Problem 5

> Refs:
>
> None

Subproblem (a)

(1)

$$dp(i,j) = \begin{cases} dp(i,j-1) + E(j,j-1) & \text{if } i < j-1 & \text{(Case 1)} \\ dp(i-1,j) + E(i-1,i) & \text{if } j < i-1 & \text{(Case 2)} \\ E(1,0) & \text{if } (i,j) = (0,1) & \text{(Case 3)} \\ E(0,1) & \text{if } (i,j) = (1,0) & \text{(Case 4)} \\ \min_{0\le k\le j-2}(dp(i,k) + E(j,k)) & \text{if } i = j-1,\, i \ne 0 & \text{(Case 5)} \\ \min_{0\le k\le i-2}(dp(k,j) + E(k,i)) & \text{if } j = i-1,\, j \ne 0 & \text{(Case 6)} \end{cases}$$

Case 1

Because $i < j-1$ and all points from S_0 to S_j must be chosen, S_{j-1} must be the second point on the way back. The path for $dp(i,j)$ should be something like this:

$$S_0 \to \text{some points} \to S_i \to S_N \to S_j \to S_{j-1} \to \text{some other points} \to S_0$$

We need to minimize the energy spent on $S_0 \to S_i$ and $S_{j-1} \to S_0$ combined. And $dp(i,j-1)$ does exactly that. Therefore $dp(i,j) = dp(i,j-1) + E(j,j-1)$.

Case 2

Because $j < i-1$ and all points from S_0 to S_i must be chosen, S_{i-1} must be the second last point on the way go. The path for $dp(i,j)$ should be something like this:

$$S_0 \to \text{some points} \to S_{i-1} \to S_i \to S_N \to S_j \to \text{some other points} \to S_0$$

We need to minimize the energy spent on $S_0 \to S_{i-1}$ and $S_j \to S_0$ combined. And $dp(i-1,j)$ does exactly that. Therefore $dp(i,j) = dp(i-1,j) + E(i-1,i)$.

Case 3 & 4

These two cases are just base cases. Their paths are:

$$S_0 \to S_N \to S_1 \to S_0 \text{ and } S_0 \to S_1 \to S_N \to S_0$$

Therefore $dp(0,1) = E(1,0)$ and $dp(1,0) = E(0,1)$.

Case 5

In this case, the path should be something like this:

$$S_0 \to \text{some points} \to S_i = S_{j-1} \to S_N \to S_j \to S_k \to \text{some other points} \to S_0$$

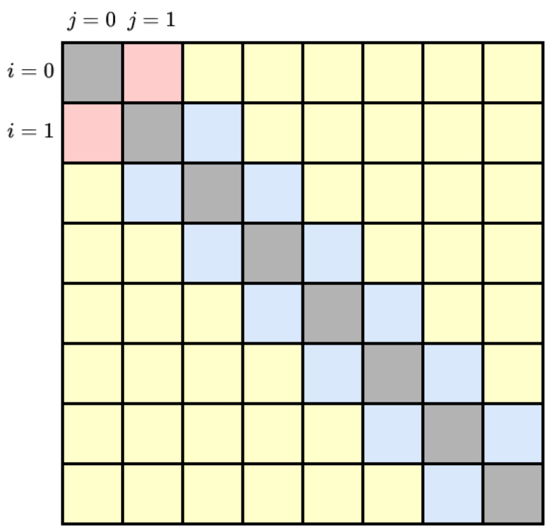
Because there are no guarantee of S_j's next point, we choose the $\min$ of all possible cases, that is, from $j-2$ to 0. Therefore $dp(i,j) = \min_{0 \le k \le j-2}(dp(i,k) + E(j,k))$

Case 6

$$S_0 \to \text{some points} \to S_k \to S_i \to S_N \to S_j = S_{i-1} \to \text{some other points} \to S_0$$

Similar to case 5, because there are no guarantee of S_i's previous point, we choose the $\min$ of all possible cases. Therefore $dp(i,j) = \min_{0 \le k \le i-2}(dp(k,j) + E(k,i))$

Time complexity



We can store $dp(i,j)$ in an 2d array like the picture above. Case 1 and 2 are the yellow ones. Case 3 and 4 are the red ones. Case 5 and 6 are the blue ones. Fill the array from left to right, then top to bottom.

