Problem 5

> Refs & people discussed with:
>
> https://en.wikipedia.org/wiki/Trinomial_triangle
> b09902100

(1) Asymptotic Notations

(a)

$$\begin{aligned} \ln n! &= \sum_{i=1}^{n} \ln i \leq \sum_{i=1}^{n} \ln n = n \ln n = \ln n^n \\ \Rightarrow \ln n! &= O(\ln n^n) \end{aligned}$$

(b)

$$\begin{aligned} n^{\ln c} &= (e^{\ln n})^{\ln c} = e^{\ln n \cdot \ln c} = c^{\ln n} \\ \Rightarrow n^{\ln c} &= \Theta(c^{\ln n}) \end{aligned}$$

(c)

Assume that $\sqrt{n} = O(n^{\sin n})$ is true, then $\exists\ n_0,\ c > 0$ such that $\forall\ n > n_0,\ \sqrt{n} \leq c \cdot n^{\sin n}$.

Let $n_1 = \lceil \frac{c^2 + n_0}{\pi} \rceil \pi$, we have:

$$\begin{aligned} n_1 &\geq \frac{c^2 + n_0}{\pi}\pi = c^2 + n_0 > n_0 \\ \sqrt{n_1} &\geq \sqrt{\frac{c^2 + n_0}{\pi}\pi} = \sqrt{c^2 + n_0} > c \\ \sin n_1 &= \sin\left(\lceil \frac{c^2 + n_0}{\pi} \rceil \pi\right) = 0 \\ c \cdot {n_1}^{\sin n_1} &= c \cdot {n_1}^0 = c \\ \Rightarrow \sqrt{n_1} &> c = c \cdot {n_1}^{\sin n_1} \end{aligned}$$

This conflicts with the assumption, therefore the assumption is false, $\sqrt{n} \neq O(n^{\sin n})$

(d)

Let $f(x) = x - (\ln x)^3$, we have:

$$
\begin{aligned}
f(x) &= x - (\ln x)^3 \\
f'(x) &= 1 - \frac{3(\ln x)^2}{x} = \frac{x - 3(\ln x)^2}{x} \\
f''(x) &= -\frac{3(2\ln x - (\ln x)^2)}{x^2} = \frac{3\ln x(\ln x - 2)}{x^2}
\end{aligned}
$$

And:

$$
\begin{aligned}
f(e^6) &= e^6 - 6^3 = (e^2)^3 - 6^3 > 0 \\
f'(e^6) &= \frac{e^6 - 3 \cdot 6^2}{e^6} > \frac{e^6 - 6^3}{e^6} > 0 \\
f''(x) &= \frac{3\ln x(\ln x - 2)}{x^2} > 0,\ \forall\, x > e^2 \\
&\Rightarrow \forall\, x > e^6,\ f(x) = x - (\ln x)^3 > 0 \\
&\Rightarrow \forall\, x > e^6,\ (\ln x)^3 < x
\end{aligned}
$$

Choose $n_0 = \lceil e^6 \rceil,\ c = 1$:

$$
\begin{aligned}
&\forall\, n > n_0,\ (\ln n)^3 < c \cdot n \\
&\Rightarrow (\ln n)^3 = o(n)
\end{aligned}
$$

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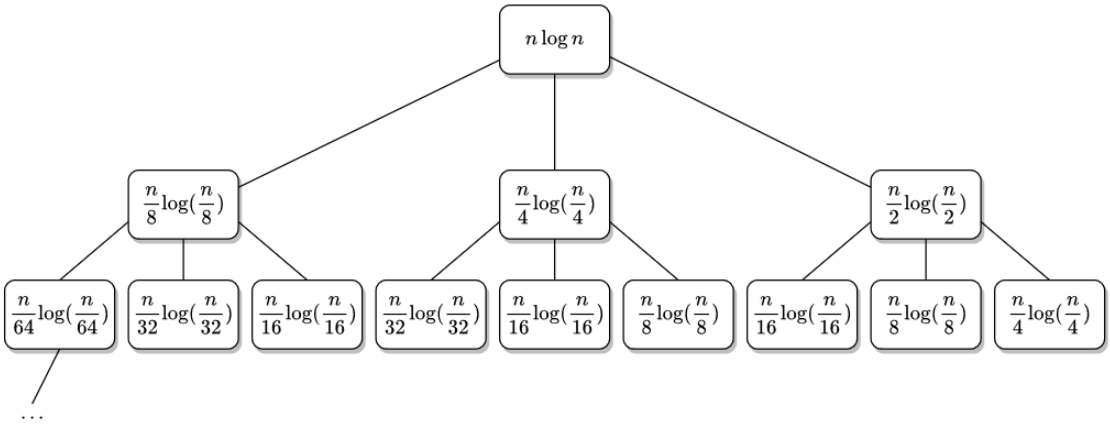
(2) Solve Recurrences

