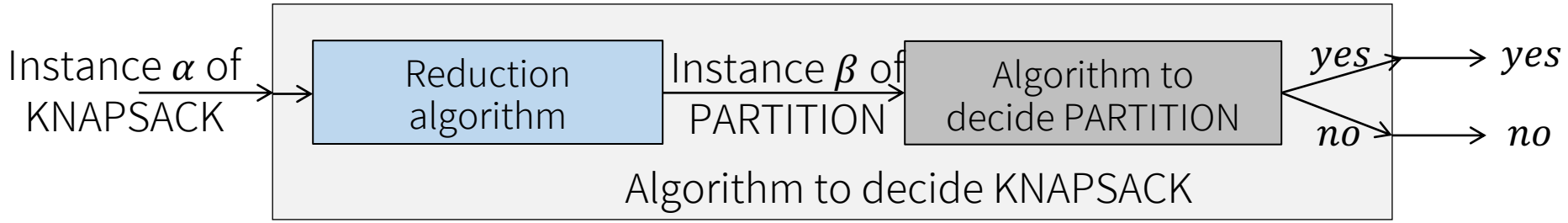
PARTITION:
Given a set of non-negative integers $\{a_1, \ldots, a_n\}$, decide if there is a subset $P \subseteq [1, n]$ such that $\Sigma_{i \in P}\, a_i = \Sigma_{i \notin P}\, a_i$

- If we can solve PARTITION, how can we use that to solve KNAPSACK?
- Polynomial-time reduction: Add $a_{n+1} = |H - 2K|$, where $H = \Sigma_{i \in 1,\ldots,n} a_i$

A KNAPSACK instance with K = 7

A PARTITION instance

練習：證明 KNAPSACK $\leq_p$ PARTITION

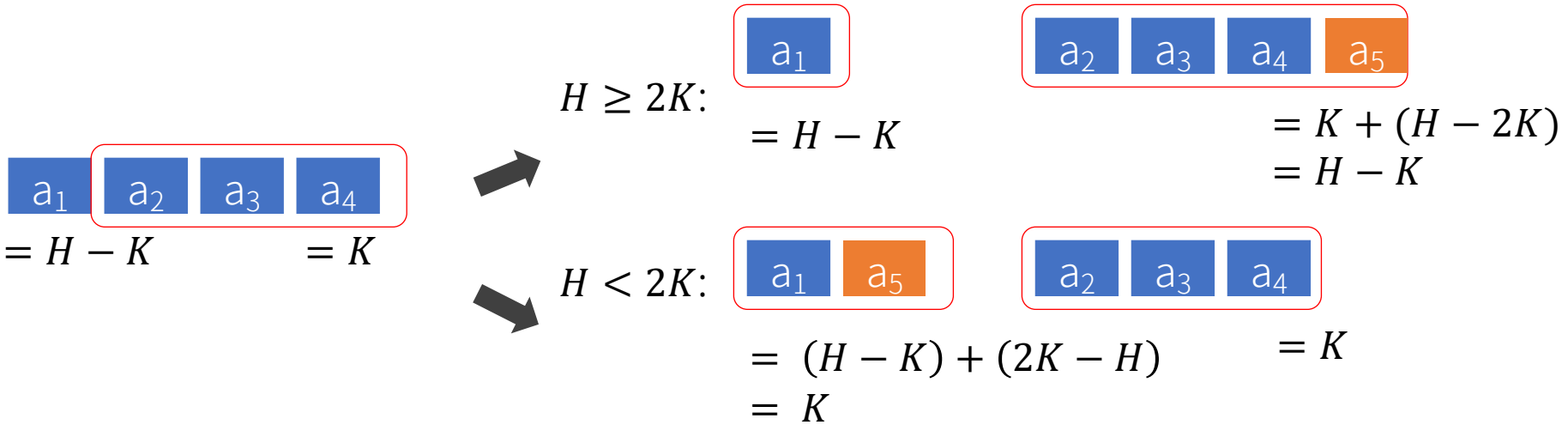


- If we can solve PARTITION, how can we use that to solve KNAPSACK?
- Polynomial-time reduction: Add $a_{n+1} = |H - 2K|$, where $H = \Sigma_{i \in 1,\ldots,n} a_i$
- Correctness proof: show that $\mathbf{PARTITION}(\{a_1, \ldots, a_n, a_{n+1}\})$ returns yes if and only if $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes

練習：證明 KNAPSACK $\leq_p$ PARTITION

- Polynomial-time reduction: Add $a_{n+1} = |H - 2K|$, where $H = \Sigma_{i \in 1,\ldots,n} a_i$
- Correctness proof: show that $\mathbf{PARTITION}(\{a_1, \ldots, a_n, a_{n+1}\})$ returns yes if and only if $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes

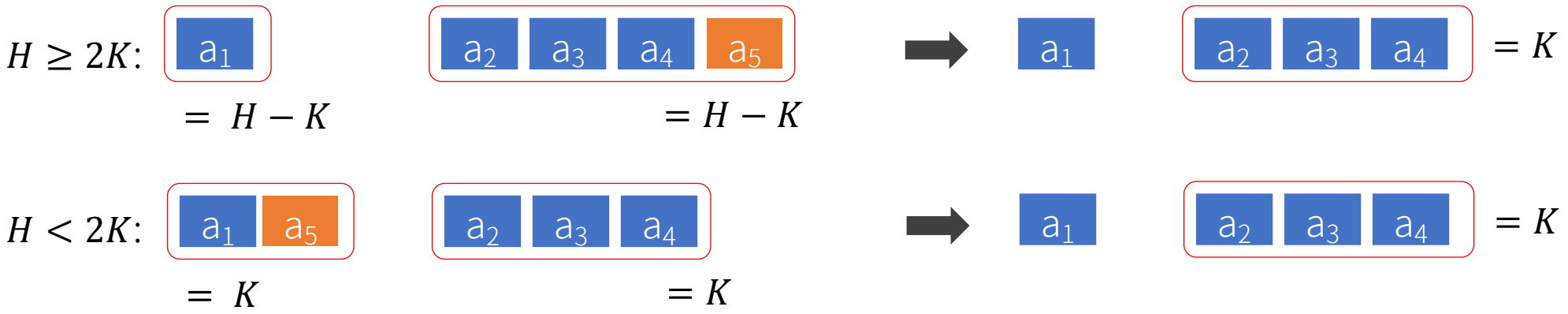
(1) If $\mathbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes -> $\mathbf{PARTITION}(\{a_1, \ldots, a_n, a_{n+1}\})$ returns yes



