Energy Landscape of Social Balance

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(Received 15 June 2009; published 4 November 2009)

We model a close-knit community of friends and enemies as a fully connected network with positive and negative signs on its edges. Theories from social psychology suggest that certain sign patterns are more stable than others. This notion of social "balance" allows us to define an energy landscape for such networks. Its structure is complex: numerical experiments reveal a landscape dimpled with local minima of widely varying energy levels. We derive rigorous bounds on the energies of these local minima and prove that they have a modular structure that can be used to classify them.

DOI: 10.1103/PhysRevLett.103.198701 PACS numbers: 89.65.−s, 89.75.Hc

The shifting of alliances and rivalries in a social group can be viewed as arising from an energy minimization process. For example, suppose you have two friends who happen to detest each other. The resulting awkwardness often resolves itself in one of two ways: either you drop one of your friends, or they find a way to reconcile. In such scenarios, the overall social stress corresponds to a kind of energy that relaxes over time as relationships switch from hostility to friendship or vice versa.

This view, now known as balance theory, was first articulated by Heider [1,2] and has since been applied in fields ranging from anthropology to political science [3,4]. In the 1950s, Cartwright and Harary converted Heider's conceptual framework to a graph-theoretic model and characterized the global minima of the social energy landscape [5]. Their tidy analysis gave no hint that the energy landscape was anything more complicated than a series of equally deep wells, each achieving the minimum possible energy. Recently, however, Antal, Krapivsky, and Redner [6] observed that the energy landscape also contains local minima, which they called *jammed states*.

Jammed states are important to understand because they can trap a system as it moves down the energy landscape. Yet little is known about their allowed energies, their structure, or how they depend on the size of the network. Even the maximum possible energy of a jammed state is not obvious: a simple argument (see below) shows that jammed states cannot be located more than halfway up the energy spectrum, but it is hard to see whether this upper bound can be achieved.

In this Letter, we prove that for arbitrarily large networks, there do indeed exist jammed states all the way up to the midpoint energy, using a construction based on highly symmetric structures first discovered by Paley in his work on orthogonal matrices [7]. We also show that jammed states have a natural modular structure. This allows us to organize the jammed states encountered by simulation and to explain why high-energy jammed states must be structurally more complex than low-energy ones.

More broadly, our work here is part of a growing line of research that employs tools from physics to analyze models of complex social systems [8–12]. Theories of signed social networks form an appealing domain for such techniques, as they are naturally cast in the framework of energy minimization.

We begin by modeling a fully connected social network as a signed complete graph on n nodes. Each edge $\{i, j\}$ of the network is labeled with either a plus or minus sign, denoted by s_{ij}, corresponding to feelings of friendship or animosity between the nodes i and j.

Up to node permutation, there are four possible signings of a triangle (Fig. 1). We view the two triangles with an odd number of plus edges as balanced configurations, since both satisfy the adages that "the enemy of my enemy is my friend," "the friend of my enemy is my enemy," and so on. Since the two triangles with an even number of plus edges break with this logic of friendship, we consider them unbalanced.

The product of the edge signs is positive for a balanced triangle and negative for an unbalanced triangle. If we sum the negative of these products and divide by the total number of triangles, we obtain a quantity U that represents the elevation, or potential energy, of a social network above the domain of all its possible sign configurations (Fig. 2). Explicitly,

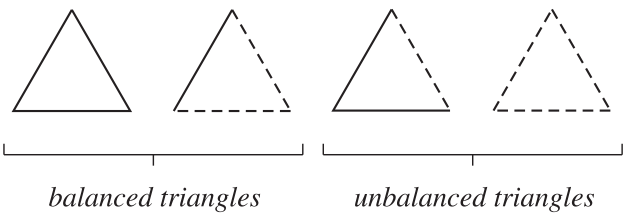

FIG. 1. Socially balanced and unbalanced configurations of a triangle. Solid edges represent friendly (+) relationships, and dashed edges hostile (−) relationships.

