Vicissitudes of Cities driven by Redistributive Growth

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Empirical evidence suggests that the evolution of urban systems is not only determined by local conditions, but also is constrained by regional status. We propose an out-of-equilibrium model of emerging cities within a given region, which explains the spatial transitions of developmental focus and urban shrinkage phenomenon in developed cities. Meanwhile the model analytically keeps the classical results such as Clark's law for urban population density, and Zipf's law for cities' rank size distributions. We derive that these classical properties are valid for developing areas, or equivalently, most of the present cities; and the second phase of our model predicts the inevitability of various urban diseases given finite regional resource.

INTRODUCTION

The rapid urbanization in the past few decades brings an utmost importance of the interdisciplinary urban studies from a complex systems perspective. Elaborately, disciplines formulated from urban observations have led to many spatial growth models that succeed in deriving analogies[1], such as urban scaling laws[2], fractality[3, 4], and city size distributions[5]. Furthermore, they give quantitative predictions broadly to urban morphology and spatial structures[6]. Existing works have gained insights underlying complex features of cities, especially those in their developing phase. The macroscopic urban aspects, i.e., the size distributions, are usually presented by multiplicative or correlated percolation[7–9] models. Their formations including the spatial explicit preferential attachment[10], or resource utilization[11] mechanisms which leads to the proportional growth of each cities, result in a Zipfian distribution of urban aggregates rank sizes. On other hand, the economies of scale provided city-wise, represented by urban scaling laws[2, 12, 13], are realized through the characterization of human interactions, especially spatial networks[2, 14, 15]. These depictions have widely crossed human actions[16–18]. These models function in specific spatial scale under certain equilibrium conditions in growth rates or optimization aims[5]. However, urban systems are dynamic, and necessarily out-of-equilibrium with limitations in economical, political, topographical, and many other sustainability goals that lead to vicissitudinary development[19, 20]. For instance, city's attractiveness is not eternal as its marginal effect for latter stage of urbanization[21, 22]. Meanwhile, the present spatial growth models still lack the ability to capture the competition among cities in both space and resource. To address this issue needs a cross-scale consideration of inter- and intra-city formation in growth dynamics, and some systematic sustainable conditions.

These concerns call for a general approach to model the spatial sprawl of emerging cities and their population under sustainable rules, regarding the competitions for space and resource. Below we propose an out-of-equilibrium spatial preferential growth model with restrictions on the maximum systematic rate to grow. In some non-spatial context[23], a finite population has been proved to put severe constraints on evolution modes, which can be specified as urban rank-size distribution here. This restriction is proved later to enhance the level of competitions, and result in realistic modern urban phenomena like dual cities[24], superior switch[25], and urban shrinkage[26], that cannot be formulated by existing growth models. The spatial aspect of our model takes the idea of diffusion-limited aggregation[7, 9]. In other words, new comers would settle near those active citizens, to be involved in the economies of scale[27].

RESULTS

The Spatial Yule Model

Our model tells how cities emerge, grow, and compete over a closed region in a 2-dimensional continuous space. As this model regards the growth of cities as how active dwellers constantly introduce new citizens, its dynamics are determined through a dearth of quantitative and spatial rules. *Quantitatively*, let there be k cities and N_i active citizens in the ith city. By active we mean in urban growth process, only these citizens attracts new comers to a nearby place in her city. Here, we take $\sum_i N_i \leq N^*$ as the satiation condition, i.e., when the population of the whole region exceeds N^*, an introduction of new active citizen deactivates a random dweller who is previously active. This mechanism keeps only up to N^* citizens registered as introducers. Therefore we say these N^* people add up to the *memory kernel*. Thus we define the model in terms of a two-phase master equation, assigning N_i to the growth rate of a city. Namely, we assume that the probability of a population increase in city i within the time interval $(t, t + dt)$ is $\beta_2 N_i dt$. We also assume that with a small probability proportional to the number of cities, $\beta_1 k dt$, a new city is created with one citizen. The