The red ones can be directly calculated in $O(1)$ time, and there are 2. The yellow ones can be generated with the box on the left (or above for the bottom left half) in $O(1)$ time, and there are $O(N^2)$ yellow boxes. The blue ones can be calculated in $O(N)$ time with all values on the left (or above for the bottom left half), and there are $O(N)$ blue boxes.

The total time complexity is $2 \cdot O(1) + O(N^2) \cdot O(1) + O(N) \cdot O(N) = O(N^2)$.

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(2)

Explanation

Because every point in between must be visited exactly once in a valid path, S_{N-1} must be on the way go or back. And because no turning back in the middle is allowed, S_{N-1} must be either the last point on the way go or the first point on the way back. Therefore the optimal path must be $dp(N-1,k)$ or $dp(k,N-1)$ for some k.

The minimal cost is:

$$C = \min_{0 \le k \le N-2} (dp(N-1,k) + E(N-1,N) + E(N,k),\ dp(k,N-1) + E(k,N) + E(N,N-1))$$

When computing $dp(i,j)$, we don't need to save the entire table. We can keep only the current row, previous row, min of $dp(k,j) + E(k,j+1)$ of each column, reducing the space complexity to only $O(N)$.

Pseudocode

```
function solve_cost(N)
    prev_row = array(N)
    cur_row = array(N)
    col_min = array(N) with value INF
    cost_min = INF
    for i from 0 to N-1
        row_min = INF
        for j from 0 to N-1
            if i == j
                continue
            if i < j
                if i < j-1 /* Case 1 */
                    cur_row[j] = cur_row[j-1]+E(j,j-1)
                else if i==0 and j==1 /* Case 3 */
                    cur_row[j] = E(1,0)
                else /* Case5 */
                    cur_row[j] = row_min
                /* Update min for case 6 */
                col_min[j] = min(col_min[j], cur_row[j] + E(i, j+1))
            else
                if j < i-1 /* Case 2 */
                    cur_row[j] = prev_row[j] + E(i-1, i)
                else if i==1 and j==0 /* Case 4 */
```

```
                    cur_row[j] = E(0,1)
                else /* Case 6 */
                    cur_row[j] = col_min[j]
                /* Update min for case 5 */
                row_min = min(row_min, cur_row[j] + E(i+1,j))

            prev_row[j] = cur_row[j]
            if i == N-1
                cost_min = min(cost_min, cur_row[j] + E(N-1,N) + E(N,j))
            if j == N-1
                cost_min = min(cost_min, cur_row[j] + E(i,N) + E(N,N-1))
    return cost_min
```

Rewrite case 5 and 6 into:

$$dp(i,j) = \begin{cases} \min_{0\le k\le i-1}(dp(i,k) + E(i+1,k)) & \text{if } i = j-1,\, i \ne 0 \\ \min_{0\le k\le j-1}(dp(k,j) + E(k,j+1)) & \text{if } j = i-1,\, j \ne 0 \end{cases}$$

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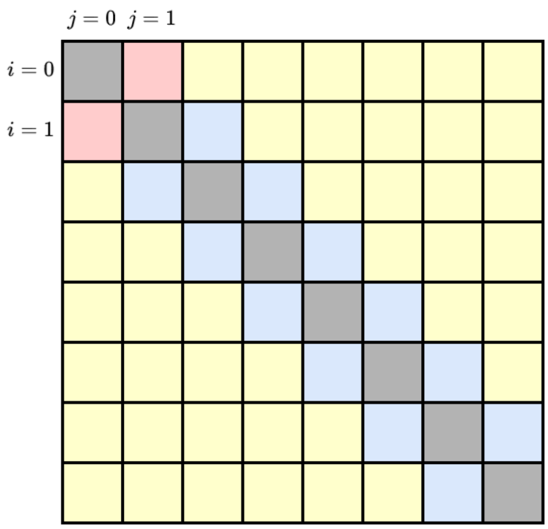
(3)



Since every yellow boxes must come from the left (or above for the bottom left half), and red boxes are base cases, only blue boxes' previous condition needs extra spaces to keep track of. And because there are $O(N)$ blue boxes, we only need $O(N)$ extra space to track their previous conditions.