(a)

$$
\begin{aligned}
T(n) &= 2T(n-1) + 1 \\
&= 2(2T(n-2) + 1) + 1 = 2^2 T(n-2) + 3 \\
&= 2^2(2T(n-3) + 1) + 3 = 2^3 T(n-3) + 7 \\
&\cdots \\
&= 2^k T(n-k) + (2^k - 1) \\
&= 2^{n-2} T(n - (n-2)) + 2^{n-2} - 1 \\
&= 2^{n-1} - 1 \\
\Rightarrow T(n) &= \Theta(2^n)
\end{aligned}
$$

(b)

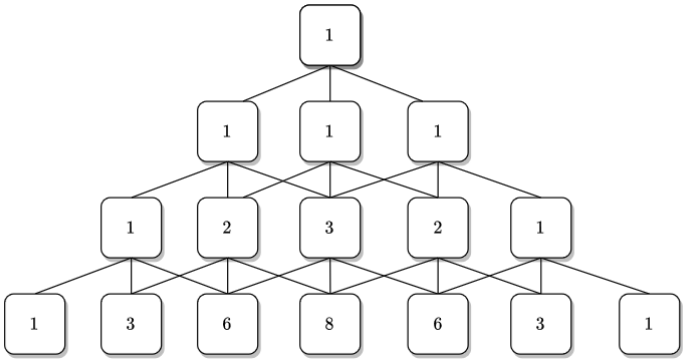
The recursion tree looks like this:



Let R_k be the sum of k-th row of the recursion tree.

$$
\begin{aligned}
R_1 &= n \log n \\
R_2 &= \frac{n}{2}\log(\frac{n}{2}) + \frac{n}{4}\log(\frac{n}{4}) + \frac{n}{8}\log(\frac{n}{8}) \\
&= \frac{7}{8}n\log n - \frac{(2^2 \cdot 1 + 2 \cdot 2 + 1 \cdot 3)\log 2}{8}n \\
&= \frac{7}{8}n\log n - \frac{11\log 2}{8}n \\
&\leq \frac{7}{8}n\log n \\
\\
R_3 &= \frac{n}{4}\log(\frac{n}{4}) + 2 \cdot \frac{n}{8}\log(\frac{n}{8}) + 3 \cdot \frac{n}{16}\log(\frac{n}{16}) + 2 \cdot \frac{n}{32}\log(\frac{n}{32}) + \frac{n}{64}\log(\frac{n}{64}) \\
&= (\frac{7}{8})^2 n\log n - \frac{154\log 2}{64}n \\
&\leq (\frac{7}{8})^2 n\log n \\
\\
R_k &\leq (\frac{7}{8})^{k-1} n\log n
\end{aligned}
$$

To prove the coefficient of $n \log n$, we combine the same terms in each row, and look only at the coefficient:



This triangle is called "trinomial triangle".
The i-th term in j-th row (both starts from 0) is the coefficient of x^i in $(1+x+x^2)^j$.

Using $f(i,j)$ to denote the i-th term in j-th row in trinomial triangle, we can rewrite the $n\log n$ term in k-th row (starts from 1) as:

$$\begin{aligned}\sum_{i=0}^{2(k-1)}(f(i,k)\frac{n}{2^{3(k-1)-i}}\log n) &= \sum_{i=0}^{2(k-1)}\frac{f(i,k)\cdot 2^i}{8^{k-1}}n\log n \\ &= \frac{\sum_{i=0}^{2(k-1)}f(i,k)\cdot 2^i}{8^{k-1}}n\log n \\ &= \frac{(1+2+2^2)^{k-1}}{8^{k-1}}n\log n \\ &= (\frac{7}{8})^{k-1}n\log n\end{aligned}$$

Then:

$$\begin{aligned}T(n) &\le \sum_{k=1}^{\infty}R_k \\ &\le \sum_{k=1}^{\infty}(\frac{7}{8})^{k-1}n\log n \\ &= \sum_{k=0}^{\infty}(\frac{7}{8})^{k}n\log n \\ &= \frac{1}{1-\frac{7}{8}}n\log n \\ \Rightarrow T(n) &= O(n\log n)\end{aligned}$$

And:

$$\begin{aligned}T(n) &\ge R_1 \\ &= n\log n \\ \Rightarrow T(n) &= \Omega(n\log n)\end{aligned}$$

Therefore, $T(n) = \Theta(n \log n)$.

