The three-state Potts model

1 Introduction

A large amount of information about the behaviour of real, physical systems can be extracted from simple models in statistical physics, because of the phenomenon of *universality*: near a second order phase transition, the behaviour of the system only depends on a few characteristics like the dimensionality and global symmetries of the system. Because of this, very simple models like the Ising model remain of great importance.

The Potts model is the simplest extension of the Ising model in statistical mechanics. The Hamiltonian looks very similar to that of the Ising model:

$$H = -J \sum_{ij} \delta(\sigma_i, \sigma_j) + h \sum_i \sigma_i \,, \tag{1}$$

where the first sum is over all nearest neighbours on the lattice and the second sum over all lattice points, and h is an external magnetic field. The difference is that the spins σ_i can now take on q different values, where q is some integer ≥ 2. The 2-state Potts model ($q = 2$) is just the Ising model.

In this project we will investigate the 3-state Potts model in 2 dimensions. Like the Ising model, it has a second order phase transition separating a low-temperature ordered phase from a high-temperature disordered phase. It is relevant for systems with a three-fold symmetry, including quantum chromodynamics.

2 Simulation

We will be using the Metropolis algorithm to simulate the 3-state Potts model with zero external magnetic field ($h = 0$) on a $N \times N$ lattice with periodic boundary conditions. The main procedure will be very similar to that used for the Ising model, with the following differences:

1. Each spin can now take 3 different values. It may be most convenient to choose these values to be $\exp(2n\pi i/3), n = 0, 1, 2$.

2. When updating the spin on a site, choose randomly between the two other possible spin values and calculate the old and new energies using (1).

After each sweep through the entire lattice, calculate the magnetisation $\mu = \sum_i \sigma_i / N^2$ and the total energy E.

Choose $J = 1$ and a range of temperatures, starting at $T = 0.6$ and going up to $T = 1.5$. For each temperature (and lattice size) you will first need to perform a series of thermalisation sweeps in order for the system to adjust and settle down to the thermal behaviour at the given temperature. You can find out how many sweeps are required by monitoring the time evolution of μ. The values of μ and E during thermalisation will be thrown away. Subsequently, you should perform as many sweeps at each temperature as you need to get a precise determination of the quantities you will be calculating (see below).

In order to study the behaviour of a statistical mechanics system numerically near a phase transition, it is necessary to perform simulations at several different volumes, and extrapolate to the infinite volume limit. You may start on a relatively small volume like 20^2, and proceed to larger volumes $(40^2, 60^2, 100^2)$ later as required.

3 Analysis

For each temperature and lattice volume, you should calculate the following quantities:

1. Magnetisation $\langle \mu \rangle$;
2. Average energy $\langle E \rangle$;
3. Magnetic susceptibility $\chi = N^2(\langle \mu^2 \rangle - \langle \mu \rangle^2)$;
4. Specific heat $C_{\mathrm{v}} = (\langle E^2 \rangle - \langle E \rangle^2)/T^2$

You should also estimate the statistical uncertainties of these quantities. To do this, use the *jackknife method*:

1. From your ensemble of n measurements of your basic quantities, create n ensembles with $n - 1$ measurements each by removing one measurement.
2. For each quantity $K = (\langle \mu \rangle, \langle E \rangle, \chi, C_{\mathrm{v}})$ determine the average K_i on each of these n ensembles.
3. The uncertainty in the quantity K is given as

$$(\Delta K)^2 = \sum_{i=1}^{n} K_i^2 - \frac{1}{n}\Big(\sum_{i=1}^{n} K_i\Big)^2$$

Plot all the quantities as a function of T including the errors (you can use the MatLab function `errorbar` to do this).

3.1 Critical temperature

From the data collected above, you may now determine the critical temperature T_c. In the infinite volume limit, the magnetisation $\langle\mu\rangle(T)$ will be zero for all $T > T_c$, but on a finite volume $\langle\mu\rangle \neq 0$ for all T. The value will however drop sharply near T_c, which translates into a peak in the susceptibility χ and the specific heat C_v. Use this to determine T_c for each lattice size.

3.2 Critical exponents

Once the critical temperature has been found, we may define the *reduced temperature* τ,

$$\tau = (T - T_c)/T_c\,. \tag{2}$$

Near the critical temperature, the system will exhibit *universal critical behaviour*, which can be described by a number of *critical exponents*:

$$C_\mathrm{v} \propto |\tau|^{-\alpha} \quad \chi \propto |\tau|^{-\gamma}\,, \quad \langle\mu\rangle \propto (-\tau)^{\beta}(\tau < 0)\,. \tag{3}$$

These relations (and others like them) are valid for *all* systems with the same dimensions and symmetries provided the interaction is short range. To determine the exponents α, β, γ you may plot the quantities $C_\mathrm{v}, \chi, \langle\mu\rangle$ vs τ on a logarithmic scale for the various lattice sizes. For small τ these curves should become straight lines, with the slope given by the critical exponent.