Generalized Yule Model

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Outline

1. Yule Model
2. Why Yule Model is important?
3. Generalizations:
   - Death
   - Nonlinearity
   - Fractionality

Papers:

- Lansky, P, Sacerdote (2014), arXiv:1309.7823 [math.PR],
- Pachon, P, Sacerdote (2015), arXiv:1503.06150 [math.PR],
- Lansky, P, Sacerdote (2015), soon on the arXiv [math.PR].

Have a look also at the references therein!

Yule Model

Remarks

- Large scale hypertexts, such as the WWW, are examples of *scale-free networks* in the real-world.
- When the interest focuses on the *popularity of a webpage*, its connectivity (i.e. the number of its in-links) is usually considered.
- Mainly, scale-free networks derive by *preferential attachment models*: new webpages are constantly added to the network with a given probability and the number of in-links of each webpage grows *proportionally* to the number of in-links already possessed by that webpage.
- With these assumptions, the limiting distribution of the number of in-links for a webpage chosen uniformly at random *exhibits a power-law tail*.

Yule Model

Construction

* The Yule model evolves in continuous time.
* The Yule model is defined through Yule processes of different parameters as described in the following.

- A first Yule process denoted by $\{N_\beta(T)\}_{T\geq 0}$, $\beta > 0$, accounts for the growth of the number of vertices.
- As soon as the first vertex is created, a second Yule process, $\{N_\lambda(T)\}_{T\geq 0}$, $\lambda > 0$, starts describing the creation of in-links to the vertex.
- The evolution of the number of in-links for the successively created vertices, proceeds similarly. Specifically, for each of the subsequent created vertices, an independent copy of $\{N_\lambda(T)\}_{T\geq 0}$, modeling the appearance of the in-links is initiated.

Yule Model

Historical context

* Udny Yule in 1925 observed that the distribution of species per genus in the evolution of a biological population typically presents a **power law** behavior.

- He defined a model of evolution by considering two independent linear birth processes: one for species with rate $\lambda > 0$, one for new genera created at rate $\beta > 0$.
- At time $T = 0$ the process starts with a single genus composed by a single species.
- As time goes on, new genera (each composed by a single species) develop as a linear birth process of parameter β, and simultaneously and independently new species evolve as a linear birth process with rate λ.
- Since a new genus appears with a single species, then each time a genus is born, a linear birth process with rate λ starts.

Yule Model

Yule–Simon distribution

Let $\mathcal{N}_T$ be the size of a genus chosen uniformly at random at time T. Then, if $\rho = \lambda/\beta$,

$$\lim_{T \longrightarrow \infty} \mathbb{P}(\mathcal{N}_T = k) = \rho \frac{\Gamma(k)\Gamma(1+\rho)}{\Gamma(k+1+\rho)} = \rho B(k, 1+\rho), \qquad k \geq 1.$$

It is well known that the right tail of the above pmf decays as a power-law. For large k,

$$\rho B(k, 1+\rho) \approx k^{-(\rho+1)}$$

Why Yule Model is important?

Relation with discrete-time models

Simon Model (1955) It describes the growth of a text that is being written such that a word is added at each discrete time $t \geq 1$ following the next two conditions: for $\alpha \in (0,1)$,

1. $\mathbb{P}[(t+1)$th word has not yet appeared at time $t] = \alpha$
2. $\mathbb{P}[(t+1)$th word has appeared k times at time $t] = (1-\alpha)\frac{k\vec{N}_{k,t}}{t}$,

where $\vec{N}_{k,t}$ is the number of *different* words that have appeared exactly k times at time t.

Why Yule Model is important?