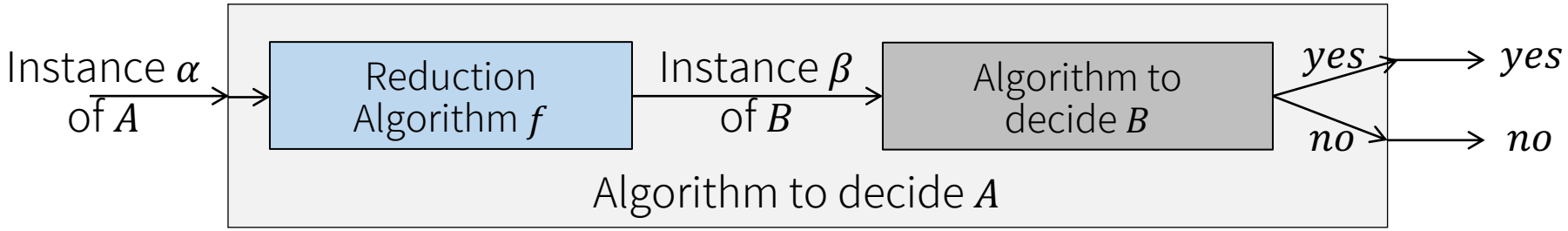
練習：證明 KNAPSACK $\leq_p$ PARTITION

- Polynomial-time reduction: Add $a_{n+1} = |H - 2K|$, where $H = \Sigma_{i \in 1,\ldots,n} a_i$
- Correctness proof: show that $\textbf{PARTITION}(\{a_1, \ldots, a_n, a_{n+1}\})$ returns yes if and only if $\textbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes

(2) If $\textbf{PARTITION}(\{a_1, \ldots, a_n, a_{n+1}\})$ returns yes -> $\textbf{KNAPSACK}(\{a_1, \ldots, a_n\}, K)$ returns yes



Applications of reduction: proving limits



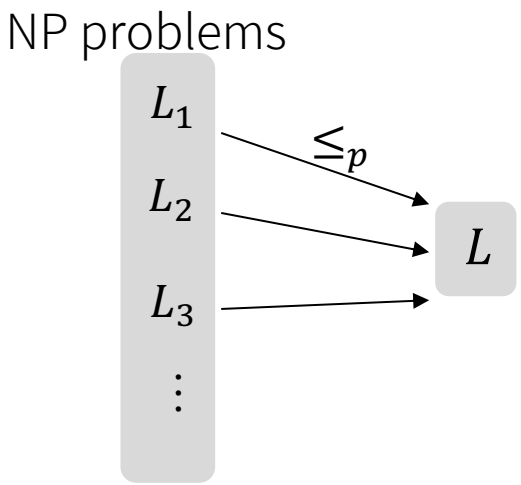
- A common usage in NP-completeness proofs
  - Known: A is NPC
  - Goal: prove that B is at least as hard as A (i.e., NP-hard)
  - Approach: show that $A \leq_p B$ by constructing a polynomial-time reduction algorithm to convert every α to β

Q: 如果 reduction 不是 polynomial-time，這個證明流程還成立嗎？

Proving NP-completeness

Proving NP-Completeness

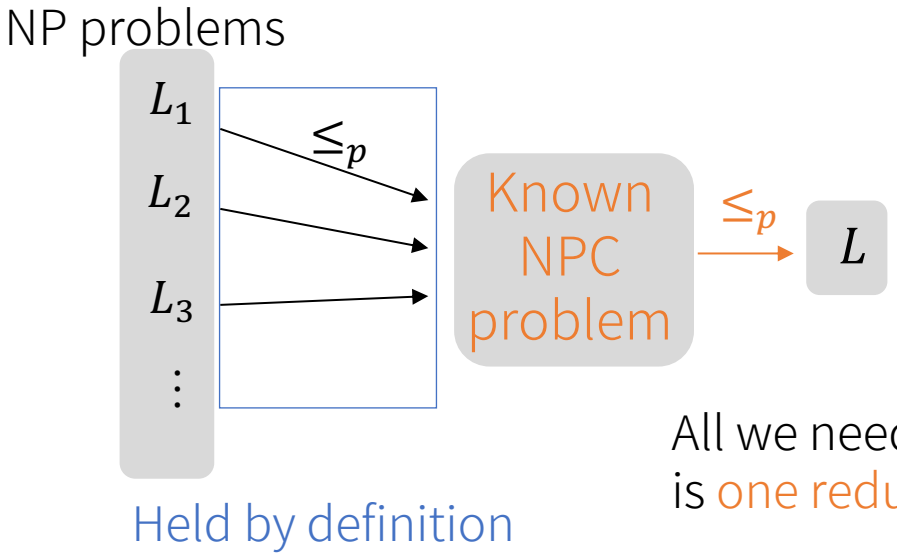
- Class NP-Complete (NPC): class of decision problems in both NP and NP-hard
- In other words, a decision problem L is NP-complete if
  1. $L \in$ NP
  2. $L \in$ NP-Hard (that is, $L' \leq_p L$ for every $L' \in$ NP)



Q: How to prove NP-Hard (every NP problem reduces to L)?

Proving NP-Completeness

- Class NP-Complete (NPC): class of decision problems in both NP and NP-hard
- In other words, a decision problem L is NP-complete if
  1. $L \in$ NP
  2. $L \in$ NP-Hard (that is, $L' \leq_p L$ for every $L' \in$ NP)
- Note, if $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$

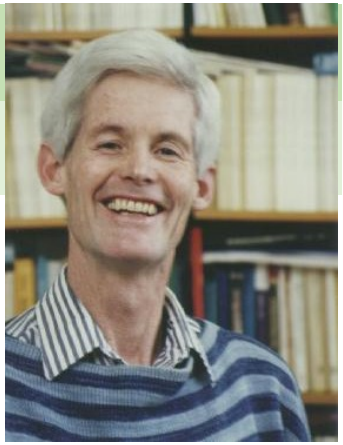


The first NP-Complete problem: Circuit-Satisfiability Problem

CIRCUIT-SAT = {<C>: C is a satisfiable Boolean combinational circuit}

Cook's Theorem

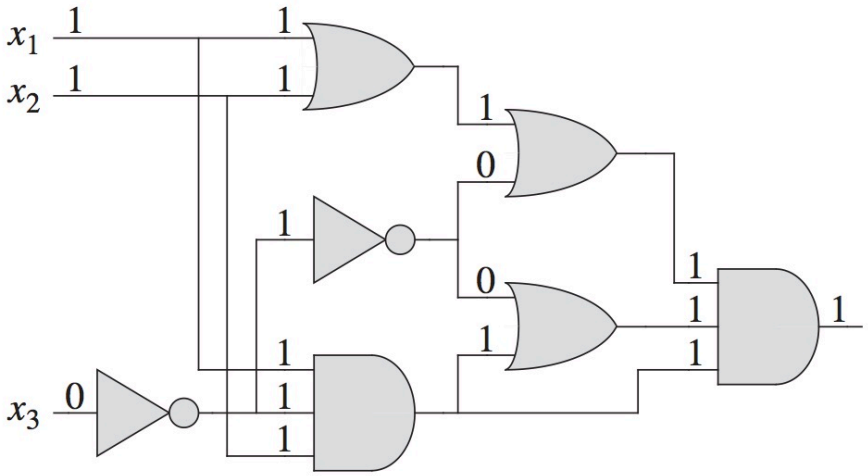
The Circuit-Satisfiability Problem (CIRCUIT-SAT) is NP-complete