The optimal path can be obtained by:

1. Solve the minimal cost
   - While solving minimal cost, use an extra array `parent` to store where each blue boxes comes from.
   - Also while solving minimal cost, when `cost_min` is updated, record which (i, j) combination it is, and return it.
2. Create a list `path` containing only S_N

3. While $(i, j) \neq (1, 0)$ or $(0, 1)$
   - While $|i - j| > 1$
     - If $i > j$, add i to `path` front, and decrease it by 1.
     - If $j > i$, add j to `path` back, and decrease it by 1.
   - Use the array `parent` to obtain previous condition, and add the larger one(between original i, j) to `path`'s front/back.
4. Add 1 to `path` front if $(i, j) = (1, 0)$. Add 1 to `path` back if $(i, j) = (0, 1)$.
5. Add 0 to front and back.
6. The list `path` is the optimal path.

Space complexity is $O(N)$ $(\text{solving minimal cost}) + O(N)$ $(\text{extra variables used}) = O(N)$.

Summing time complexity for each step is $O(N^2) + O(1) + O(N) + O(1) + O(1) = O(N^2)$.

Subproblem (b)

(1)

$dp(i, j, h)$: The minimal cost of $path(i, j)$ with h health remaining. $dp(i, j, h) = \infty$ if there are no possible path or $h \leq 0$.

$$dp(i,j,h) = \begin{cases} \infty & \text{if } h > H \text{ or } h \leq 0 & \text{(Impossible)} \\ dp(i, j-1, h) + E(j, j-1) & \text{if } i < j-1 & \text{(Case 1)} \\ dp(i-1, j, h+D_i) + E(i-1, i) & \text{if } j < i-1 & \text{(Case 2)} \\ E(1,0) & \text{if } (i,j) = (0,1),\, h = H & \text{(Case 3)} \\ \infty & \text{if } (i,j) = (0,1),\, h \neq H & \text{(Impossible)} \\ E(0,1) & \text{if } (i,j) = (1,0),\, h = H - D_1 & \text{(Case 4)} \\ \infty & \text{if } (i,j) = (1,0),\, h \neq H - D_1 & \text{(Impossible)} \\ \min_{0 \leq k \leq j-2}(dp(i,k,h) + E(j,k)) & \text{if } i = j-1,\, i \neq 0 & \text{(Case 5)} \\ \min_{0 \leq k \leq i-2}(dp(k,j,h+D_i) + E(k,i)) & \text{if } j = i-1,\, j \neq 0 & \text{(Case 6)} \end{cases}$$

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(2)

We can think of case 1, 2, 5, 6 as adding one more path to a path. This added point is marked red below:

$$S_0 \to \text{some points} \to S_i \to S_N \to \color{red}{S_j} \to S_{j-1} \to \text{some other points} \to S_0$$
$$S_0 \to \text{some points} \to S_{i-1} \to \color{red}{S_i} \to S_N \to S_j \to \text{some other points} \to S_0$$
$$S_0 \to S_N \to S_1 \to S_0$$
$$S_0 \to S_1 \to S_N \to S_0$$
$$S_0 \to \text{some points} \to S_i = S_{j-1} \to S_N \to \color{red}{S_j} \to S_k \to \text{some other points} \to S_0$$
$$S_0 \to \text{some points} \to S_k \to \color{red}{S_i} \to S_N \to S_j = S_{i-1} \to \text{some other points} \to S_0$$

Case 1

Because the added point is on the way back, the health is the same, therefore
$dp(i,j,h) = dp(i,j-1,h) + E(j,j-1)$.

Case 2

Because the added point is on the way go, the health in previous state is higher, therefore
$dp(i,j,h) = dp(i-1,j,h+D_i) + E(i-1,i)$.

Case 3 & 4

These two are base cases, and the path is shown above.