Simon model as a random graph

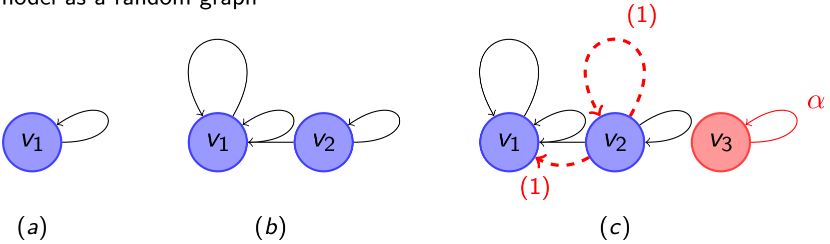

Figure : Construction of the random graph G_α^t associated to Simon model. (a) Begin at time 1 with one single vertex and a directed loop. (b) Suppose some time has passed, in this case, the picture corresponds to a realization of the process at time $t = 4$. (c) Given G_α^4 form G_α^5 by either adding with probability α a new vertex v_3 with a directed loop, or adding a directed edge with probability given by

$$\mathbb{P}(v \longrightarrow v_j) = \frac{(1-\alpha)\vec{d}(v_j, t)}{t}, \qquad 1 \leq j \leq t.$$

Why Yule Model is important?

Main result on Simon model

Let $\vec{N}_{k,t}$ be the number of vertices with **in-degree** k at time $t = n(m+1)$, $n \in \mathbb{N}$, in the Simon model.

$$\frac{\vec{N}^{Simon}_{k,t}}{V_t} \quad \xrightarrow{\mathbb{P}} \quad \frac{1}{1-\alpha} \frac{\Gamma(k)\Gamma\left(1+\frac{1}{1-\alpha}\right)}{\Gamma\left(k+1+\frac{1}{1-\alpha}\right)} \sim \frac{1}{1-\alpha} k^{-1-\frac{1}{1-\alpha}},$$

Therefore it seems resaonable that Yule and Simon model are in some way connected.

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Why Yule Model is important?

Convergence of Simon model to Yule model

In order to relate Yule and Simon models, in Pachon, P, Sacerdote (2015) we identify two different processes which conditionally describe the Simon model, and identify how these are related with the two Yule processes which define a Yule model.

Two theorems allow us to recognize the first process inside a Simon model **behaving asymptotically as a Yule process with parameter** $(\mathbf{1}-\alpha)$ and the second process which **behaves asymptotically as a Yule process with parameter equal to one**. The first process models how the vertices get new in-links, thus at each moment a new vertex appears, a process starts. On the other hand, the second process is related to how the vertices appear.

Generalization: Death

- We study here the consequences of **detachment** of in-links (**death**).
- This is accomplished by considering the Yule model and replacing the linear birth process governing the growth of in-links with a linear birth-death process.
- It turns out that the introduction of the possibility of detachment of in-links in the Yule model still leads to an analytically tractable model and thus still permits to obtain exact results.

Generalization: Death

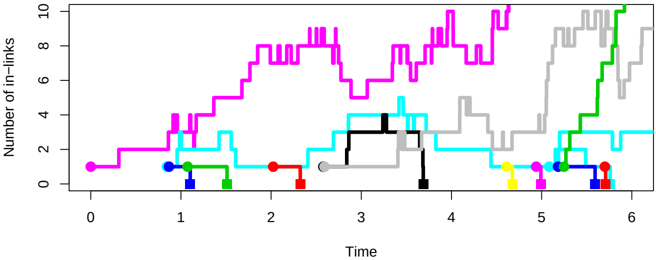

Figure : An example of a realization of the generalized Yule model with $\beta = 0.1$, $\lambda = 1.1$, $\mu = 1$ (supercritical case). The evolution of the number of in-links in different webpages is highlighted with different colors. The instants when new webpages are introduced are indicated with colored dots while the moments when webpages disappear due to the removal of the last in-link are denoted by colored squares.

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Generalization: Death

Results: Distribution, critical regime

We have

$$\mathbb{P}(\mathcal{N} = 0) = U(1, 0, \beta/\lambda),$$

and

$$\mathbb{P}(\mathcal{N} = n) = (\beta/\lambda)\,\Gamma(n)\,U(n, 0, \beta/\lambda).$$

where $U(a, b, z)$ is the confluent hypergeometric function.