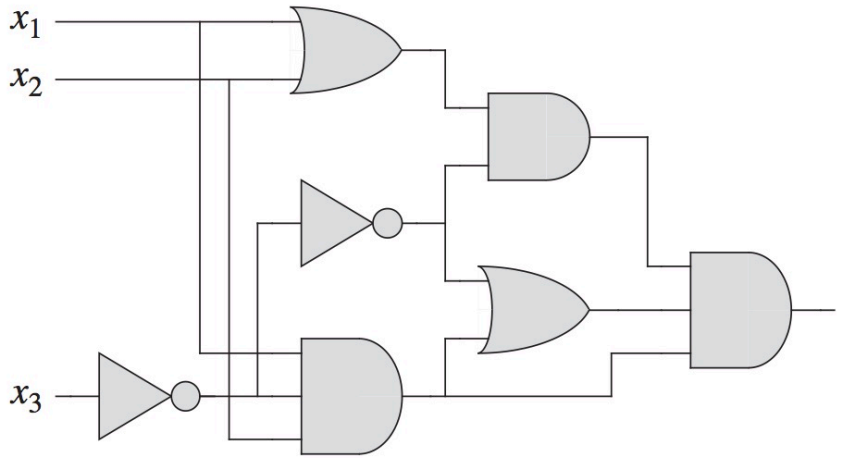


Stephen Cook
Turing Award (1982)

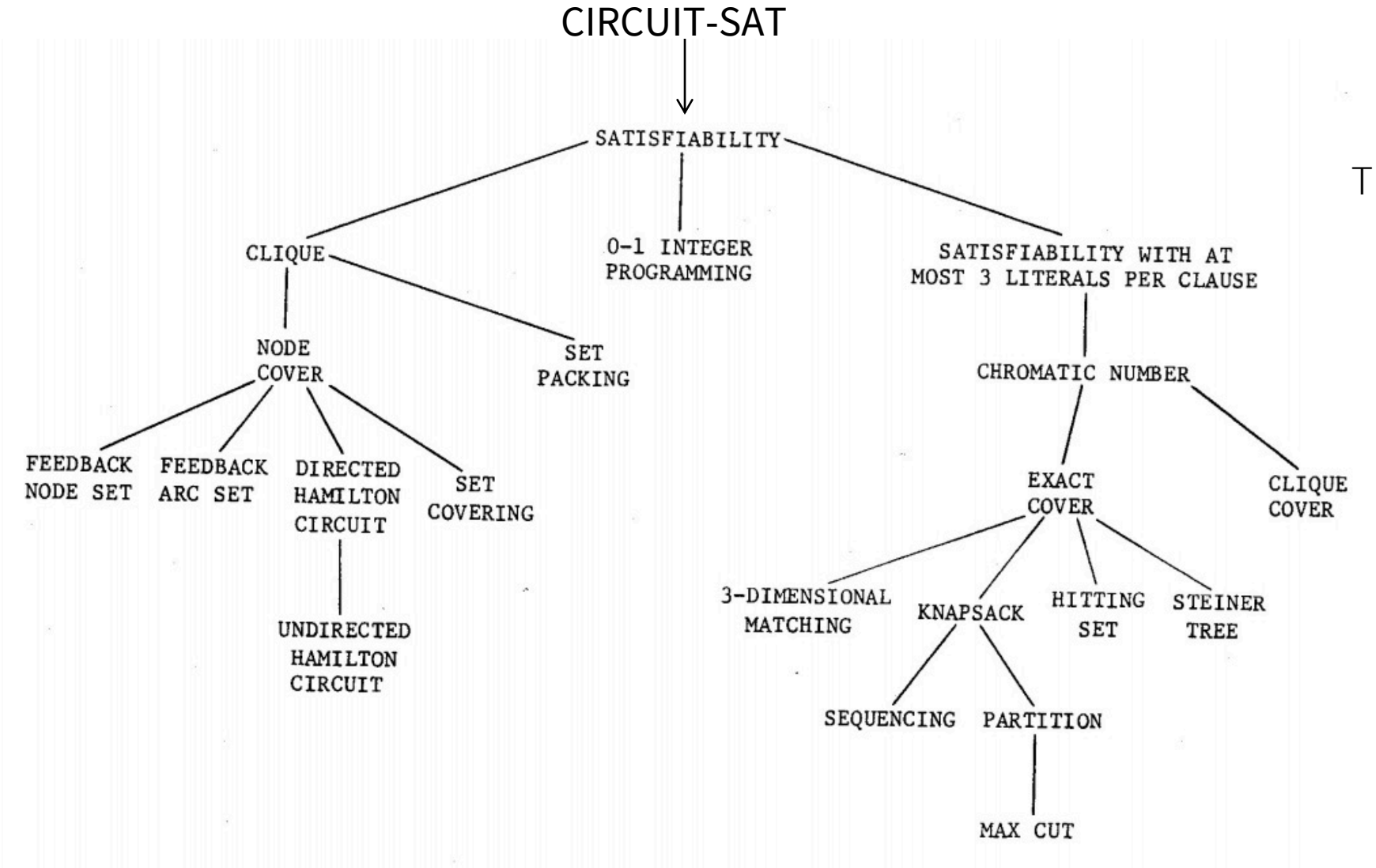
The first NP-Complete problem: Circuit-Satisfiability Problem

- Given a Boolean combinational circuit composed of AND, OR, and NOT gates, is it satisfiable?
  - Satisfiable = there exists an assignment s.t. the circuit outputs 1