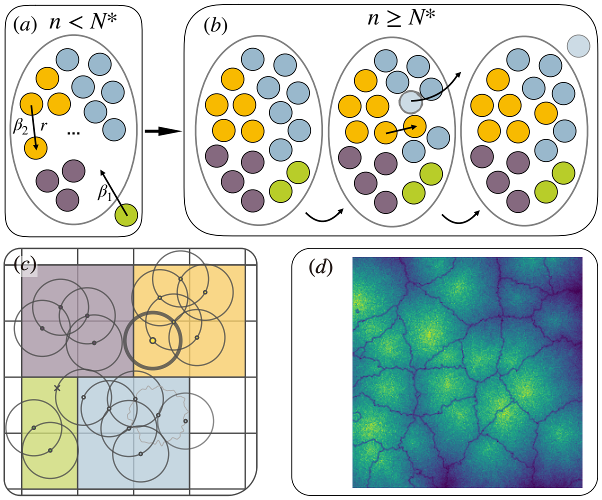


FIG. 1. (a) Status in the memory kernel at the first phase, where the total population is less than N^*. Existing citizens introduce new dwellers at the rate β_2, while each existing city (noted by nodes in different colors) introduces new cities at the rate β_1. (b) When the memory kernel is fulfilled, every introduction of new city or citizen leads to an ejection of existing active citizen currently included in the memory kernel. (c) The spatial aspect is that, an offspring citizen's placement is at distance r from the ancestral dweller. Also, when the kernel is filled, a new yellow node ejects an existed blue node, or equivalently deprive her ability to introduce. (d) A simulated result for L, r, β equals to 256, 0.5, and 4, respectively.

master equation can therefore be written as

$$\frac{\partial}{\partial t}N_i(t) = \delta_{N_i(t)} \cdot k\beta_1 + (1-\delta_{N_i(t)}) \cdot N\beta_2, \quad (1)$$

for the free growth phase, and

$$\frac{\partial N_i(t)}{\partial t} = (N^*\beta_2)[\frac{N_i(t)}{N^*} \cdot (1 - N_i(t)/(N^*-1)) \\ -(1 - N_i(t)/N^*) \cdot (1 - N_i(t)/(N^*-1))] \quad (2)$$

for the resource constraint phase. Assigning the constants β_1 and β_2 specifies the model. Competition for more active population helps us set up the constraint in systematic resource.

The *spatial* growth rules consist of three parts. We first assume the studied area is an $L \times L$ square space with grids. Through the second part we achieve the competition for space: Once a grid have held a citizen from the ith city, any citizens from other cities $j \neq i$ cannot introduce new comers on the same grid; Cities are established randomly over untaken grids; Thirdly, the coordinates of citizens belong to $\mathbb{R}^2$, and every new citizen settles at a constant distance r and an random angle θ from its introducer.

Since the setting of relative growth rate $\beta := \beta_2/\beta_1$ is parallel Yule's work on the distribution of species per genus[28], in the following text we refer our model as Spatial Yule models with constraints (SYM). A sketch for the SYM is shown in Fig. 1.

The SYM is based on mainly three realistic assumptions. Basically, to insure the appearance of free growth phase, the region is naturally assumed to be sufficiently large and active ($L \gg 1$, $N^* \gg 1$), while the impact of a citizen is restrained within the neighborhood($r \leq 1$). Secondly but most importantly, urban developments are density driven (active citizens are attractive). Literature has suggested that social ties and interactions resulted from urban density comprise an important driver of the economies of scale[22, 29, 30]. In the SYM, we further assume that only parts of the population are attractive. Such active part can be recognized as the total employed or productive people. Combined with the grid structure of the SYM, we can further analyze the density dynamics both collectively and internally, leading to Zipf's law and phenomenal urban shrinkage, respectively. Thirdly, to make it analytical, we assume a homogeneous background in two parts: New cities' establishment is a Poisson point process[31]; And active citizens at different position introduce new comers at the same rate. This diffusive setting of sequential settlements is also realistic urban growth[32]. Meanwhile, the last setting copes with Clark's law of urban population density[33].

