Problem 5

(1) Asymptotic Notations

(a)

$$\ln n! = \sum_{i=1}^{n} \ln i \le \sum_{i=1}^{n} \ln n = n \ln n = \ln n^n$$
$$\Rightarrow \ln n! = O(\ln n^n)$$

(b)

$$n^{\ln c} = \left(e^{\ln n}\right)^{\ln c} = e^{\ln n \cdot \ln c} = c^{\ln n}$$
$$\Rightarrow n^{\ln c} = \Theta(c^{\ln n})$$

(c)

Assume that $\sqrt{n} = O(n^{\sin n})$ is true, then $\exists\, n_0,\ c > 0$ such that $\forall\, n > n_0,\ \sqrt{n} \le c \cdot n^{\sin n}$.

Let $n_1 = \lceil \frac{c^2 + n_0}{\pi} \rceil \pi$, we have:

$$n_1 \ge \frac{c^2 + n_0}{\pi}\pi = c^2 + n_0 > n_0$$
$$\sqrt{n_1} \ge \sqrt{\frac{c^2 + n_0}{\pi}\pi} = \sqrt{c^2 + n_0} > c$$
$$\sin n_1 = \sin\left(\lceil \frac{c^2 + n_0}{\pi} \rceil \pi\right) = 0$$
$$c \cdot n_1{}^{\sin n_1} = c \cdot n_1{}^0 = c$$
$$\Rightarrow \sqrt{n_1} > c = c \cdot n_1{}^{\sin n_1}$$

This conflicts with the assumption, therefore the assumption is false, $\sqrt{n} \ne O(n^{\sin n})$

(d)

Let $f(x) = x - (\ln x)^3$, we have:

$$f(x) = x - (\ln x)^3$$
$$f'(x) = 1 - \frac{3(\ln x)^2}{x} = \frac{x - 3(\ln x)^2}{x}$$
$$f''(x) = -\frac{3(2\ln x - (\ln x)^2)}{x^2} = \frac{3\ln x(\ln x - 2)}{x^2}$$

And:

$$f(e^6) = e^6 - 6^3 = (e^2)^3 - 6^3 > 0$$
$$f'(e^6) = \frac{e^6 - 3 \cdot 6^2}{e^6} > \frac{e^6 - 6^3}{e^6} > 0$$
$$f''(x) = \frac{3\ln x(\ln x - 2)}{x^2} > 0, \ \forall\, x > e^2$$
$$\Rightarrow \forall\, x > e^6,\ f(x) = x - (\ln x)^3 > 0$$
$$\Rightarrow \forall\, x > e^6,\ (\ln x)^3 < x$$
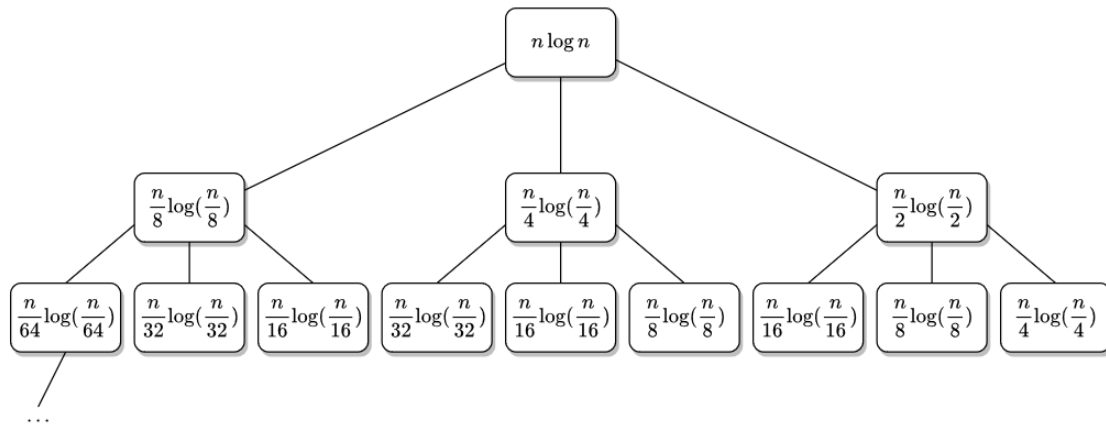
Choose $n_0 = \lceil e^6 \rceil$, $c = 1$:

$$\forall\, n > n_0,\ (\ln n)^3 < c \cdot n$$
$$\Rightarrow (\ln n)^3 = o(n)$$

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(2) Solve Recurrences

(a)

$$\begin{aligned}
T(n) &= 2T(n-1) + 1 \\
&= 2(2T(n-2) + 1) + 1 = 2^2 T(n-2) + 3 \\
&= 2^2(2T(n-3) + 1) + 3 = 2^3 T(n-3) + 7 \\
&\quad \cdots \\
&= 2^k T(n-k) + (2^k - 1) \\
&= 2^{n-2} T(n - (n-2)) + 2^{n-2} - 1 \\
&= 2^{n-1} - 1 \\
\Rightarrow T(n) &= \Theta(2^n)
\end{aligned}$$