Karp's 21 NP-complete problems

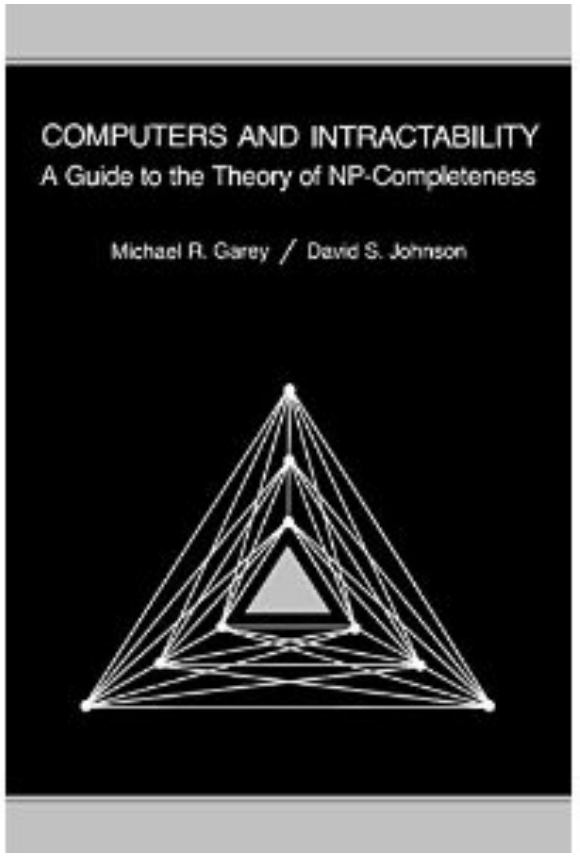


Richard M. Karp
Turing Award (1985)

More NP-complete problems

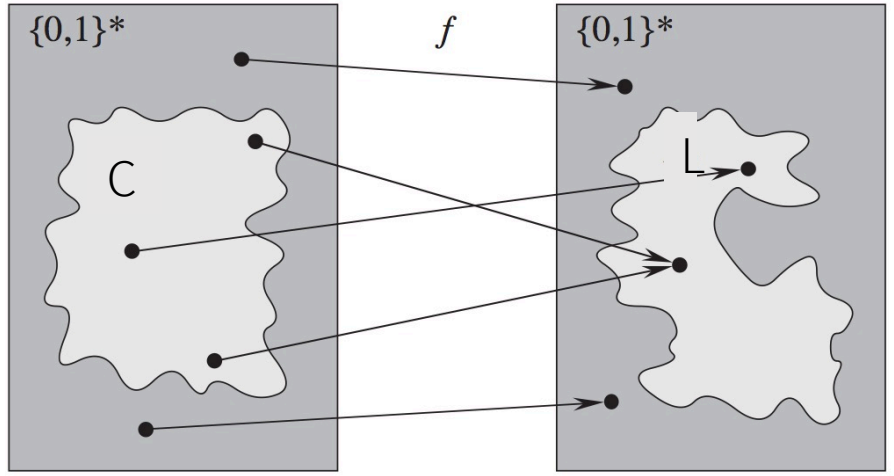
- "Computers and Intractability" by Garey and Johnson includes more than 300 NP-complete problems
  - Most cited citations in CS

1. M R Garey, D S Johnson
   Computers and Intractability: A Guide to the Theory of NP-Completeness 1979
   8562
2. T H Cormen, C E Leiserson, R L Rivest
   Introduction to Algorithms 1991
   7126



Proving NP-Completeness

$L \in$ NP-Complete $\Leftrightarrow L \in$ NP and $L \in$ NP-hard



Step-by-step approach for proving L in NPC:

1. Prove $L \in$ NP
2. Prove $L \in$ NP-hard
   - ① Select a known NPC problem C
   - ② Construct a reduction f transforming every instance of C to an instance of L
   - ③ Prove that x in C if and only if $f(x)$ in L for all x in $\{0,1\}^*$
   - ④ Prove that f is a polynomial-time transformation

Fun examples

差一點 差很多之 complexity class

猜猜以下每組中哪一個 in P (if P≠NP)？

Shortest simple path	vs.	Longest simple path
Eulerian tour	vs.	Hamiltonian cycle
LCS with two input sequences	vs.	LCS with arbitrary input sequences
Degree-constrained spanning tree	vs.	Minimum spanning tree

How about these games?

Sudoku

4	1						3	7
6	2						9	8
		5	9		2	1		
		2	7		3	4		
				5				
		1	4		6	9		
		4	3		8	6		
2	5						4	3
9	6						1	2

Candy Crush

MineSweeper

別解問題の計算量と完全性、およびパズルへの応用

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概　要

問題 Π に対する別解問題 (ASP) というのは、Π のインスタンス x とそれに対する 1 つの解 s が与えられた時に、x の s 以外の解を求める問題のことである。(新しい問題クラスとしての ASP の概念は Ueda と Nagao による。) 本論文では n 個の解が与えられた時にもう 1 つの解を求める問題 (n-ASP) について考察する。特に、対応する解への変換も多項式時間で行えるような多項式時間 parsimonious 還元に関する完全性である ASP 完全性について考察する。

応用として、本論文では 3 つの有名なパズル、スリザーリンクとカックロ (クロスサム) とナンバープレース (数独) についてその ASP 完全性 (NP 完全性を含意する) を示す。

Complexity and Completeness of Finding Another Solution and Its Application to Puzzles

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Abstract

The Another Solution Problem (ASP) of a problem Π is the following problem: for a given instance x of Π and a solution s to it, find a solution to x other than s. (The notion of ASP as a new class of problems was first introduced by Ueda and Nagao.) In this paper we consider n-ASP, the problem to find another solution when n solutions are given. In particular we consider ASP-completeness, the completeness with respect to the parsimonious reductions which allow polynomial-time transformation of solutions.

As an application, we prove the ASP-completeness (which implies NP-completeness) of three popular puzzles: Slither Link, Cross Sum, and Number Place.

isnphard.com/g/bejeweled/

Bejeweled, Candy Crush and other match-three games are (NP-)Hard!

What is this all about?