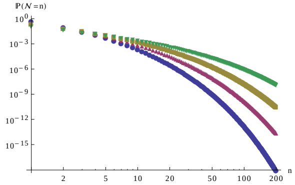

Figure : Distribution of the number of in-links for different values of the webpage rate constant, $\beta = \{1, 0.25, 0.5, 0.1\}$. Critical case, $\lambda = \mu = 1/2$; we can see that the tails decay faster than a power-law.

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Generalization: Death

Results: Distribution, subcritical regime

$$\mathbb{P}(\mathcal{N} = 0) = r_\beta(\mu, \lambda), \qquad \mathbb{P}(\mathcal{N} = n) = (\lambda/\mu)^{n+1} q_\beta^n(\mu, \lambda)$$

where

$$r_b(x, y) = c_b(x, y) {}_2F_1\left(1, \frac{b}{x-y}; 2 + \frac{b}{x-y}; \frac{y}{x}\right),$$
$$q_b^\nu(x, y) = d_b^\nu(x, y) {}_2F_1\left(\nu + 1, 1 + \frac{b}{x-y}; \nu + 1 + \frac{b}{x-y}; \frac{y}{x}\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and where

$$c_b(x, y) = \frac{b}{x-y}\Gamma\left(\frac{b}{x-y}\right)\Big/\Gamma\left(2 + \frac{b}{x-y}\right),$$
$$d_b^\nu(x, y) = \frac{b(x-y)}{x^2}\Gamma\left(1 + \frac{b}{x-y}\right)\Gamma(\nu)\Big/\Gamma\left(\nu + 1 + \frac{b}{x-y}\right).$$

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Generalization: Death

Results: Distribution, subcritical regime

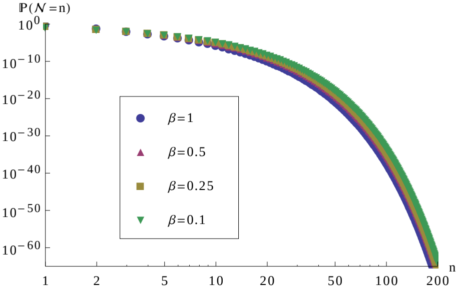

Figure : Distribution of the number of in-links for different values of the webpage rate constant, $\beta = \{1, 0.25, 0.5, 0.1\}$. Subcritical case, $\lambda = 1/4$, $\mu = 1/2$; note that the webpage rate constant β plays almost no role.

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Generalization: Death

Results: Distribution, supercritical regime

$$\mathbb{P}(\mathcal{N} = 0) = (\mu/\lambda) r_\beta(\lambda, \mu), \qquad \mathbb{P}(\mathcal{N} = n) = q_\beta^n(\lambda, \mu)$$

where

$$r_b(x,y) = c_b(x,y)\,{}_2F_1\left(1, \frac{b}{x-y}; 2 + \frac{b}{x-y}; \frac{y}{x}\right),$$

$$q_b^\nu(x,y) = d_b^\nu(x,y)\,{}_2F_1\left(\nu+1, 1+\frac{b}{x-y}; \nu+1+\frac{b}{x-y}; \frac{y}{x}\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function and where

$$c_b(x,y) = \frac{b}{x-y}\Gamma\left(\frac{b}{x-y}\right)\Big/\Gamma\left(2+\frac{b}{x-y}\right),$$

$$d_b^\nu(x,y) = \frac{b(x-y)}{x^2}\Gamma\left(1+\frac{b}{x-y}\right)\Gamma(\nu)\Big/\Gamma\left(\nu+1+\frac{b}{x-y}\right).$$

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Generalization: Death

Results: Distribution, supercritical regime

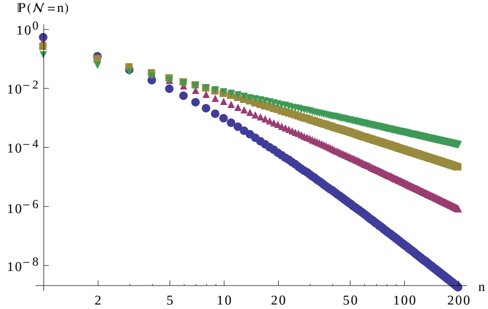

Figure : Distribution of the number of in-links for different values of the webpage rate constant, $\beta = \{1, 0.25, 0.5, 0.1\}$. Supercritical case, $\lambda = 1/2$, $\mu = 1/4$; the tails decay as power-laws.

