Problem 1

> *References:*
>
> *None*

1.

In the k-th iteration of the while loop, $sum = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

$\Rightarrow$ total iteration time x satisfies $\frac{x(x-1)}{2} < n \leq \frac{x(x+1)}{2} \Rightarrow$ time complexity $x = \Theta(\sqrt{n})$

2.

In the k-th iteration, $m = 2^{2^{k-1}}$

$\Rightarrow$ total iteration time x satisfies $2^{2^{x-2}} < n \leq 2^{2^{x-1}} \Rightarrow$ time complexity $x = \Theta(\sqrt{n})$

3.

For $n > 87506055$, total operation $x = 1 + 4 + \cdots + 4^{n-k} + 4^{n-k} \cdot 3 + \cdots + 4^{n-k} \cdot 3^k$, where $k = 87506055$

$\Rightarrow$ time complexity $x = \Theta(4^n)$

4.

$\because f(n), g(n)$ are both positive $\therefore max(f(n),\ g(n)) \leq f(n) + g(n) \leq 2 \cdot max(f(n),\ g(n))$

$\Rightarrow f(n) + g(n) = \Theta(max(f(n),\ g(n)))$

5.

$f(n) = O(i(n)) \Rightarrow \exists\, c_1 > 0,\ n_1 > 0\ s.t.\ \forall\, n > n_1,\ f(n) \leq c_1 \cdot i(n)$

$g(n) = O(j(n)) \Rightarrow \exists\, c_2 > 0,\ n_2 > 0\ s.t.\ \forall\, n > n_2,\ g(n) \leq c_2 \cdot j(n)$

Let $n' = max(n_1,\ n_2),\ c' = c_1 \cdot c_2$

Multiplying the first two lines we have

$\forall\, n > n',\ f(n)\cdot g(n) \le c' \cdot i(n)\cdot j(n) \Rightarrow f(n)\cdot g(n) = O(i(n)\cdot j(n))$

6.

$f(n) = O(g(n)) \Rightarrow \exists\, c_1 > 0,\ n_1 > 0\ s.t.\ \forall\, n > n_1,\ f(n) \le c_1 \cdot g(n)$

$\Rightarrow \forall\, n > n_1,\ 2^{f(n)} \le 2^{c_1\cdot g(n)} = 2^{c_1}\cdot 2^{g(n)}$

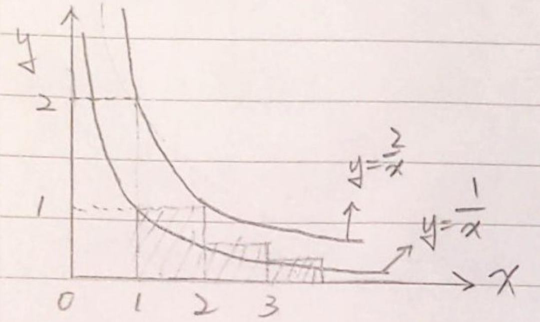
choose $n' = n_1,\ c' = 2^{c_1}$

$\Rightarrow \forall\, n > n',\ 2^{f(n)} \le c' \cdot 2^{g(n)} \Rightarrow 2^{f(n)} = O(2^{g(n)})$

7.

$$\sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n} 1\cdot\frac{1}{k} \ge \int_1^{n+1}\frac{1}{x}dx = \ln(n+1) > \ln(n) = \frac{\lg n}{\lg e} = \ln 2\cdot \lg n \quad \forall n>1 \quad —①$$

$$\frac{\frac{1}{k}}{\frac{1}{k+1}} = \frac{k+1}{k} = 1+\frac{1}{k} \le 2 \ \forall k \ge 1 \Rightarrow \frac{1}{k+1}\cdot 2 \ge \frac{1}{k} \ \forall k\ge 1$$



$$\Rightarrow \sum_{k=1}^{n}\frac{1}{k} \le \int_1^{n+1}\frac{2}{x}dx = 2\ln(n+1)$$

$$2\ln(n+1) = 2\cdot\frac{\ln(n+1)}{\ln(n)}\cdot \ln(n)$$

$$\le 2\cdot\frac{\ln 3}{\ln 2}\cdot\ln 2\cdot\lg n \quad \forall n\ge 2 \quad —②$$