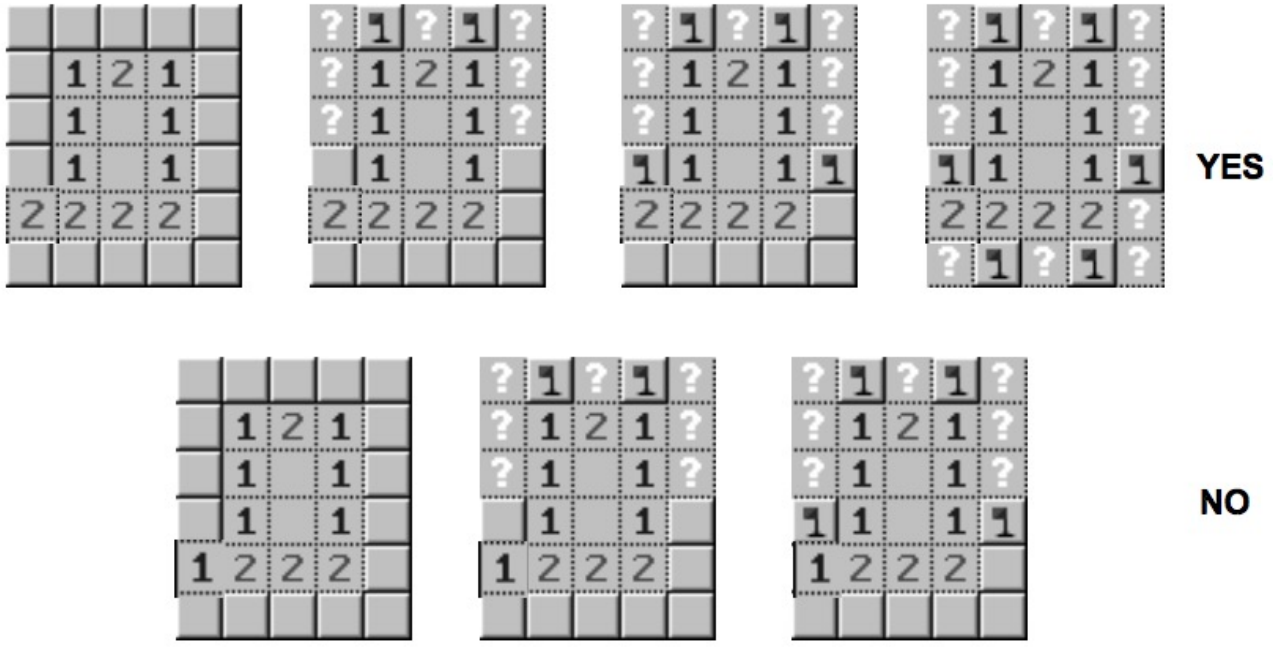
This is an implementation of the reduction provided in the paper *Bejeweled, Candy Crush and other Match-Three Games are (NP)-Hard* which has been accepted for presentation at the *2014 IEEE Conference on Computational Intelligence and Games (CIG 2014)*. To find more about what NP-Hard means you can read this blog post or the corresponding page on Wikipedia.

About the authors

We are an Italian group of three people: Luciano Gualà, Stefano Leucci, and Emanuele Natale. We had the weird idea to spend our weekends proving that Candy Crush Saga is NP-Hard. We also thought that it was nice to put online an implementation of our hardness reduction... so here it is!

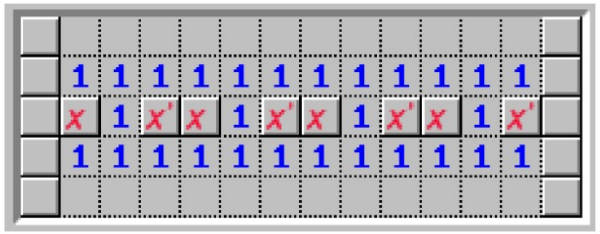
Minesweeper consistency problem is also NP-complete

- **Minesweeper consistency problem**: Given a state of what purports to be a Minesweeper game, is it logically consistent?