Case 5

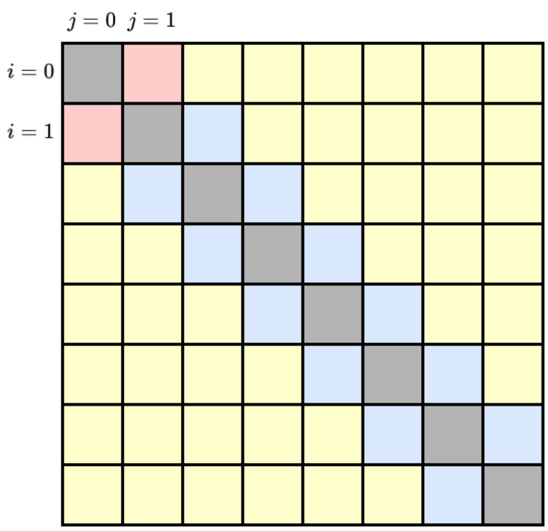
Because the added point is on the way back, and there are no guarantee of k,
$dp(i,j,h) = \min_{0\le k\le j-2}(dp(i,k,h) + E(j,k))$

Case 6

Because the added point is on the way go, and there are no guarantee of k,
$dp(i,j,h) = \min_{0\le k\le i-2}(dp(k,j,h+D_i) + E(k,i))$

Time Complexity

We can use an array $DP[N][N][H]$ to store the value of $dp(i,j,h)$. Fill the array from a large h to smaller h. And for each h, fill the 2d array row by row.



In each layer, there are $O(N)$ boxes (blue ones) that need $O(N)$ time to fill, and $O(N^2)$ boxes (red and yellow ones) that need $O(1)$ time to fill. There fore each layer takes $O(N^2)$ time.

The entire $DP[N][N][H]$ has H layers of this kind of 2d array, so total time complexity is
$H \cdot O(N^2) = O(HN^2)$.

Problem 6

> Refs:
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> None

(1)

- No assumption: 98141210
- Under assumption 3: 981120

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(2)

Algorithm

1. Sort P_i from large to small.
2. Concatenate P_i in this sorted order, first element in the leftmost place.

Proof

Suppose S is an optimal solution, S_x is the x-th digit (from left to right).

Assume that $\exists\ i, j$ such that $i < j$ and $S_i < S_j$, because P_i is exactly one digit long, we can swap S_i, S_j and get S'. Since $S'_i = S_j > S_i$ and $S'_x = S_x \ \forall\ 1 \leq x < i, S' > S$. This contradicts with the assumption that S is an optimal solution, therefore $\forall\ i < j,\ S_i \geq S_j$.

Time Complexity

Time complexity of each step:

1. Sort in $O(n \log n)$.
2. Concatenate in $O(n)$.

Total time complexity $O(n \log n) + O(n) = O(n \log n)$.

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(3)

Algorithm

1. Sort P_i by:
   1. Compare their first (leftmost) digit, the larger one goes first.
   2. If the same, compare their next digit. If there are no second digit, compare as if it's the same as first digit. The larger one does first.
   3. If they are still not separated, repeat previous step until all digits are compared.
   4. If all digits are compared, their order will remain the same.

2. Concatenate P_i in this sorted order, first element in the leftmost place.

Proof

Suppose S is an arbitrary arrangement, S_x is the x-th preference value (from left to right) concatenated, $S_x[d]$ is the d-th digit from left to right.

Assume that $\exists\, i, j$ such that $i < j$ but S_i should be after S_j according to the compare method above.

First consider when $j = i + 1$, that is, when they are next to each other. S_i and S_j should have a common prefix of length $l \geq 0$. There are three cases:

- If $l < \min\{len(S_i), len(S_j)\}$, then $S_i[l+1] < S_j[l+1]$. Let the number after swapping be S'. Because S and S' are the same until $S_i[l]$, since $S_i'[l+1] = S_j[l+1] > S_i[l+1]$, $S' > S$.
- If $l = len(S_i)$, then $l > 0$ and $S_j[l+1] > S_i[1] = S_j[1]$. Let the number after swapping be S'. Because S and S' are the same until $S_i[l]$, since $S_i'[l+1] = S_j[l+1] > S_i[1] = S_j[1]$, $S' > S$.
- If $l = len(S_j)$, then $l > 0$ and $S_i[l+1] \leq S_i[1] = S_j[1]$. Let the number after swapping be S'. Because S and S' are the same until $S_i[l]$, since $S_j'[1] = S_i[1] \geq S_i[l+1]$, $S' \geq S$.

In all three cases, any unordered neighboring pair gives a higher S after swapping. Only if they are sorted by the compare method above, there will be no unordered neighboring pair. Therefore this sorting yields the maximum satisfying value.

Time Complexity

Time complexity of each step:

1. The compare function runs in $O(1)$ time (because P_i is capped at 1000), therefore the sort runs in $O(n \log n) \cdot O(1) = O(n \log n)$ time.
2. Concatenating runs in $O(n)$ time.