In the numerical experiments, we conclude that $L = 256$ is sufficiently large for the free growth phase to emerge. So the truly worth-tuning parameters are only three. β contributes to the Zipf's coefficient and later defined turnover rate[34]; r contributes to fractality of urban envelops and the time to fill the whole space; And N^* is the severeness of resource competition which results in the fluctuation of turnover rate.

The free growth phase

Depending on the relative importance of city sprawl, emergence rate, and economic constraints, SYM predicts the existence of three phases: freely growth phase, economic constraint phase, and spatial constraint phase. We focus on the first two phases, which correspond with cities' memory. In the freely growth phase, cities grow desolately, without being controlled by total resource and space. We now describe its two important properties, stately (1) Zipf's law[35] for rank size distribution of cities' population, and (2) Clark's law for mono-centric cities' population density.

The populations of cities typically decay proportionally to the inverse of their ranks[35]. This is referred as Zipf's law of cities' population sizes, i.e., the populations of cities distribute as a power of ranks, $f_r(r) \sim r^{-(1+\beta)}$. Recall that the number of individuals in the system at time t, $N_i(t)$, has a geometric distribution[36], $P(N_i(t) = n) = e^{-\beta_1 t}(1 - \exp(-\beta_1 t))^{n-1}$, and the second assumption that the number of cities will grow exponentially at rate $k\beta_1$, if we pick a random city, the time since its first appearance waits exponential times with parameter β_1. Thus the distribution of population of a random city is

$$f(n) = \frac{\Gamma(1+1/\beta)\Gamma(n)}{\beta\Gamma(n+1+1/\beta)} \approx Cn^{-1-1/\beta}, \text{ as } n \to \infty, \quad (3)$$

where $\Gamma(\cdot)$ is the gamma function. This implies a Zipf's relationship with $n(\text{rank}) \sim rank^{-\beta}$. Noticing that β takes value from all positive real number in SYM, we can derive arbitrary scaling behaviors by switching β. According to some existing studies[37], the power law dependence of population frequency is 2.03 ± 0.05 for the world, indicating the average relative emerging rate of cities is around 1.03.

A small β means the emergence of cities is fast, which corresponds to a large spatial scale. The problem of determining the relative speed of city's generation, is very reminiscent of some problems encountered in gas physics. It is interesting to investigate the number of cities in a given regions of the same population. Some groups tend to form new cities to have sufficient infrastructures and less diversity of urban output ($\beta > 1$) and some cities may go otherwise ($\beta < 1$). This value is actually a reflection of the intensity of regional industry. The experiments have confirmed our analytic results for the first phase in SYM. A simulated validation for this result can be reflected in Fig.2(a).

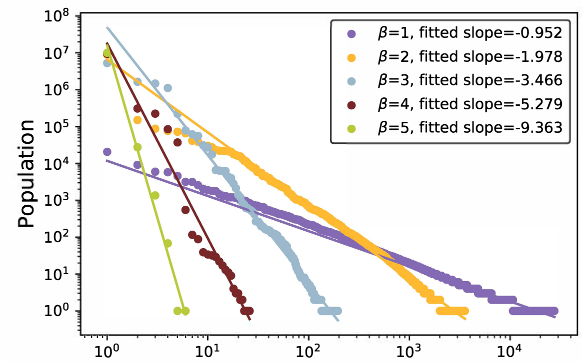
Population density evolves as two-dimensional diffusion within a city[38], where we can focus the density's growth on each axis from the oldest citizens of a city. Let $\rho(d)$ denote the active population density of places of the distance d from a city's center, and t_n as the time for the n'th citizen to generate, we have

$$\rho_{t_{n+1}}(d) = (\rho_{t_n}(d-r) + \rho_{t_n}(d+r))/2. \quad (4)$$

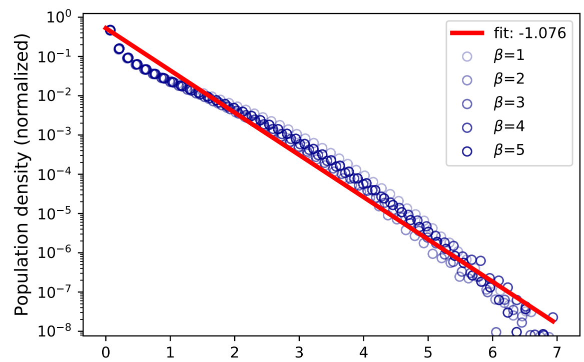
By re-scaling time as $\tau_n = t_n \cdot (k\beta_1 + N\beta_2)/T$, for a sufficient large T, this equation4 results in a exponential decay of density