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Generalization: Death

Results: Fitting real data

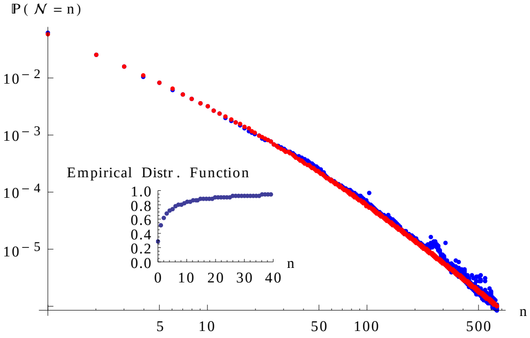

Figure : Fit of the empirical probability mass function for the number of in-links in the WWW. Data are taken from Web Data Commons, University of Manheim. It can be seen that the model (in red) fits the data even for small values of n $((\lambda; \mu; \beta) = (4; 3.8; 0.272))$. In the inset, the empirical distribution function calculated on the data.

Generalization: Fractionality/Nonlinearity

We consider a Yule-like model composed by

- a homogeneous Yule process (linear birth process) of rate $\beta > 0$ for the development of webpages; at most this hypothesis can be relaxed to mixed Poisson processes;
- independent copies of a fractional nonlinear birth process $N_{\mathrm{nl}}^{\nu}(t)$, $t \geq 0$, for the development of in-links for each webpage;
- rates λ_k, $k = 1, 2, \dots$, for the fractional nonlinear birth process such that explosions are not allowed, that is we admit only a finite number of jumps for any finite time. For this it is sufficient to assume that $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$.

Note that here we are neglecting detachment of in-links (deaths).

Call ${}_t\mathcal{N}_{\mathrm{nl}}^{\nu}$ the number of in-links of a webpage chosen uniformly at random at time t and define $\mathcal{N}_{\mathrm{nl}}^{\nu} = \lim_{t\to\infty} {}_t\mathcal{N}_{\mathrm{nl}}^{\nu}$.

Generalization: Fractionality/Nonlinearity

The distribution of ${}_t\boldsymbol{\mathcal{N}}_{\mathsf{nl}}^{\boldsymbol{\nu}}$ is derived as

$$\mathbb{P}({}_t\mathcal{N}_{\mathsf{nl}}^{\nu} = n) = \mathbb{E}_{\mathcal{T}}\mathbb{P}\big(N_{\mathsf{nl}}^{\nu}(t) = n | N_{\mathsf{nl}}^{\nu}(\mathcal{T}) = 1\big), \qquad n \geq 1.$$

where $\mathcal{T}$ has cumulative distribution function

$$\mathbb{P}(\mathcal{T} \leq y) = \frac{\mathrm{e}^{\beta y} - 1}{\mathrm{e}^{\beta t} - 1}, \qquad y \in [0, t],$$

that is $\mathcal{Q} = t - \mathcal{T}$, the random evolution time of the conditioned fractional nonlinear birth process ${}_t\mathcal{N}_{\mathsf{nl}}^{\nu}$, is distributed as a truncated exponential random variable.