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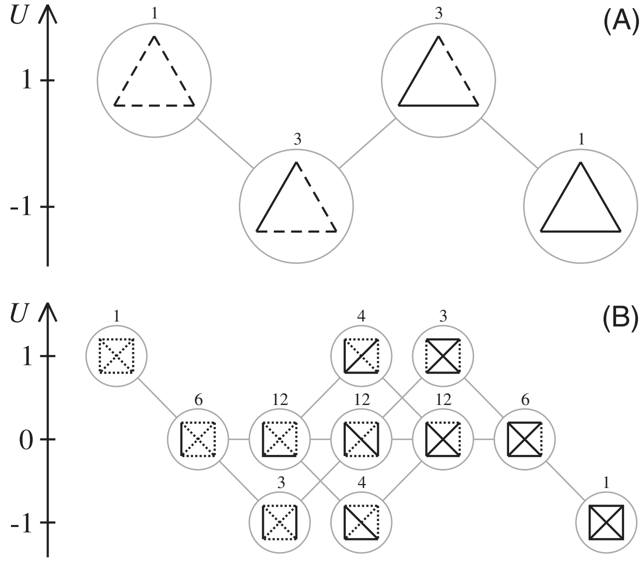

FIG. 2. The energy landscapes of signed complete social networks on (a) 3 and (b) 4 nodes. For simplicity, each set of sign configurations identical up to node permutation is represented by a single configuration; the number above each configuration indicates its multiplicity. Lines between circles join networks differing by a single sign flip. No jammed states occur for these small networks; they appear only when $n = 6$ or $n \geq 8$. Strict jammed states occur when $n = 9$ and $n \geq 11$ [6].

$$U = -\frac{1}{\binom{n}{3}} \sum s_{ij} s_{jk} s_{ik} \tag{1}$$

where the sum is over all triangles $\{i, j, k\}$ of the network.

The configuration in which all node pairs are friends has the lowest possible energy $U = -1$. Hence, no additional structure is necessary to define the global minima; they are just the sign configurations for which $U = -1$. Cartwright and Harary [5] identified all such ground states, finding that they consist of two warring factions: internally friendly cliques with only antagonistic edges between them. (The all-friends configuration represents the extreme in which one clique is the empty set.)

To define the concept of a local minimum, however, we need to specify what it means for two states to be adjacent. The most natural choice is to define two sign configurations to be adjacent if each can be reached from the other by a single sign flip. Then a jammed state, as defined by Antal *et al.*, is a sign configuration for which all adjacent sign configurations have higher energy [6]. Here, however, we will slightly vary their terminology by calling this a "strict jammed state," reserving the term "jammed state" for the weaker concept of a sign configuration with no adjacent sign configurations of lower energy.

Our first result is that jammed states cannot have energies above zero. To see this, note that every edge of a jammed state takes part in at least as many balanced triangles as unbalanced triangles. It is therefore found in $(n-2)/2$ unbalanced triangles if n is even and $(n-3)/2$ unbalanced triangles if n is odd. Thus, summing over all edges and dividing by 3 to avoid triple counting yields $U \leq -\frac{1}{3}\binom{n}{2}[(n-2-\frac{n-2}{2}) - \frac{n-2}{2}]/\binom{n}{3} = 0$ if n is even and $U \leq -(n-2)^{-1}$ if n is odd.

Are there jammed states that achieve this upper bound on U? One possible way to address this question is through computational searches. For example, suppose that from a random initial configuration, we select and switch single signs uniformly at random from the set of unbalanced edges (an edge is defined as unbalanced if more than half the triangles that include it are unbalanced). We continue switching signs until the network reaches a local minimum of U. Extensive searches of this form reveal only two small examples of zero-energy jammed states: a configuration on 6 nodes, consisting of a 5-cycle of positive edges and all other edges negative, and a more complex configuration on 10 nodes. Even on 10 nodes, only about 7 in 10^8 searches end up at zero-energy jammed states, and no such states were found on larger numbers of nodes. The failure of this approach to produce even moderately sized examples is consistent with findings of Antal *et al.* [6], who showed that such local search methods reach jammed states with a probability that decreases to 0 extremely rapidly as a function of the network size n.