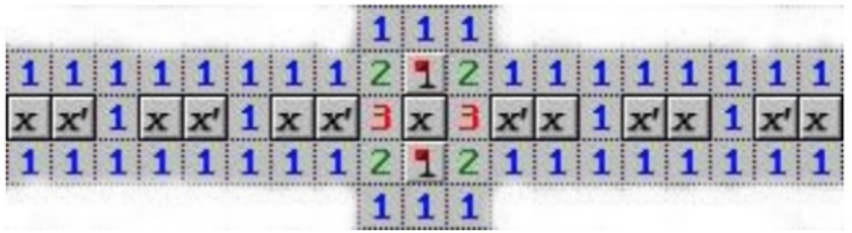


http://www.cs.princeton.edu/~wayne/cs423/lectures/np-complete-4up.pdf

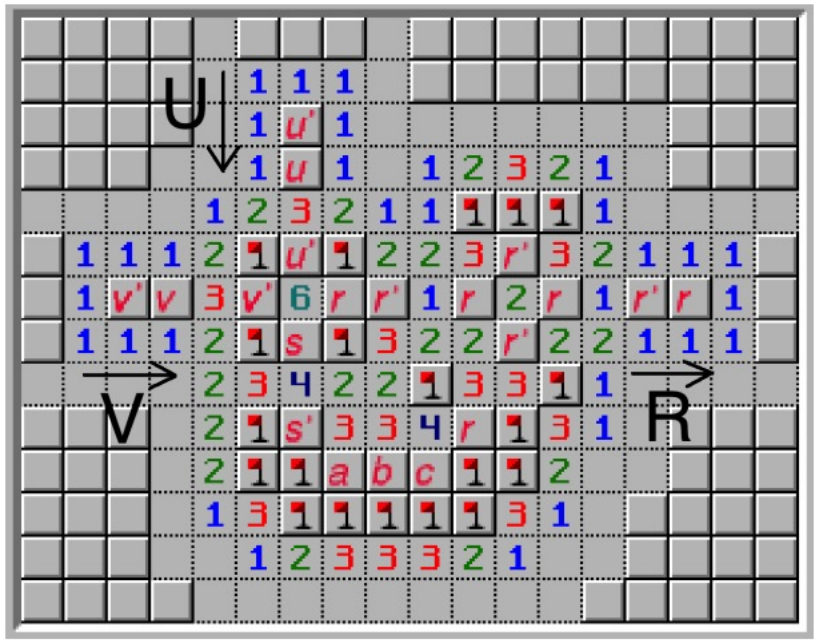
Wire



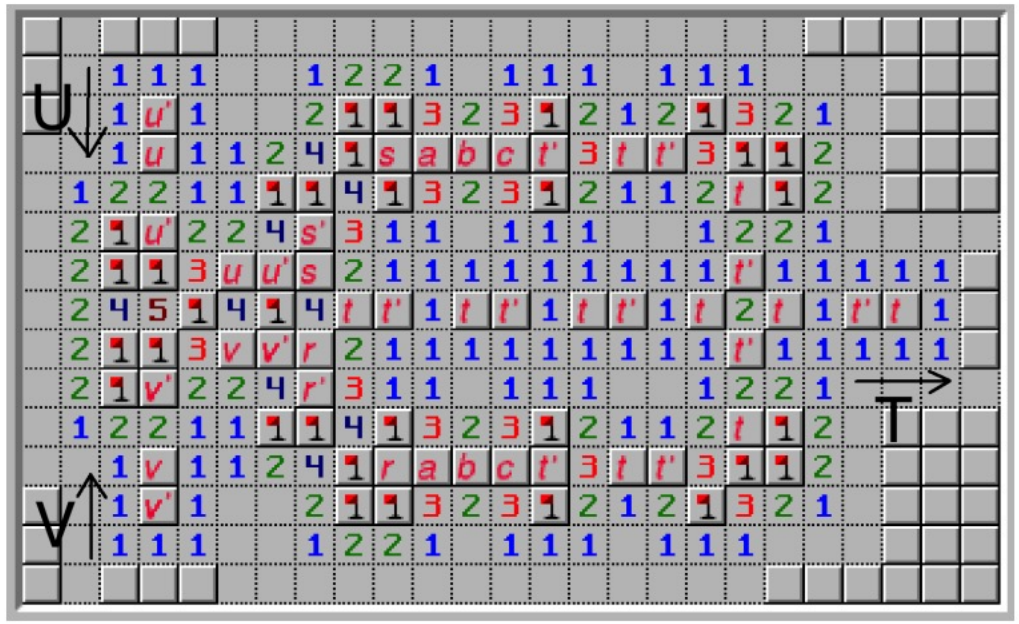
NOT Gate



OR Gate



AND Gate



http://web.math.ucsb.edu/~padraic/ucsb_2014_15/ccs_problem_solving_w2015/NP3.pdf

Exercise

True/False: If A can be reduced to B in polynomial time ($A \leq_p B$), then A is no harder than B
True/False: if $A \leq_p B$ and $B \leq_p C$, then $A \leq_p C$
True/False: NP-Complete problems can be reduced to each other in polynomial time.
True/False: if $A \leq_p B$ and B is in NPC, then A is in NPC too.
True/False: if $A \leq_p B$ and A is in NPC, then B is in NPC too.

Appendix: Polynomial-time Solving & Polynomial-time Verification

Chap. 34.1, 34.2

Abstract problems

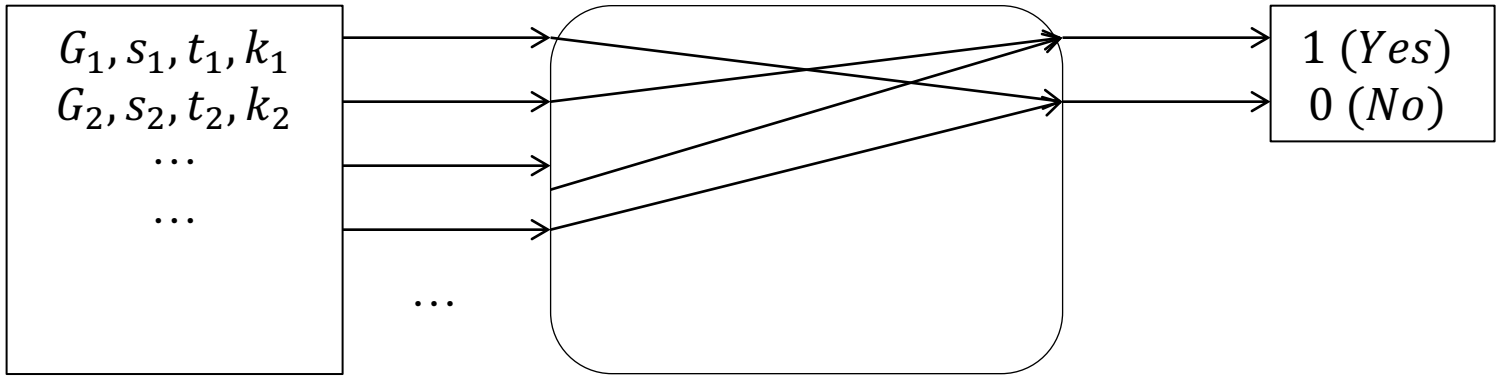
- I: the set of problem instances
- S: the set of problem solutions
- Q: an abstract problem, defined as a binary relation on I and S

Example of a decision problem, $\mathrm{SHORTEST-PATH}$:

$I: < G, src, dest, k >$
所有可能的圖、起訖點、長度上限

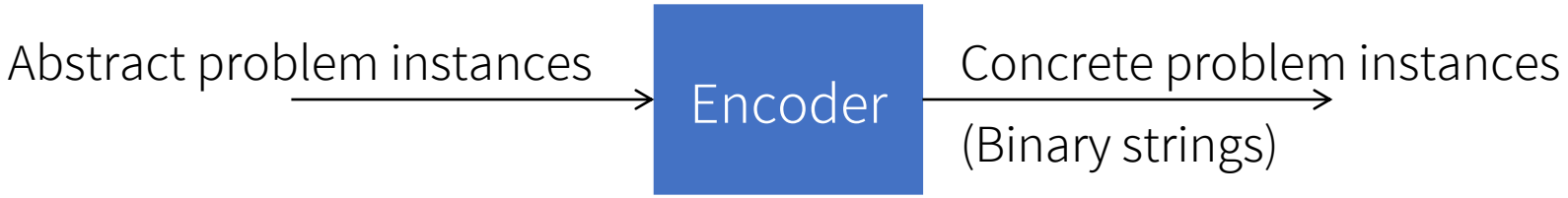
Q: SHORTEST−PATH
是否找得到最短路徑長度 $\leq k$？

S: $\{0,1\}$



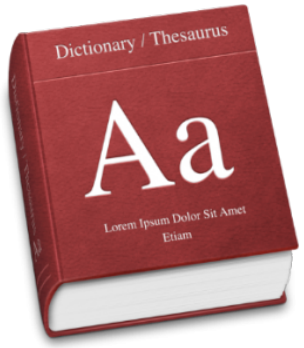
Encoding of problem instances

- Convert an abstract problem instance into a binary string that is fed to a computer program



- A concrete problem is polynomial-time solvable if there exists an algorithm that solves any concrete instance of length n in time $O(n^k)$ for some constant k
  - Solvable = can produce a solution

Formal-language theory



- An Alphabet Σ = a finite set of symbols
- A Language L over Σ = any set of strings made up from alphabet Σ
- Notations
  - ε: empty string
  - ϕ: empty language
  - Σ^*: the language of all strings over Σ

Q: In English, what is the alphabet Σ and the language L?