$$\rho(d) \sim e^{-\alpha d}. \quad (5)$$

This is a reinvention of Clark's law in empirical urban studies[33]. Beyond solitary growth, we analyze the competitiveness of land for different cities. The population within an edging block of city j is estimated by $e^{(T-T_j)} \int_d^{d+1} \rho(r)dr/(2\pi d)$, where T_j is the emerging time of city j. We also have the waiting time $T_{n+1} - T_n \sim 1/n$, and the total population approximation $e^{\beta_1+\beta_2}$, combining which we derive the population of edging blocks if time and the urban radius are given. Since the attractiveness of large urban center is larger, the edging population of large cities is actually smaller than minor cities. We validate our prediction with simulations in Fig.2(b). Investigating r, we conclude that the metropolis areas over the world have very different densities. In SYM, it determines the sprawl of a city with given population. It can also be taken as the area proportion for a city in the studied region. On the other hand, it is also constraint of regional growth controlling the expected allowance of cities.



(a) Rank



(b) Distance to the urban center

FIG. 2. **(a)** The distribution of population among cities. In the simulation we take $N^* = 10^5$ and alternate βs. The realistic Zipf's coefficient is reproduced when $\beta \approx 1$. The theoretical predictions of the slopes are $-\beta$, and is well approximated when βs are small. Larger β reduce the chance of latter city's emergence. Thus the spatial aspects of the SYM strengthen inequality. This result confirms that Zipf's law is valid for growing urban systems where all cities share the same rate to grow. From the other master equation we analyze that this observation vanishes if total growing force is finite. **(b)** The population distribution as a function of distance from a district's center. The vertical axis is logarithmic processed, which represents the exponential decaying of population distribution. Regardless of the finite-sample effect, we fit the middle part of spatial population density to the exponential distribution with a slope of -1.076.

The economic constraint phase

The multi-dimensional coincidence between the exponents derived in our model and the universal exponents in empirical evidences of population distribution indicates that only two observation scales lead to the behaviors of regional dynamics. This means that the actual urban growth has not yet reached the constrained cases. However, preventive measures are still necessary. Thus we bring a general constraining parameter N^* to further discuss the second phase of SYM, the economy constraint phase, i.e., the total population reaches N^*, the size of memory kernel. Such setting is the abstract of many real-life rules set by global organizations such as the allowance of carbon emissions or sustainable development projects. In each city, a proportion of population are coin holders, labeled as memorized. Here, $\sum_{i=1} m_i(t) = N^*$ for t that is sufficiently large. If in some period, the minor cities generate more offspring than major ones and the superiority of remaining population within the memory kernel changes, minor city will increase its ranking, as the growing rate for each city i is actually $m_i\beta_2$. On the dynamics within the memory kernel, for each city, m_i acts as a random walk with absorption wall 0, since no offspring will be expected if no nodes are left in the kernel. This result also works for single block case within a city. Denote the population with a block j of city i as m_{ij}. According to[36], we use a result for branching process that a block loses its vitality if the population goes downhill under a threshold

$$\rho_{threshold} = k\beta_1/(2\beta_2) + N^*/2. \tag{6}$$

This value shall be regarded as the sign for *urban shrinkage*, for the edging blocks have lower density according to equation5 thus have an exponentially higher probability to be languished. In other words, urban shrinkage shall be reasoned by limited systematic resources in the given region.