With only these data, the chances of finding a larger collection of jammed states at $U = 0$ may seem slim. However, we now show how an infinite collection of zero-energy jammed states can be identified through a direct combinatorial construction. This construction is motivated by the two small examples found through computational searches: when we reexamined the zero-energy jammed states on 6 and 10 nodes, we noticed that the positive edges formed so-called Paley graphs [13] on 5 and 9 nodes. This beautiful connection turns out to be general: a family of arbitrarily large jammed states with $U = 0$ may be derived from the undirected Paley graphs.

Briefly, an undirected Paley graph P_q can be constructed on a set of q nodes, where q is a prime of the form $q = 4k + 1$ for some integer k. To do so, we index the nodes with the integers $0, \ldots, q-1$ and then connect each v and w in this node set with an edge if there is an x in $\{0, \ldots, q-1\}$ such that $(v - w) \bmod q = x^2 \bmod q$. To construct the jammed state with $U = 0$ from P_q, we give plus signs to the edges of P_q and minus signs to the edges of its complement. We then add a node v_n, where $n = q + 1$, and link it to all nodes of P_q with negative edges. (Paley graphs also exist if q is a prime power, but then one needs to work over the finite field of order q.)

We now show that this new signed complete graph has zero energy. Clearly, this is equivalent to the condition that each edge is in exactly $\frac{n-2}{2}$ balanced triangles. To check the latter claim, we make use of two known properties of Paley graphs: (i) P_q is $2k$-regular, and (ii) for any two nodes v and w of P_q, there are k nodes adjacent to v but not w, and k nodes adjacent to w but not v [13].

Now, if $\{v, w\}$ is a negative edge in P_q, then it forms balanced triangles with all nodes x in P_q that are linked by a positive edge to exactly one of v or w. By property (ii), there are $2k = \frac{q-1}{2} = \frac{n-2}{2}$ such nodes, so $\{v, w\}$ is in exactly $\frac{n-2}{2}$ balanced triangles. Similarly, if $\{v, w\}$ is a positive edge in P_q, then it forms unbalanced triangles with all nodes x of P_q that are linked via a positive edge to exactly one of v or w. Again, these nodes account for $2k = \frac{n-2}{2}$ unbalanced triangles, so $\{v, w\}$ is in exactly $\frac{n-2}{2}$ balanced triangles. Finally, since P_q is $2k$-regular, there are exactly $2k$ nodes in P_q adjacent via positive edges to each node w of P_q. Hence, each negative edge $\{v_n, w\}$ is also in exactly $\frac{n-2}{2}$ balanced triangles.

The above construction is related to a result by Seidel regarding two-graphs [14]. Using the theory of two-graphs, one can also construct infinite families of strict jammed states that approach $U = 0$ from below as n grows large. Such constructions can be carried out using bilinear forms modulo 2 [14], and projective planes in finite vector spaces [15].

Given the conceptual complexity of these constructions of high-energy jammed states, and the computational difficulty in finding such states via search, it is natural to ask why it is harder to construct jammed states closer to $U = 0$ than at lower energies. We now explain this by formulating a measure of the complexity of different jammed states. This will establish a precise sense in which higher-energy jammed states are structurally more complex than lower-energy jammed states, through a result showing that every signed complete graph has a natural decomposition into internally balanced modules.

The statement of this *edge balance decomposition* is as follows. Consider the subgraph K consisting of all nodes in the network, together with those edges that appear only in balanced triangles. Then (i) K is a union of disjoint cliques $\{C_a\}$ (possibly including single-node cliques), and (ii) for every pair of cliques C_a and C_b, every edge between C_a and C_b is involved in the same number of balanced triangles. In the spirit of (i), we call each clique of the partition a *balanced clique*.