Generalization: Fractionality/Nonlinearity

Distribution of ${}_t\mathcal{N}^{\nu}_{\mathrm{nl}}$

Working out the above formula we get ($n \geq 1$)

$$\begin{aligned}
&\mathbb{P}({}_t\mathcal{N}^{\nu}_{\mathrm{nl}} = n) \\
&= \frac{1}{1-e^{-\beta t}} \left[\beta^{\nu} \frac{\prod_{r=1}^{n-1} \lambda_r}{\prod_{r=1}^{n} (\beta^{\nu} + \lambda_r)} - \beta \int_t^{\infty} e^{-\beta y} \mathbb{P}(N^{\nu}_{\mathrm{nl}}(y) = n) \, \mathrm{d}y \right].
\end{aligned}$$

Thus ($\rho_r^{-1} = \beta^{\nu} / \lambda_r$)

$$\begin{aligned}
\mathbb{P}(\mathcal{N}^{\nu}_{\mathrm{nl}} = n) &= \beta^{\nu} \frac{\prod_{r=1}^{n-1} \lambda_r}{\prod_{r=1}^{n} (\beta^{\nu} + \lambda_r)} \\
&= \frac{\rho_n^{-1}}{\prod_{r=1}^{n} (\rho_r^{-1} + 1)} \qquad n \geq 1.
\end{aligned}$$

Generalization: Fractionality/Nonlinearity

Distribution of ${}_t\mathcal{N}_{\mathrm{nl}}^{\nu}$

When the rates are all different we further get

$$\mathbb{P}({}_t\mathcal{N}_{\mathrm{nl}}^{\nu} = n) = \frac{1}{1-e^{-\beta t}} \prod_{h=1}^{n-1} \lambda_h \sum_{m=1}^{n} \frac{1}{\prod_{l=1, l\neq m}^{n} (\lambda_l - \lambda_m)} \times \sum_{r=0}^{\infty} \frac{\gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^{\nu}} \right)^r, \qquad n \geq 1,$$

or, equivalently but highlighting the asymptotics, as

$$\mathbb{P}({}_t\mathcal{N}_{\mathrm{nl}}^{\nu} = n) = \frac{1}{1-e^{-\beta t}} \prod_{h=1}^{n-1} \lambda_h \sum_{m=1}^{n} \frac{1}{\prod_{l=1, l\neq m}^{n} (\lambda_l - \lambda_m)} \left[\frac{\beta^{\nu}}{\beta^{\nu} + \lambda_m} - \sum_{r=0}^{\infty} \frac{\Gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^{\nu}} \right)^r \right], \qquad n \geq 1.$$

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Generalization: Fractionality/Nonlinearity

Remarks

Remark: In the non fractional case, that is $\nu = 1$, we can further simplify the above probability mass function obtaining, after some steps,

$$\mathbb{P}({}_t\mathcal{N}^1_{\mathrm{nl}} = n)$$
$$= \frac{\beta}{1-e^{-\beta t}} \prod_{r=1}^{n-1} \lambda_r \sum_{m=1}^{n} \frac{1}{\prod_{l=1, l\neq m}^{n} (\lambda_l - \lambda_m)} \left(\frac{1-e^{-(\beta+\lambda_m)t}}{\beta + \lambda_m} \right), \quad n \geq 1,$$

Generalization: Fractionality/Nonlinearity

Remarks

Remark: For linear rates, $\lambda_r = \lambda r$, $r \geq 1$, the fractional nonlinear birth process $N^{\nu}_{\mathrm{nl}}(t)$ coincides with the fractional Yule process $N^{\nu}_{\mathrm{lin}}(t)$. The distribution can be written, for $n \geq 1$, as

$$\mathbb{P}({}_t\mathcal{N}^{\nu}_{\mathrm{lin}} = n) = \frac{1}{1-e^{-\beta t}} \left[\frac{\beta^{\nu}}{\lambda} \frac{\Gamma\left(\frac{\beta^{\nu}}{\lambda}+1\right)\Gamma(n)}{\Gamma\left(\frac{\beta^{\nu}}{\lambda}+n+1\right)} \right.$$
$$\left. - \sum_{j=1}^{n} \binom{n-1}{j-1} (-1)^{j-1} \sum_{r=0}^{\infty} \frac{\Gamma(\nu r+1, \beta t)}{\Gamma(\nu r+1)} \left(-\frac{\lambda}{\beta^{\nu}} m\right)^{r} \right]$$