(c)

Choose $n_1 = e,\ c_1 = 1,\ \epsilon = 0.5$

$$\begin{aligned} &\forall\ n > n_1,\ n \log n \geq n \cdot 1 = c_1 \cdot n \\ &\Rightarrow n \log n = \Omega(n^1) = \Omega(n^{(\log_4 2)+\epsilon}) \end{aligned}$$

Choose $n_2 = 2,\ c_2 = 2$

$$\forall\ n > n_2,\ 4 \cdot \frac{n}{2} \log \frac{n}{2} = 2n(\log n - \log 2) \leq 2n \log n = c_2 \cdot n \log n$$

By case 3 of master theorem, $T(n) = \Theta(n \log n)$

(d)

$$\begin{aligned} n = 2^m \Rightarrow T(2^m) &= 2^{m/2} T(2^{m/2}) + 2^m \\ F(m) = T(2^m) \Rightarrow F(m) &= 2^{m/2} F(\frac{m}{2}) + 2^m \end{aligned}$$

Let $\lg x = \log_2 x$.

Claim: $F(m) \leq (2 \lg m) 2^m\ \forall\ m \geq 2$

For $m = 2, F(2) = T(4) = 2 \cdot T(2) + 4 = 6 \leq 2 \cdot 2^2 = 8$
If it's true for $m = k/2$:

$$\begin{aligned} F(k) &= 2^{k/2} F(\frac{k}{2}) + 2^k \\ &\leq 2^{k/2} (2(\lg k - \lg 2) 2^{k/2}) + 2^k \\ &= (2 \lg k) 2^k - 2^k \\ &< (2 \lg k) 2^k \end{aligned}$$

By induction, $F(m) \leq (2 \lg m) 2^m\ \forall\ m \geq 2 \Rightarrow F(m) = O((\log m) 2^m)$.

Claim: $F(m) \geq (\lg m) 2^m\ \forall\ m \geq 2$

For $m = 2, F(2) = T(4) = 2 \cdot T(2) + 4 = 6 \geq 2^2 = 4$
If it's true for $m = k/2$:

$$
\begin{aligned}
F(k) &= 2^{k/2}F(\frac{k}{2}) + 2^k \\
&\geq 2^{k/2}((\lg k - \lg 2)2^{k/2}) + 2^k \\
&= (\lg k)2^k
\end{aligned}
$$

By induction, $F(m) \geq (\lg m)2^m \ \forall \ m \geq 2 \Rightarrow F(m) = \Omega((\log m)2^m)$.

$\Rightarrow F(m) = \Theta((\log m)2^m) \Rightarrow T(2^m) = \Theta((\log m)2^m) \Rightarrow T(n) = \Theta(n \log \log n)$

Problem 6

> Refs & people discussed with:
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> b09902100

(1)

Use merge sort to sort the sequence B and start with $|I(B)| = 0$.
During the merging of two subarrays A_1, A_2, assuming A_1 is the one at front, each time when an element from A_2 is being merged, increase $|I(B)|$ by "$\#$ of elements in A_1 not merged yet".
After merge sort is done, $|I(B)|$ is also calculated.

(2)

The time complexity of merge sort $T(N)$ has this recurrence relation:
$T(N) = 2T(\frac{N}{2}) + O(N)\ \forall\ N \geq 2$.
The extra calculation done for each merge is $O(N)$, because "$\#$ of elements in A_1 not merged yet" can be acquired by using a variable to store length of A_1 and substract by 1 when an element from A_1 is merged.

Adding this cost to the original recurrence relation yields my algorithm's recurrence relation:
$T(N) = 2T(\frac{n}{2}) + 2 \cdot O(N) = 2T(\frac{n}{2}) + O(N)$.

Because the recurrence relation is the same as merge sort, my algorithm also has the same time complexity $O(N \log N)$.

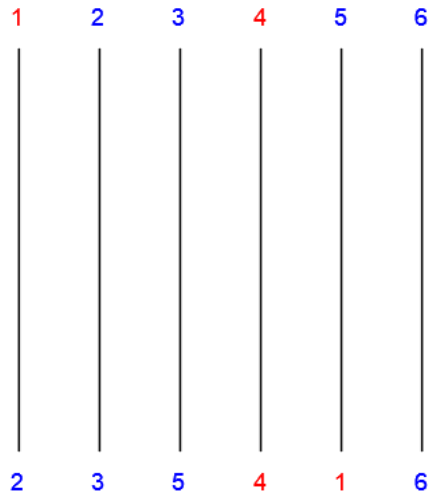
(3)

When an exchange happens during bubble sort, because it reverse a neighboring inversion, number of inversions decreases by 1. And since the number of inversions in a sorted array is 0, the total number of exchanges during bubble sort must be $|I(S)|$.