①: $\sum_{k=1}^{n}\frac{1}{k} \ge \ln 2\cdot\lg n \ \ \forall n>1 \Rightarrow \sum_{k=1}^{n}\frac{1}{k} = \Omega(\lg n)$

②: $\sum_{k=1}^{n}\frac{1}{k} \le 2\cdot\ln 3\cdot\lg n \ \ \forall n\ge 2 \Rightarrow \sum_{k=1}^{n}\frac{1}{k} = O(\lg n)$

$\Rightarrow \sum_{k=1}^{n}\frac{1}{k} = \Theta(\lg n)$

8.

$$\lg(n!) = \lg(n \times (n-1) \times \cdots \times 1) = \sum_{k=1}^{n} \lg k \le \sum_{k=1}^{n} \lg n = n \cdot \lg n \quad \text{— ①}$$

$$\lg(n!) = \sum_{k=1}^{n} \lg k = \left(\sum_{k=2}^{n} \lg k\right) + \lg 1 = \sum_{k=2}^{n} \lg k > \int_{1}^{n} \lg x \, dx \quad \text{— ②}$$

$$\int_{1}^{n} \lg x \, dx = \ln 2 \cdot \int_{1}^{n} \ln x \, dx = \ln 2 \cdot (n \ln n - n + 1)$$

$$\forall n \ge e^{2}, \quad 2 \cdot (n \ln n - n) = n \ln n + n(\ln n - 2) \ge n \ln n > n \lg n$$

$$\Rightarrow \int_{1}^{n} \lg x \, dx = \Omega(n \lg n) \quad \text{— ③}$$

$$\text{①} \Rightarrow \lg(n!) = O(n \lg n)$$

$$\text{②} + \text{③} \Rightarrow \lg(n!) = \Omega(n \lg n)$$

$$\Rightarrow \lg(n!) = \Theta(n \lg n)$$

9.

let $m = \lfloor \lg n \rfloor$

let $a_1 = n$, $a_{k+1} = \lfloor \frac{a_k}{2} \rfloor$, then $a_{m+1} = 1$

let $b_k = a_{m+2-k}$, then $b_1 = 1$, $b_{m+1} = n$, $b_k = \lfloor \frac{b_{k+1}}{2} \rfloor$

and also $2^{k-1} \le b_k \le 2^k$

let $f_k = f(b_k)$, then $f_1 = f(1) = 1$

$$f_{m+1} = 2f_m + b_{m+1} \lg(b_{m+1})$$

$$2f_m = 4f_{m-1} + b_m \lg(b_m) \cdot 2$$

$$\vdots$$

$$+)\ 2^{m-1} f_2 = 2^m f_1 + b_2 \lg(b_2) \cdot 2^{m-1}$$

$$f_{m+1} = 2^m f_1 + \sum_{k=2}^{m+1} b_k \cdot \lg(b_k) \cdot 2^{m+1-k}$$

$$= 2^m + 2^{m+1} \cdot \sum_{k=2}^{m+1} b_k \lg(b_k) \cdot 2^{-k}$$

$$\sum_{k=2}^{m+1} 2^{k-1} \cdot \lg(2^{k-1}) \cdot 2^{-k} \le \sum_{k=2}^{m+1} b_k \lg(b_k) \cdot 2^k \le \sum_{k=2}^{m+1} 2^k \cdot \lg(2^k) \cdot 2^k$$

$$\sum_{k=2}^{m+1} \frac{1}{2}(k-1) = \frac{1}{2} \frac{m(m+1)}{2} = \Omega(m^2) \qquad \sum_{k=2}^{m+1} k = \frac{(m+1)(m+2)}{2} - 1 = O(m^2)$$

$$\Rightarrow 2^m(1+2\cdot\Omega(m^2)) \le f_{m+1} = f(n) \le 2^m(1+2\cdot O(m^2)$$

$$f(n) = 2^m(1+2\cdot\Theta(m^2)) = \Theta(2^m \cdot m^2) = \Theta(n(\lg n)^2)$$