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Generalization: Fractionality/Nonlinearity

Remarks

The limiting distribution as

$$\mathbb{P}(\mathcal{N}^{\nu}_{\text{lin}} = n) = \lim_{t\to\infty} \mathbb{P}({}_t\mathcal{N}^{\nu}_{\text{lin}} = n) = \frac{\beta^\nu}{\lambda} \frac{\Gamma\left(\frac{\beta^\nu}{\lambda}+1\right)\Gamma(n)}{\Gamma\left(\frac{\beta^\nu}{\lambda}+n+1\right)}, \qquad n \geq 1.$$

The above probability mass function is the usual Yule or Yule–Simon distribution of parameter $\rho^{-1} = \beta^\nu/\lambda$.

Thus there is a problem of **model identification**.

We remark however that from the finite time distribution it is clear the contribution to the asymptotics and the finite time correction. The latter depends indeed on the fractional parameter ν.

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Generalization: Fractionality/Nonlinearity

Mean of ${}_t\mathcal{N}_{\mathrm{nl}}^{\nu}$

Theorem: If the rates λ_r, $r \geq 1$, are all different we have, for $t \geq 0$,

$$\mathbb{E}\, {}_t\mathcal{N}_{\mathrm{nl}}^{\nu} = 1 + \sum_{k=1}^{\infty} \left[1 - \frac{1}{1-e^{-\beta t}} \sum_{m=1}^{k} \left(\prod_{\substack{l=1 \\ l \neq m}}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \times \sum_{r=0}^{\infty} \frac{\gamma(\nu r + 1, \beta t)}{\Gamma(\nu r + 1)} \left(-\frac{\lambda_m}{\beta^{\nu}} \right)^{r} \right].$$

Remark: When $\nu = 1$, that is in the classical non-fractional case, the mean value simplifies to

$$\mathbb{E}\, {}_t\mathcal{N}_{\mathrm{nl}}^{1} = 1 + \sum_{k=1}^{\infty} \left[1 - \frac{\beta}{1-e^{-\beta t}} \sum_{m=1}^{k} \left(\prod_{\substack{l=1 \\ l \neq m}}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \frac{1 - e^{-(\beta + \lambda_m)t}}{\beta + \lambda_m} \right].$$

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Generalization: Fractionality/Nonlinearity

Mean of ${}_t\boldsymbol{\mathcal{N}}_{\mathsf{nl}}^{\nu}$

Remark: The expected value of the limiting random variable $\mathcal{N}_{\mathsf{nl}}^{\nu}$ can be determined directly by letting $t \to \infty$:

$$\mathbb{E}\,\mathcal{N}_{\mathsf{nl}}^{\nu} = 1 + \sum_{k=1}^{\infty} \left[1 - \sum_{m=1}^{k} \left(\prod_{\substack{l=1 \\ l \neq m}}^{k} \frac{\lambda_l}{\lambda_l - \lambda_m} \right) \frac{\beta^{\nu}}{\beta^{\nu} + \lambda_m} \right].$$

It is interesting to note that the effect of the fractional parameter ν which is heavily present in the mean value for any fixed time t, practically vanishes (except a change in the parametrization) in the limiting mean value.

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Generalization: Fractionality/Nonlinearity

Models with saturation

An interesting behaviour that could possibly lead to a more realistic scenario is when the number of in-links for each webpage **has intrinsically a fixed value to which it saturates**. Examples of choice could be:

- We consider the rates as the weights of a discrete finite measure on the finite set $\{1, 2, ..., N-1\}$;
- Rates specialized as

$$\lambda_j = \eta \, (j/N)^{\omega_1} \left((N-j)/N\right)^{\omega_2} = \lambda j^{\omega_1} (N-j)^{\omega_2},$$

$(\omega_1, \omega_2) \in [0, \infty) \times (0, \infty)$, $\eta > 0$, where $\lambda = \eta / N^{\omega_1 + \omega_2}$.