(4)

1. Organize constraints as tuples. (i, j) means when i should go to j.
2. Loop through all starting points. If point i isn't constrained, constrain it to the first finish point that hasn't been designated.

For example, in this picture the constraints are $(1, 5),\ (4, 4)$. The rest of starting points will be assigned a finish point in order, that is, adding these constraints: $(2, 1),\ (3, 2),\ (5, 3),\ (6, 6)$.



3. Sort these tuples by the first element (starting point) with quick sort.
4. Apply the algorithm described in (1) to calculate the number of inversions.
5. The minimum number of horizontal lines needed is the number of inversions.

(5)

When a horizontal line is added, the corresponding pair of lines exchange their finish point. This is similar to an exchange during bubble sort. We can think of a "constraint" as the initial position of an element in an unsorted array.

To use the minimal number of horizontal lines is to use the minimal number of exchanges, therefore other unconstrained starting points should have their finish points in the same relative order. And to calculate number of exchanges, we can apply the algorithm used to calculate the number of inversion, solving this problem in $O(N \log N)$ time.

Problem 7

> Refs & people discussed with
>
> https://ithelp.ithome.com.tw/articles/10221041
> b09902011 b09902100

(1)

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(2)

Algorithm

```
function DP(f, N)
    max_ending = array(N)
    max_p = array(N)
    max_ending[2] = f[2]
    max_st[2] = 2
    sol = 2
    for i from 3 to N
        max_ending[i] = f[i]
        max_p[i] = i
        if (max_ending[i-1] + f[i] > max_ending[i])
            max_ending[i] = max_ending[i-1] + f[i]
            max_p[i] = max_p[i-1]
        if (max_ending[i] > max_ending[sol])
            sol = i
    return max_ending[sol], max_p[sol], sol

function solve(f, N)
    f_neg = array(N)
    sum_f = 0
    for i from 1 to N
        sum_f += f[i]
        f_neg = -f[i]
    max_sum, max_st, max_ed = DP(f, N)
    min_sum_neg, min_st, min_ed = DP(f_neg, N)
    if (max_sum ≥ sum_f + min_sum_neg)
        st, ed = max_st, max_ed
    else
        st, ed = min_ed+1, min_st-1
        if (st > N)
            st -= N
```

```
        return st, ed
```

The solution should be one of these two cases:

- If the solution doesn't include `f[1]`, the problem becomes finding maximum subarray of `f[2:N]`.
- If the solution does include `f[1]`, the rest of array should be the minimum subarray of `f[2:N]`, and the problem becomes finding minimum subarray of `f[2:N]`.

For finding maximum subarray:

1. Use an array `max_ending`, where `max_ending[i]` is the maximum sum of subarray ending at `f[i]`.
   And an array `max_p` to indicate the start of this subarray.
2. Build `max_ending` bottom up with this recurrence relation `max_ending[i] = max(f[i], max_ending[i-1]+f[i])`.
3. While building `max_ending`, use `sol` to store the index of maximum value in `max_ending`.
4. After the array is built, `sol` is the end of maximum subarray. Use `max_p` to get the start of maximum subarray.

Finding minimum subarray is essentially the same thing. We just use an array `f_neg` where each element is added a negative sign, then find the maximum subarray of `f_neg`.

Finally, we compare the answer of these two cases and choose the larger one.