To prove part (i) of the decomposition, one can show that if some connected component of K is not a clique, then this component contains edges $\{i, j\}$ and $\{i, k\}$ sharing a node i that are both found only in balanced triangles, and such that $\{j, k\}$ is in at least one unbalanced triangle (involving a fourth node ℓ). But then the set of four nodes $\{i, j, k, \ell\}$ would have three of its four triangles balanced, which is not possible for any sign pattern.

To prove part (ii) of the decomposition, one can show that if there were cliques C_a and C_b such that two different edges between them were involved in different numbers of unbalanced triangles, then there would be two such edges $\{i, j\}$ and $\{i, k\}$ sharing a node i in C_a, such that for some other node ℓ, the triangle $\{i, j, \ell\}$ is balanced but the triangle $\{i, k, \ell\}$ is not. But since $\{j, k\}$ is inside the clique C_b, all the triangles involving it are balanced, and so the four-node set $\{i, j, k, \ell\}$ would have three of its four triangles balanced, which again is not possible for any sign pattern.

We now return to the question that we posed above: why is it harder to construct jammed states near $U = 0$ than at substantially lower energies? We can close in on an elementary answer by computing an upper bound on the allowed energy of a jammed state as a function of the number of balanced cliques it contains. We find that as the energy approaches $U = 0$ from below, the number of cliques in the decomposition must grow unboundedly in n, the number of nodes in the network.

First, observe that for a fixed number of balanced cliques m, the fewest number of edges are in balanced cliques—and hence the most edges are available for inclusion in unbalanced triangles—when the n nodes of the network are equally distributed among the m balanced cliques. We can verify this using Lagrange multipliers: we seek to minimize $\sum_i \binom{c_i}{2}$ relative to the constraints $\sum_i c_i = n$, $c_i > 0$, where c_i is the number of nodes in the ith balanced clique. This implies $\frac{d}{dc_i}\binom{c_i}{2} = \lambda$ for all c_i, where λ is some constant. The derivative of the gamma function extension of $\binom{c_i}{2}$ is monotonically increasing on $c_i > 0$, so we invert it to find all c_i equal to the same function of λ.

Hence, no jammed state with n nodes and m balanced cliques can have greater energy than one in which the nodes are equidistributed among the balanced cliques and each edge spanning two balanced cliques participates in $\frac{n-2}{2}$ unbalanced triangles. This implies an upper bound on U of

$$U_n^{UB}(m) = -1 + 2\frac{\frac{1}{3}\binom{m}{2}\left(\frac{n}{m}\right)^2\frac{n-2}{2}}{\binom{n}{3}} = -\frac{n-m}{m(n-1)}. \quad (2)$$

For example, $\lim_{n\to\infty} U_n^{UB}(3) = -1/3$, whereas the corresponding tight upper bound (also verified by Lagrange multipliers) is $\lim_{n\to\infty} U = -\lim_{n\to\infty}\{[\binom{n}{3} - \binom{\frac{n}{3}}{3}^3] - \binom{\frac{n}{3}}{3}^3\}/\binom{n}{3} = -5/9$.

We can see directly from (2) that as we approach $U = 0$ from below, jammed states with n nodes and m or fewer balanced cliques no longer appear above $U_n^{UB}(m)$. In other words, jammed states disappear as the energy is raised in order of least to greatest complexity. Finally, at $U = 0$, the condition $U_n^{UB}(m) = 0$ implies that $m = n$, as we would expect since every edge must be in exactly $\frac{n-2}{2}$ balanced triangles.