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Generalization: Fractionality/Nonlinearity

A model with saturation

Here we specialize the birth rates as $\lambda_r = \lambda r(N - r)$, $1 \leq r \leq N$.

This corresponds to $(\omega_1, \omega_2) = (1, 1)$, $\lambda = \eta/N^2$. N represents the saturation level.

If $\mathcal{N}_{s_2}^{\nu}$ is the size of a randomly chosen webpage for $t \to \infty$ for this model allowing saturation, we have $(\rho = \lambda/\beta^{\nu})$

$$\mathbb{P}(\mathcal{N}_{s_2}^{\nu} = n) = \rho^{n-1} \frac{\Gamma(n)\Gamma(N)}{\Gamma(N - n + 1)} \frac{1}{\prod_{r=1}^{n}(1 + \rho r(N - r))}, \quad 1 \leq n \leq N.$$

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Generalization: Fractionality/Nonlinearity

A model with saturation

By working out the preceding formula, for $1 \leq n \leq N$, the explicit expression of the probability mass function for the random number of in-links in a webpage chosen uniformly at random can be written as

$$\mathbb{P}(\mathcal{N}_{s_2}^{\nu} = n) = \rho^{-1} \frac{\Gamma(n)\Gamma(N)}{\Gamma(N-n+1)} \frac{\Gamma\left(\frac{1}{2}\sqrt{N^2+\frac{4}{\rho}}+1-\frac{N}{2}\right)}{\Gamma\left(\frac{1}{2}\sqrt{N^2+\frac{4}{\rho}}+1-\frac{N}{2}+n\right)}$$
$$\times \frac{\Gamma\left(\frac{1}{2}\sqrt{N^2+\frac{4}{\rho}}+\frac{N}{2}-n\right)}{\Gamma\left(\frac{1}{2}\sqrt{N^2+\frac{4}{\rho}}+\frac{N}{2}\right)}.$$

Generalization: Fractionality/Nonlinearity

In the next slides various plots of the probability mass function $\mathbb{P}(\mathcal{N}^{\nu}_{s_2} = n)$, $n \geq 1$ for different values of the characterizing parameters, N (threshold at which saturation occurs), and ρ (which takes into account the webpage appearing rate β, the in-links appearing rate λ, and the parameter of fractionality ν).

Interestingly, the probability mass function of $\mathcal{N}^{\nu}_{s_2}$ can be **either convex or concave** depending on the values of the parameters.

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Generalization: Fractionality/Nonlinearity

Graphs

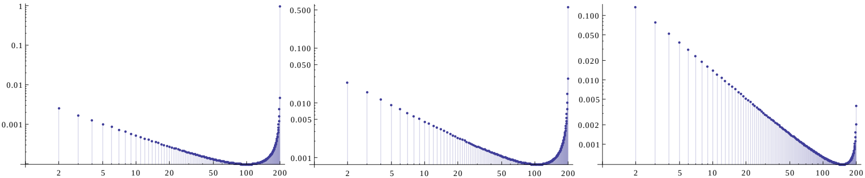

Figure : Various plots of $\mathbb{P}(\mathcal{N}_{s_2}^{\nu} = n)$. From left to right, the parameters are respectively $(\rho, N) = \{(1, 200), (0.1, 200), (0.01, 200)\}$.

Generalization: Fractionality/Nonlinearity

Graphs

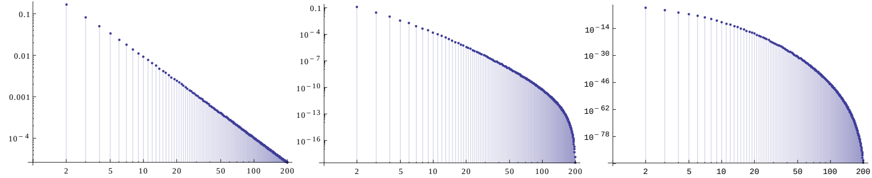

Figure : Various plots of $\mathbb{P}(\mathcal{N}^{\nu}_{s_2} = n)$. From left to right, the parameters are respectively $(\rho, N) = \{(0.005, 200), (0.001, 200), (0.0001, 200)\}$.