- $\Sigma = \{a, b, c, \ldots, z\}$
- L = a set of English words

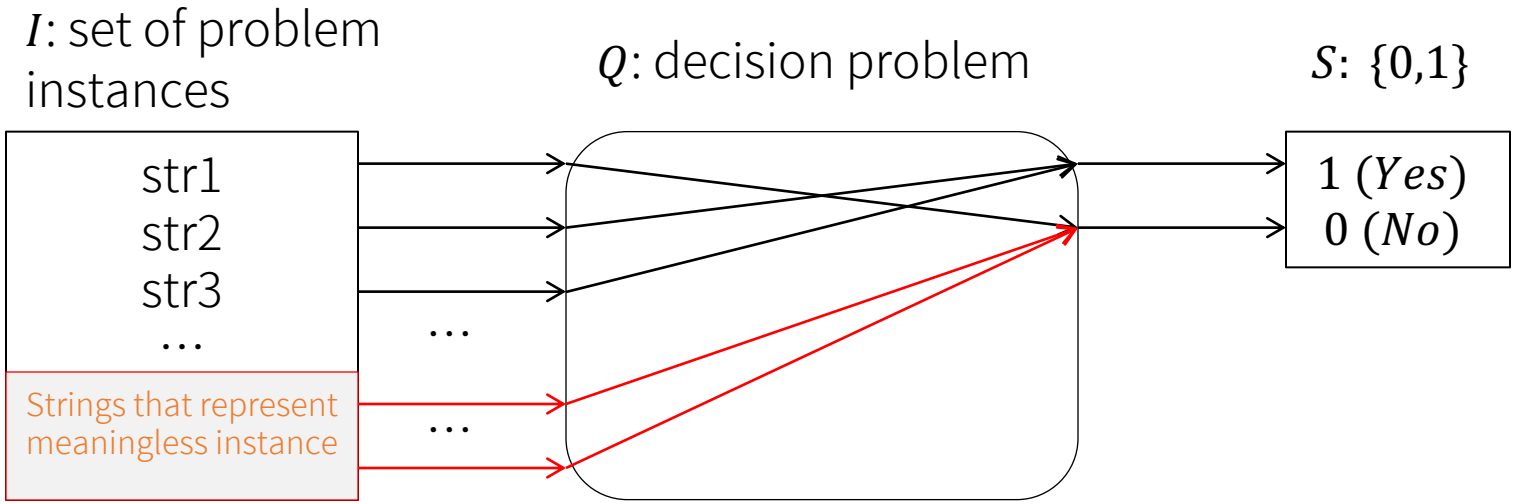
Q: In the language of binary representation of prime numbers, what is the alphabet Σ and the language L?

- $\Sigma = \{0,1\}$
- $L = \{10, 11, 101, 111, \ldots\}$

Q: If $\Sigma = \{0,1\}$, what is Σ^*?

- $\Sigma^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\}$

Representation of a decision problem

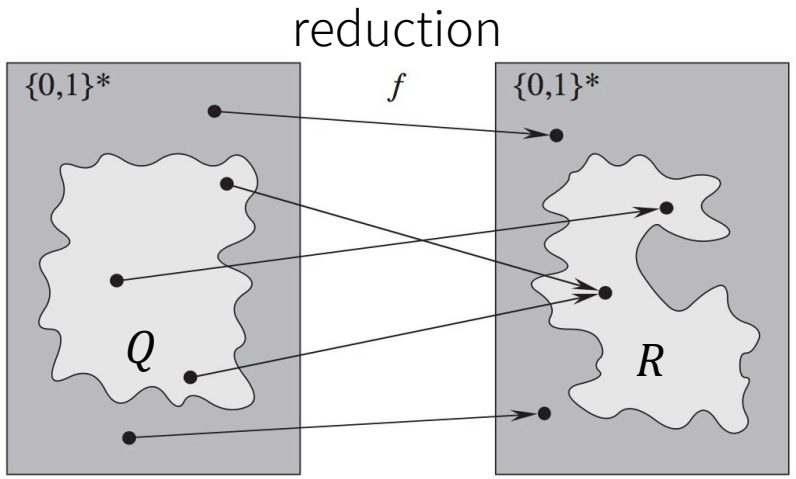


- I = the set of problem instances = Σ^*, where $\Sigma = \{0, 1\}$
- Q = a decision problem = a language L over $\Sigma = \{0, 1\}$ such that $L = \{x \in \Sigma^*: Q(x) = 1\}$
  - 以答案為 1 的 instances 定義 decision problem Q!
  - $L = \{str2, str3\}$ in this simple example

Decision problem

A decision problem Q can be defined as
a language L over $\Sigma = \{0, 1\}$ such that $L = \{x \in \Sigma^* : Q(x) = 1\}$

- 以答案為 1 的 instances 定義 decision problem Q
- Q can be reduced to R means $x \in Q \Leftrightarrow f(x) \in R$



Definition of the complexity class P in language-theoretic framework

- An algorithm A accepts a string $x \in \{0,1\}^*$ if $A(x) = 1$
- An algorithm A rejects a string $x \in \{0,1\}^*$ if $A(x) = 0$
- An algorithm A accepts a language L means…
  - A accepts every string $x \in L$
  - 對所有在 L 中的 string 都要能正確地輸出 YES
  - 對不在 L 中的 string 可能輸出 NO, 也可能無法判斷（e.g., loop forever)
- An algorithm A decides a language L if A accepts L and A rejects every string $x \notin L$
  - 對所有的string都要能正確地輸出YES/NO

Definition of the complexity class P in language-theoretic framework

- Class P: a class of decision problems solvable in p-time
  - That is, given an instance x of a decision problem Q, its solution $Q(x)$ (i.e., YES or NO) can be found in polynomial time
- Hence, an alternative definition of class P:

Definition of Class P

$P = \{L \subseteq \{0,1\}^* |$ there exists an algorithm that decides L in p-time$\}$

Theorem 34.2

P is also the class of language that can be accepted in polynomial time.
$P = \{L \subseteq \{0,1\}^* | L$ is accepted by a polynomial-time algorithm$\}$