The kernel mechanism also plays a role at the cross-city scale: The preference of larger cities is easier to fail. The competition for coins in SYM receives more than pure birth settings because the sum of fortune is given as N^*. In other words, SYM system doesn't suffer from inflation. To test this interpretation, we analyze the superior switching rate, defined as the average frequency of timesteps in a realization that the second largest city surpasses the largest in active population within the memory kernel. We conduct numerical experiments, and receive power law dependence between the frequency and the simulating steps, shown in Fig.3. Moreover, the switching is more likely to happen with a memory kernel, i.e., switching with memory kernel decay slower in probability. It is also a clear result since a growing society (a society without a memory kernel) suffer less from inter-specific competition.

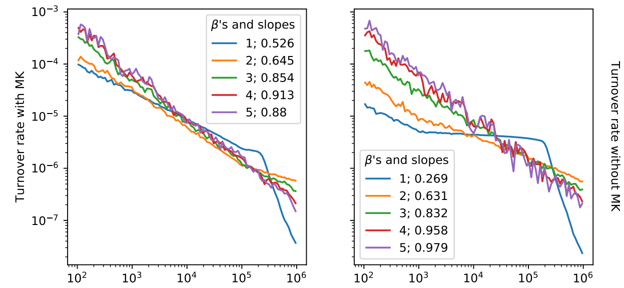


FIG. 3. The change rate statistics with and without a memory kernel. The kernel keeps turning over more often. With same β's, a kernel-based SYM's decay in turnover rate is smaller. These results validate our prediction that with finite resource, advantages are more likely to be kept.

The last property of SYM is the fractality of urban envelop, stately, the length of urban edges vary with the used measurement. Inspired by multi-player interaction in fractal financial market[39], we interpret that fractal urban boundary is driven by the competition for land at cities' edges. In SYM, the uncertain competition for space lies in parameter r. A larger r indicates larger randomness and brings an extra advantage for minor cities, resulting in a larger fractal dimension. We apply the box counting technique to calculate the fractal dimension of urban envelops, and receive an stable output of $d_f = 1.2 \pm 0.05$ with $r = 0.5$, similar to empirical results[30]. We also find larger d_f's for greater r. These results validate our hypothesis that fractal edges coexist with spatial competition.

DISCUSSION

This *Letter* concludes the urban system dynamics in only three key components, and receives fruitful results. The SYM leads a way in the adaptation of realistic conditions in statistical physical modeling, by regardless of the whole present population within the system, and considering only the active part of them. SYM explains existing properties, such as Zipf's and Clark's law, and also predicts regional trends in a probabilistic perspective. We analytically derive Zipf's law of global population, Clark's law for urban density, and the fractal behaviors of urban edges. Due to the simplicity of SYM, we investigate the future phase transition of urban development in great details, and explain the dilemmas of the present stage of urbanization through the competitions for systematic resource and space. The assumptions of the SYM are well-held if sufficient divergence of meta-population across the world is considered. Simulations of this model can be adjust to heterogeneous geographical circumstances by applying the growing rate on each block to the product of inherent dynamic $m_{ij} \cdot \beta_2$ and the local characteristic c_{ij} to better suit for realistic conditions.

The memory kernel mechanism leads to a straightforward corollary that the construction of infrastructure is the reason of population's spatial transitions, as only those who are recorded in the kernel are considered as productive people that attract new-comers to his city. This result provides a bottom-up explanation of transition of urban centers with stochastic spatial shifts of cities' memorized people. It also tells that the economic growth is the basis of growth potentials. Under the circumstances of preferential attraction, if the size of the memory kernel cannot grow fast enough to match with population, the concentration of production will go far from tolerance.

Taking the productive aspect together in the memory kernel helps to talk about many other properties like the age structure. The stationary age can be calculated as the average time for a new city to emerge is $(\beta_2 N^* + \beta_1 k)^{-1}$, which equals to the average losing age of the whole kernel. This result gives an instruction of the length of workable age in a given social urban system.

Although our results are not all analytically proved, we believe it is a essential step to strip out the power of urban dynamics. The model is non-commuting, but the community structure is naturally embedded. For further consideration, we can extend the model by adding links as the volume of exploration and preferential return between cities[40]. The model can further be extended with multi-dimensional memory kernel, allowing one citizen to be introduced if different factors (i.e., the existing citizens in different dimension of kernel) agree to allow her in the system.

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