Interestingly enough, when $\rho = 1/N$, the distribution seems to exhibit a **perfect power-law behaviour**. This fact is further investigated graphically in the next slide for different values of the parameter ρ.

Generalization: Fractionality/Nonlinearity

Graphs

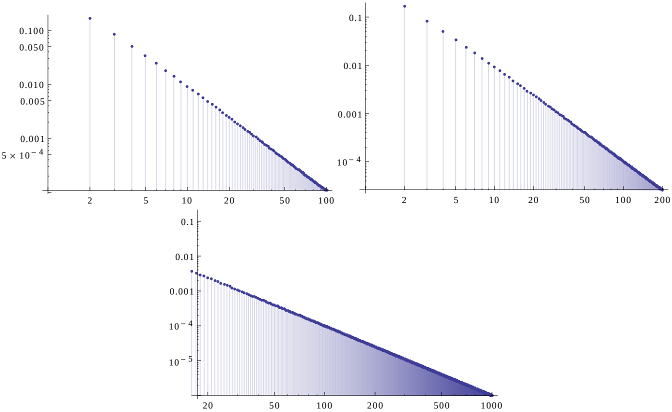

Figure : Plots of $\mathbb{P}(\mathcal{N}^{\nu}_{s_2} = n)$ when $\rho = 1/N$. Counterclockwise, starting from the upper-left plot. The parameters are $(\rho, N) = \{(0.01, 100), (0.001, 1000), (0.005, 200)\}$.

Generalization: Fractionality/Nonlinearity

Graphs

Preliminary results on the probability $\mathbb{P}(\mathcal{N}_{s_2}^{\nu} = N)$ that the uniformly selected webpage (vertex) is saturated, with $\rho = (1/N)^{\alpha}$.

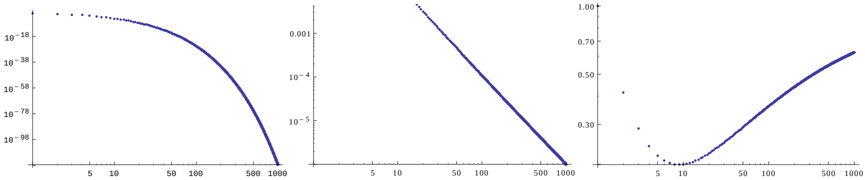

Figure : Plots of $\mathbb{P}(\mathcal{N}_{s_2}^{\nu} = N)$. Left to right, $\alpha = \{1.5, 1, 0.5\}$ with respect to increasing values of N.

Generalization: Fractionality/Nonlinearity

Graphs

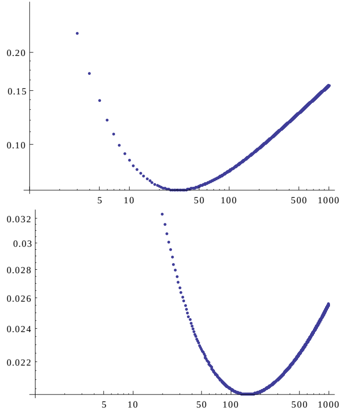

Figure : Plots of $\mathbb{P}(\mathcal{N}_{s_2}^{\nu} = N)$. Left to right, Top to bottom, $\alpha = \{0.7, 0.75, 0.8, 0.85\}$. The minimum moves towards ∞ when $\alpha \to 1$.

References

- Lansky, P, Sacerdote (2014), arXiv:1309.7823 [math.PR],
- Pachon, P, Sacerdote (2015), arXiv:1503.06150 [math.PR],
- Lansky, P, Sacerdote (2015), soon on the arXiv [math.PR].

Have a look also at the references therein!