In addition to illuminating a fundamental progression within the energy spectrum of the jammed states, the edge balance decomposition also provides a partition of the set of $2^{\binom{n}{2}}$ possible sign configurations which proves useful for classifying jammed states. Consistent with Antal *et al.* [6], our numerical simulations of small networks (generally $n < 2^{10}$) turned up an enormous number of three-clique jammed states. Less frequently, we encountered jammed states with five, six, and seven cliques, and rarely did we find jammed states with more than seven cliques (Fig. 3). This numerical evidence leads us to suspect that the most

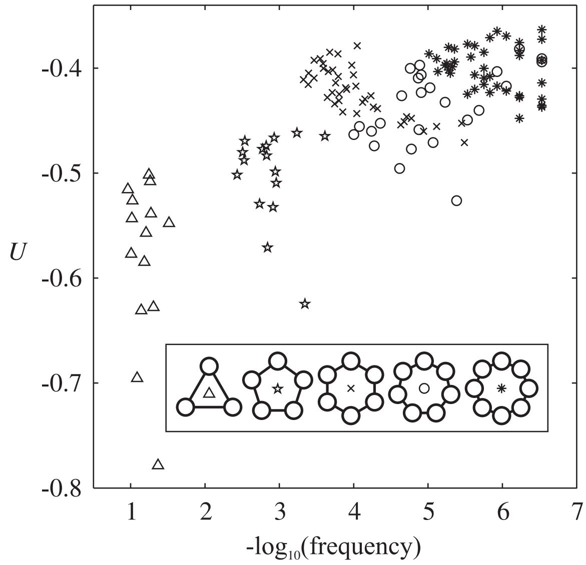

FIG. 3. Jammed states for networks with $n = 26$ nodes, distinguished according to their energy, frequency of occurrence, and clique structure. The different data symbols show the number of balanced cliques in a given state (see inset for key). We find that jammed states with higher energies are not only rarer (as shown by Antal *et al.* [6])—they also have inherently greater structural complexity, as measured by their number of balanced cliques. To find these states, we evolved 10^8 social networks to energy minima via the Markov process described in the text, assuming that each edge was initially unfriendly. For simplicity, only jammed states with eight or fewer balanced cliques are shown (these comprised >99.99% of all jammed states found). Jammed states with two and four balanced cliques are impossible. Analogous distributions for other n and other initial sign patterns are similar, and increasing the number of trial networks does not significantly change the distribution.

common jammed states found in sign patterns arising from local search have only a few balanced cliques and hence would be easily classified by the edge balance decomposition. (That said, it is possible to *construct* strict jammed states with m balanced cliques for all odd m in the large-n limit; whether such a construction exists for even $m > 6$ remains open).

In future work, it could be interesting to explore the model above using tools from other parts of physics [16–20]. For example, the social balance model may be viewed as a generalized Ising model [18] or Z_2 gauge theory [19] on the complete graph. It is also similar to spin-glass models [16,17] where nodes in a network are likewise joined by edges of mixed signs, and U measures the average frustration of the system. This line of work includes results on spin-glass systems with three-way interactions [20], such as occur in Eq. (1). One potential obstacle to making this link is that in spin-glass models, adjacency between configurations is defined by changes in the signs of nodes (due to spin flips) while edge signs remain fixed, whereas here it is the signs of the edges that vary as one moves across the landscape. This could possibly be addressed using transformations that interchange the roles of nodes and edges; however, when the complete graph is transformed in this way, the resulting network has a complex structure that may hinder analysis.

Taken together, the results presented here yield a first look at the energy landscape for completely connected social networks in which opportunities for greater relational consistency and cooperation are the driving forces for change. For now, our understanding of the landscape is confined to a few results about its local and global minima. The challenge for the future is to understand its large-scale structure, perhaps even including a characterization of the pathways leading out of the deepest minima—those corresponding to the most entrenched conflict—and toward states of reconciliation.

Research supported in part by the John D. and Catherine T. MacArthur Foundation, a Google Research Grant, a Yahoo Research Alliance Grant, and NSF Grants No. CCF-0325453, No. BCS-0537606, No. IIS-0705774, and No. CISE-0835706.

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