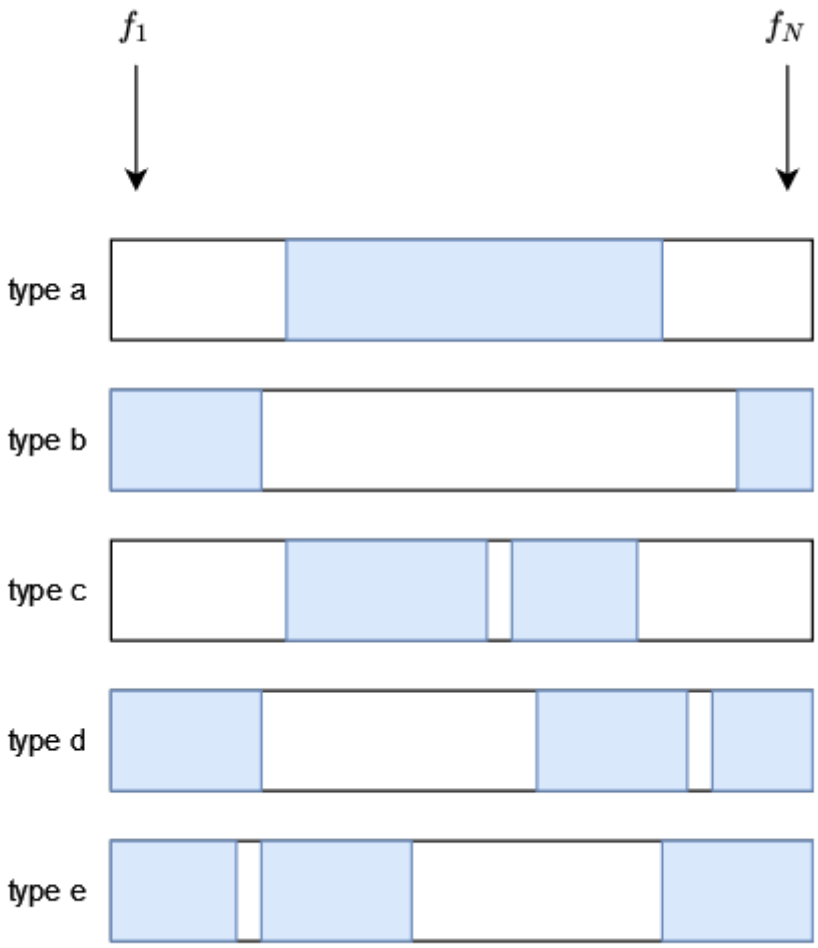
Time and space complexity

Each iteration of `for` in `DP()` takes constant time, so the entire `for` loop is takes $O(N)$ time, and `DP()` takes $O(N) + O(1) = O(N)$ time. Extra space used in `DP()` are `max_ending`, `max_p`, and `sol`, so it has $O(N)$ space complexity.

Time complexity of `solve()` is $2 \cdot T(DP()) + O(N) + O(1) = 3 \cdot O(N) + O(1) = O(N)$. Space complexity of `solve()` is $2 \cdot S(DP()) + O(N) + O(1) = 3 \cdot O(N) + O(1) = O(N)$

(3)

The solution should be one of these five types:



The blue part marks the philosophers given tasty dishes.

1. For type a and b, simply apply the algorithm from the previous subproblem and save them as `sol_a` and `sol_b`. This step is done in $O(N)$ time and space.
    - While calculating type a, keep the array `max_ending` and rename it to `max_ending_fw`.
    - While calculating type b, also build an array `min_sub_fw` where `min_sub_fw[i]` is the sum of maximum subarray of `f_neg[2:i]` by saving the answer after each iteration.
    - Both of these adds only constant time to each iteration of `for`, therefore time complexity is still $O(N)$. `max_ending_fw` and `min_sub_fw` both take $O(N)$ space, so space complexity is still $O(N)$.
2. Do step 1 in the other direction, that is from `f[N]` to `f[1]`, and build `max_ending_bw` and `min_sub_bw`. This step is also done $O(N)$ time and space.
3. For type c, do the following:
    1. Go through each case that `f[i]` is skipped for `i` from `3` to `N-2`. The corresponding subarray sum is `max_ending_fw[i-1] + max_ending_bw[i+1]`, that is "maximum sum of subarray ending at `f[i-1]`" plus "maximum sum of subarray starting at `f[i+1]`".
    2. Store the maximum value of that, and it will be the maximum type c solution `sol_c`.
    3. All of this is done in $O(N)$ time and $O(1)$ space.
4. For type d, do the following:
    1. The target is to find minimum sum for the white part. We do this by calculating the maximum sum of that in `f_neg`.

2. Go through each case that `f[i]` is skipped for `i` from `3` to `N`. The corresponding subarray sum is `f_neg[i] + min_sub_fw[i-1]`.
3. Store the maximum value of that as `sol_tmp_d`. The sum of blue part `sol_d`, which is what we really want, is calculated by `sol_d = sum(f)+sol_tmp_d`.
4. All of this is done in $O(N)$ time and $O(1)$ space.

5. For type e, because it's just type d mirrored, so the same steps can be applied with `min_sub_bw`, giving us `sol_e`. So this is also done in $O(N)$ time and $O(1)$ space.
6. Finally, find the maximum value from these five solutions. All steps together are done in $O(N)$ time and space.

To know the exact segment, we can use an extra array for each `max_*`, `min_*` array, where `i`-th element indicates the other end of the subarray that has this sum. This can be maintained similar to `max_p` in the previous subproblem, and the total time and space complexity don't change.