Total time complexity $O(n \log n) + O(n) = O(n \log n)$.

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(4)

Algorithm

1. Let $m = 0$.
2. Loop through P_i to record how many times each digit has appeared as a_{digit}. Also let $b = a_1 + a_4 + a_7, c = a_2 + a_5 + a_8$.
3. Let $b' = b \bmod 3, c' = c \bmod 3$.
   - If $b' > c'$, remove the least $b' - c'$ digits that satisfied $d \equiv 1 \pmod 3$ and change the corresponding a_d.
   - If $c' > b'$, remove the least $c' - b'$ digits that satisfied $d \equiv 2 \pmod 3$ and change the corresponding a_d.
4. Loop d from 9 to 0, concatenate a_d digits of d to back of m.
5. m is the maximum satisfying number under assumption 1 & 3.

Proof

According to subproblem (2), larger digits should be on the left. To make it divisible by 3, if d is not divisible by 3, group them by their remainder. From each group, repeatedly choose the largest 3 out. If some digits are left, choose one $d \equiv 2 \pmod 3$ with one $d \equiv 1 \pmod 3$.

Because the number of digits is maximized and larger digits are on the left, m is maximized.

Time Complexity

Time complexity of each step:

1. $O(1)$
2. $O(n)$
3. $O(1)$
4. $O(\sum a_d) = O(n)$
    - All cases are $O(a_d)$.

Total time complexity is $O(1) + O(n) + O(1) + O(n) = O(n)$.

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(5)

98653

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(6)

Algorithm

Let $f(A, k)$ be the maximum satisfying value of preference values A_i with k courses selected. $merge(x, y)$ is the maximum value of merging x and y in to a single number, while preserving order of digits from the same source.

The maximum satisfying value given P_i and M_i is $\max_{0 \le j \le k}(merge(f(P, j), f(M, k - j)))$.

$f(A, k)$ is done by:

1. Let $r = 0, s = 0$.
2. Loop i from 1 to k.
    1. Choose the largest digit from $A[1 : 1 + len(A) - k - s]$, concatenate the digit d to the back of r.
    2. Increase s by the number of elements in front of d. Remove d and every digit in front of it from A.
3. $r = f(A, k)$

$merge(x, y)$ is done by:

1. Let $r = 0$.
2. While x and y are not empty, choose the larger first bit and concatenate it to the back of r, then remove that digit from the source number.

3. Concatenate all digits left to the back of r.
4. $r = merge(x, y)$.

Proof

$f(A, k)$

This algorithm tries to maximize the first digit, then the second digit and so on.

Suppose an optimal solution S has $S[1] \neq \max A[1 : 1 + len(A) - k]$, because we can remove only up to $len(A) - k$ elements, $S[1] \in A[1 : 1 + len(A) - k]$. Therefore $S[1] < \max A[1 : 1 + len(A) - k]$, and choosing $\max A[1 : 1 + len(A) - k]$ for $S[1]$ should be the optimal solution. Other digits can be proved using the same method.

$merge(x, y)$

If during the merge process, when comparing the first digit left, the smaller one is chosen first, it must not be the optimal solution. Because choosing the larger one first gives a larger answer.

$\max_{0 \leq j \leq k}(merge(f(P, j), f(M, k - j)))$

The maximum satisfying value must be consist of j values from P and $k - j$ values from M, therefore we chould choose the maximum from each case of j.

Time Complexity

$f(A, k)$

In each step:

1. $O(1)$
2. $O(k) \cdot O(n)$
   1. $O(n)$
   2. $O(n)$

Total time complexity is $O(kn)$.

$merge(x, y)$

Total time complexity is $O(len(x) + len(y))$.

$\max_{0 \leq j \leq k}(merge(f(P, j), f(M, k - j)))$

Total time complexity is:

$$
\begin{aligned}
&\sum_{0 \leq j \leq k} O(jn) + O((k - j)n) + O(j + (k - j)) \\
=&\sum_{0 \leq j \leq k} O(kn) + O(k) \\
=&\sum_{0 \leq j \leq k} O(kn) \\
=&O(k^2 n) \\
=&O(kn^2) \text{ (because } k < n)
\end{aligned}
$$