(b)

Let R_k be the sum of k-th row of the recurrsion tree.

$$R_1 = n \log n$$

$$
\begin{aligned}
R_2 &= \frac{n}{2}\log\left(\frac{n}{2}\right) + \frac{n}{4}\log\left(\frac{n}{4}\right) + \frac{n}{8}\log\left(\frac{n}{8}\right) \\
&= \frac{7}{8}n\log n - \frac{(2^2 \cdot 1 + 2 \cdot 2 + 1 \cdot 3)\log 2}{8}n \\
&= \frac{7}{8}n\log n - \frac{11\log 2}{8}n \\
&\leq \frac{7}{8}n\log n
\end{aligned}
$$

$$
\begin{aligned}
R_3 &= \frac{n}{4}\log\left(\frac{n}{4}\right) + 2 \cdot \frac{n}{8}\log\left(\frac{n}{8}\right) + 3 \cdot \frac{n}{16}\log\left(\frac{n}{16}\right) + 2 \cdot \frac{n}{32}\log\left(\frac{n}{32}\right) + \frac{n}{64}\log\left(\frac{n}{64}\right) \\
&= \left(\frac{7}{8}\right)^2 n\log n - \frac{154\log 2}{64}n \\
&\leq \left(\frac{7}{8}\right)^2 n\log n
\end{aligned}
$$

$$R_k \leq \left(\frac{7}{8}\right)^{k-1} n\log n$$

Then:

$$T(n) \leq \sum_{k=1}^{\infty} R_k$$

$$\leq \sum_{k=1}^{\infty} \left(\frac{7}{8}\right)^{k-1} n \log n$$

$$= \sum_{k=0}^{\infty} \left(\frac{7}{8}\right)^{k} n \log n$$

$$= \frac{1}{1 - \frac{7}{8}} n \log n$$

$$\Rightarrow T(n) = O(n \log n)$$

And:

$$T(n) \geq R_1$$
$$= n \log n$$
$$\Rightarrow T(n) = \Omega(n \log n)$$

Therefore, $T(n) = \Theta(n \log n)$.

(c)

Choose $n_1 = e$, $c_1 = 1$, $\epsilon = 0.5$

$$\forall\, n > n_1,\ n \log n \geq n \cdot 1 = c_1 \cdot n$$
$$\Rightarrow n \log n = \Omega(n^1) = \Omega(n^{(\log_4 2) + \epsilon})$$

Choose $n_2 = 2$, $c_2 = 2$

$$\forall\, n > n_2,\ 4 \cdot \frac{n}{2} \log \frac{n}{2} = 2n(\log n - \log 2) \leq 2n \log n = c_2 \cdot n \log n$$

By case 3 of master theorem, $T(n) = \Theta(n \log n)$

(d)

$$n = 2^m \Rightarrow T(2^m) = 2^{m/2} T(2^{m/2}) + 2^m$$
$$F(m) = T(2^m) \Rightarrow F(m) = 2^{m/2} F\left(\frac{m}{2}\right) + 2^m$$

Let $\lg x = \log_2 x$.

Claim: $F(m) \leq (2 \lg m) 2^m\ \forall\, m \geq 2$

For $m = 2$, $F(2) = T(4) = 2 \cdot T(2) + 4 = 6 \leq 2 \cdot 2^2 = 8$

If it's true for $m = k/2$:

$$F(k) = 2^{k/2} F\left(\frac{k}{2}\right) + 2^k$$
$$\leq 2^{k/2}(2(\lg k - \lg 2)2^{k/2}) + 2^k$$
$$= (2\lg k)2^k - 2^k$$
$$< (2\lg k)2^k$$

By induction, $F(m) \leq (2\lg m)2^m \ \forall \ m \geq 2 \Rightarrow F(m) = O((\log m)2^m)$.

Claim: $F(m) \geq (\lg m)2^m \ \forall \ m \geq 2$

For $m = 2$, $F(2) = T(4) = 2 \cdot T(2) + 4 = 6 \geq 2^2 = 4$

If it's true for $m = k/2$:

$$F(k) = 2^{k/2} F\left(\frac{k}{2}\right) + 2^k$$
$$\geq 2^{k/2}((\lg k - \lg 2)2^{k/2}) + 2^k$$
$$= (\lg k)2^k$$

By induction, $F(m) \geq (\lg m)2^m \ \forall \ m \geq 2 \Rightarrow F(m) = \Omega((\log m)2^m)$.

$\Rightarrow F(m) = \Theta((\log m)2^m) \Rightarrow T(2^m) = \Theta((\log m)2^m) \Rightarrow T(n) = \Theta(n \log \log n)$