GW Frequency at ISCO Using Two-Body Decomposition

The radius of the innermost stable circular orbit (ISCO) for a non-rotating black hole of mass $M_{\rm BH}$ with a test particle orbiting it is given by

$$r_{\rm ISCO} = \frac{6\,G\,M_{\rm BH}}{c^2}. \tag{1}$$

Kepler's third law for two bodies with masses M_1 and M_2 is

$$T^2 = \frac{4\,\pi^2}{G\,(M_1 + M_2)}a^3, \tag{2}$$

where T is the orbital period and a is the semi-major axis. If we first transform coordinates using the two-body decomposition and evaluate this at the ISCO we get

$$T^2 = \frac{4\,\pi^2}{G\,(M_{\rm tot} + \mu)}r_{\rm ISCO}^3, \tag{3}$$

where

$$M_{\rm tot} = M_1 + M_2 \qquad \text{and} \qquad \mu = \frac{M_1\,M_2}{M_1 + M_2}. \tag{4}$$

Substituting (1) in (3) we get

$$T^2 = \frac{4\,\pi^2}{G\,(M_{\rm tot} + \mu)}\left(\frac{6\,G\,M_{\rm tot}}{c^2}\right)^3 = \frac{6^3\,4\,\pi^2\,G^2}{c^6}\frac{M_{\rm tot}^3}{M_{\rm tot} + \mu}. \tag{5}$$

Now, T is the orbital period, which is twice the period of the emitted gravitational waves at ISCO,

$$T = 2\,T_{\rm ISCO} \implies T = \frac{2}{f_{\rm ISCO}}. \tag{6}$$

So, plugging that into the above expression we get

$$\frac{4}{f_{\rm ISCO}^2} = \frac{6^3\,4\,\pi^2\,G^2}{c^6}\frac{M_{\rm tot}^3}{M_{\rm tot} + \mu} \implies f_{\rm ISCO}^2 = \frac{c^6}{6^3\,\pi^2\,G^2}\frac{M_{\rm tot} + \mu}{M_{\rm tot}^3}. \tag{7}$$

Taking the square root of both sides yields

$$f_{\rm ISCO} = \frac{c^3}{6^{3/2}\,\pi\,G}\left(\frac{M_{\rm tot} + \mu}{M_{\rm tot}^3}\right)^{1/2}. \tag{8}$$

Now, this quantity was evaluated at the location of the orbiting bodies (assumed to be at redshift z). The frequency we actually observe will be redshifted and will appear longer by a factor of $(1 + z)$. So our final result is

$$\boxed{f_{\rm ISCO,obs} = \frac{c^3}{6^{3/2}\,\pi\,G}\left(\frac{M_{\rm tot} + \mu}{M_{\rm tot}^3}\right)^{1/2} \times \frac{1}{1 + z}.} \tag{9}$$

We note that in the limit of $M_{\rm tot} \gg \mu$, this result agrees with Eq. (12) in Sesana, et al. (2005).