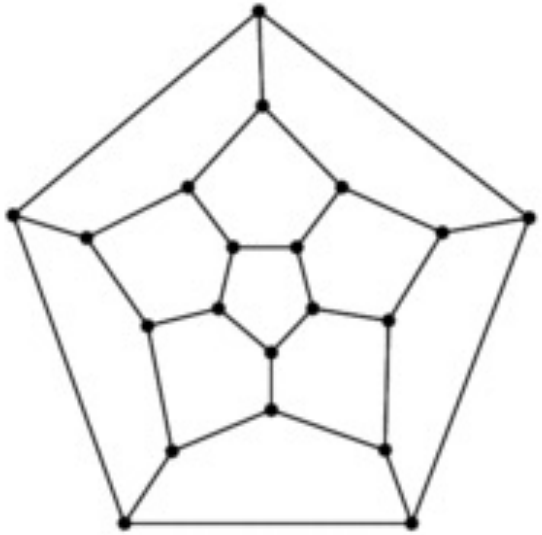
Hamiltonian-Cycle Problem

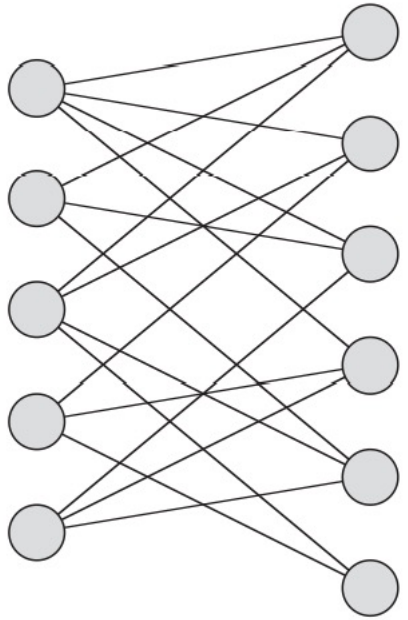
$\mathrm{HAM\text{-}CYCLE} = \{\langle G \rangle \mid G \text{ has a Hamiltonian cycle}\}$

Q: Can you find a cycle visiting each vertex exactly once in the examples?

Q: Is this language decidable?

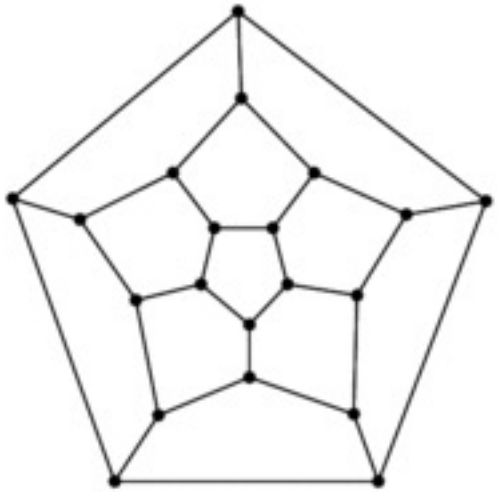
Q: Is this language decidable in polynomial time?





Polynomial-time Verification: Hamiltonian-cycle problem

- Suppose a nice TA gives you a certificate – a vertex sequence that forms a Hamiltonian cycle in a given graph G
- With this certificate, how much time does it take to verify that G indeed contains a Hamiltonian cycle?

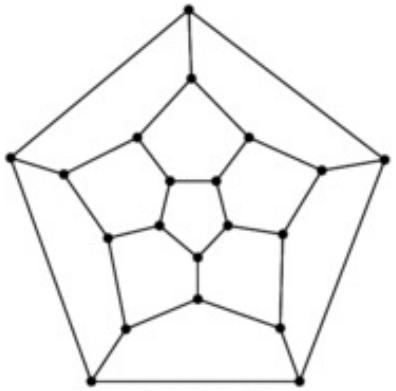


Verification algorithms

- Verification algorithms verify memberships in language with the help of certificates

$$\text{HAM}-\text{CYCLE} = \{\langle G \rangle \mid G \text{ has a Hamiltonian cycle}\}$$

Input x



Certificate y

Verification algorithm: Is y a Hamiltonian cycle in the graph (encoded in x)?

YES

(x is in HAM-CYCLE)

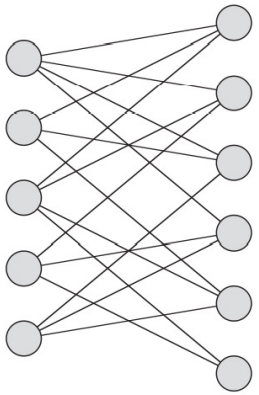
There exists a certificate for each YES instance

Verification algorithms

- Verification algorithms verify memberships in language

$$\text{HAM}-\text{CYCLE} = \{\langle G \rangle \mid G \text{ has a Hamiltonian cycle}\}$$

Input x



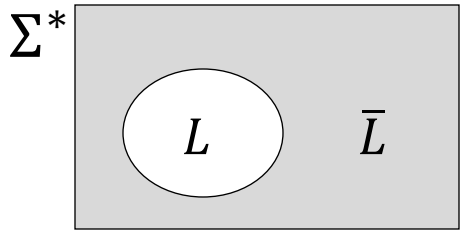
Certificate y ??

Verification algorithm: Is y a Hamiltonian cycle in the graph (encoded in x)?

NO

No conclusion

Co-NP

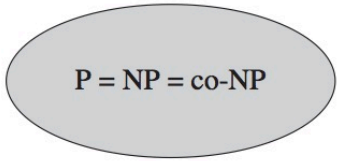


- Class co-NP: class of decision problems whose complement are in NP
  - Complement of $L = \bar{L} = \Sigma^* - L$
- $\mathbf{HAM{-}CYCLE}$: Given a graph G, is there a simple cycle that visits each vertex on G exactly once?
- $\overline{\mathbf{HAM{-}CYCLE}}$: Given a graph G, does G contain no Hamiltonian cycle?
- It's still an open problem whether $\mathbf{HAM{-}CYCLE} \in$ co-NP
  - Equivalently, whether $\overline{\mathbf{HAM{-}CYCLE}} \in$ NP
  - 是否能提供 certificate 證明 graph G 沒有 Hamiltonian cycle?

P, NP, and co-NP

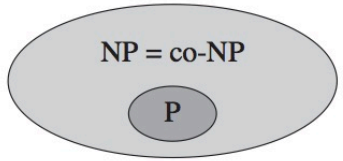
- Class P: class of decision problems solvable in $O(n^k)$
  - $O(n^k)$ means polynomial in the input size (k is a constant)
- Class NP: class of decision problems verifiable in $O(n^k)$
- Class co-NP: class of decision problems whose complement are in NP

If P = NP = co-NP



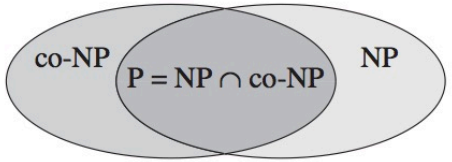
(a)

If P ≠ NP and NP = co-NP



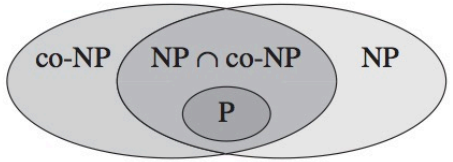
(b)

If P = NP ∩ co-NP



(c)

Most likely case!



(d)