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# Spatial Yule Model for Urban Growth with Shared Resources

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ABSTRACT

We study the competition of cities at a highly urbanized stage through a spatial version of the Yule model with a fixed carrying capacity. The capacity naturally determines two phases of the model, wherein the natural growth phase, we derive and observe the finite Zipf scaling of urban size distributions and the exponential decay of population density in each city; the resource constraint phase corresponds to the stage that the population of the urban system reaches the fixed capacity; thus migration among cities rather than urbanization contributes to the population dynamics of cities. In this phase, we derive the likelihood of a switch in the ranking of the cities. Our model sheds light on the post-urbanization era's inter-city competition patterns and population dynamics.
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## 1. Introduction

Urbanization is rapidly increasing around the world [1,2], with 65.22% of the global population living in cities in 2022 and the United Nations projecting that number to exceed 80% by 2050. This trend is driven by the advantages of innovation and productivity found in urban areas compared to suburbs, leading to a continuous influx of people into cities [3]. The rapid urbanization process has led to significant population decrease in rural areas [4]. As a result, there is a main contradiction in balancing a country's development between urban and rural areas [5,6].

In developed countries where the urban population has already exceeded 80%, the main contradiction is among cities rather than between urban and rural areas. In these countries, rural areas are highly automated and efficient, and population growth is slow and characterized by migration between cities. For example, the proportion of the urban population in the United Kingdom has exceeded 80% since around 1900 and has only grown to 84.5% in 2021 [7], while the proportion of London's population over the UK population has grown from 11.8% in 1981 to 14% in 2021 [8]. Due to political and globalization processes, large cities like London attract talent and have more developmental advantages in business, education, and globalization resources. Such advantages inherit natural heterogeneity at the later stages of urbanization, where the dominant trend of population flow is from minor cities to major cities.

Many models have been proposed to explain urbanization processes [9–16], with most addressing the attractiveness of cities through the 'preferential attachment' mechanism. These models successfully explain urban growth patterns, including the size distribution of urban clusters and the spatial decay of population density. For example, [17] adopt a spatial version of Krapivsky–Redner network growth redirection model, to give rise to a power-law Zipf-like rank-size distribution of cities, which is widely accepted as the global consistent law of urban population. However, recent empirical research has shown deviations in the size distribution of cities from Zipf's law, especially for some largest cities, in well-urbanized regions [18–22]. One possible explanation for this deviation is that cities are preferred heterogeneously over

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time based on industrial specialization. However the quantitative descriptions of the urban growth trajectories are lacking because of the scarcity of instances.

To address these issues, we propose a generalized urban growth model that follows a preferential attachment mechanism while also placing constraints on the total urbanized population shared by all cities in the region. In this model, urban cluster sizes are redistributed based on individual random choices of working chances to settle in large cities. The bounded urbanized population shared in the region represents the intrinsic, but limited benefit of living in cities, including concentrated governmental investment for high working efficiency. This setting allows us to derive competition among cities for people during the model's second phase, while the first phase reformulates Zipf's law of city size distributions.

## 2. Model

To provide a quantitative description of urban growth, we define $(n_1, n_2, \ldots, n_m)$ as the population of cities 1, 2, …, m, initially set to $(1, 0, 0, \ldots, 0)$, where $m$ is large enough to represent the total number of potential cities. The total population $N(t)$ is the sum of each city's population, $\sum_{i=1}^{m} n_i(t)$. Each city is composed of its citizens and the land on which they settle. To model individual settlement actions, we use a $L \times L$ square grid-cell region. The governance of city $i$ is constrained by the grids occupied by its $n_i$ citizens, which will grow as the population increases. Once a grid cell is occupied by a citizen of city $i$, it cannot be occupied by a citizen of another city $j \neq i$.

The distribution of clusters $(n_1, n_2, \ldots, n_m)$ can change due to three microscopic factors: (i) At a rate of $\beta_1 n_c(t)$, a new city's first citizen is generated and assigned to a grid cell at random. Here, $n_c(t)$ refers to the number of existing cities, or the number of $n_i(t)$'s where $n_i(t) > 0$. If the grid cell does not belong to any existing city, the new city is added to the system with its first citizen in the selected cell. Otherwise, the potential city is not added to the system. (ii) The urbanization process introduces more people into the cities. At a rate of $\beta_2 N(t)$, an existing citizen $x$ of some city $i$ gives birth to a new citizen $x'$. The newly born citizen is placed at a distance of $r$ from the parent citizen toward a random angle $\theta$, where $r$ is a constant of 0.5 unless specified, and $\theta$ is chosen uniformly from $\theta \in [0, 2\pi)$. If the selected location of the newly born citizen $x'$ does not contain any existing citizens of other cities, $x'$ survives and becomes a member of city $i$. Otherwise, $x'$ is not added to the system. (iii) The migration process occurs when the total number of citizens reaches a given bound $N^*$. An existing citizen is randomly selected to move to another existing city. During the migration process, a citizen may change her/his citizenship, i.e., the index of the city she/he belongs to. Additionally, a citizen who changed her citizenship from city $i$ to city $j$ can give birth to $j$'s new citizens. The migrating citizen resettles near the corresponding individual at its $(r, \theta)$.

If both $N^*$ and $L$ tend toward infinity, the quantitative characteristics of our model become analogous to the model originally proposed by Yule and Simon, which describes the distribution of species per genus. In Yule's model, the asymptotic growth of the number of species is directly proportional to the number of groups. Our model, on the other hand, presents a more contemporary competition between clusters. The survival of newly emerging clusters is heavily influenced by spatial constraints and the intrinsic competition among people. As the total population of the urban system approaches the upper limit, the competitive advantage of cities weakens, leading to an increase in individual migrations and more intense competition between cities. As a result, we refer to our model as the Spatial Yule Model. A sketch of the model is presented in Fig. 1.

## 3. Results

Given $L$ is large enough, the model indicates that the population growth of each city can be divided into two phases:

$$\frac{\partial n_i}{\partial t} = \begin{cases} \delta_{n_i} n_c(t)\beta_1 + (1 - \delta_{n_i(t)}) n_i(t)\beta_2, & N(t) < N^*, \\ \beta_2 n_i(t) - (\beta_1 n_c(t) + (N^* - n_i(t))\beta_2), & N(t) = N^*, \end{cases} \tag{1}$$

where $\delta$ is the indicator function, when the population of city $i$, $n_i(t) > 0$, then $\delta_{n_i} = 0$; and when $n_i(t) = 0$, $\delta_{n_i} = 1$. The free growth phase is when $N(t) < N^*$, the increment of urbanized population are mostly from rural areas, and the urbanization in this phase is weakly dependent on space and resource. And the resource constraint phase is when $N(t) = N^*$, the governmental investment cannot support more individuals to settle in cities, and the migration is the only mechanism that redistribute urban population. To summarize, the model is regulated by three tunable parameters. The individual exploration distance $r$, the total population of the region $N^*$, and the relative growth rate $\beta = \beta_2/\beta_1$. Here, $\beta$ can also be interpreted as the reinvestment ratio, at which rate the central governmental decide to reinvest to start another urban cluster from the start.

### 3.1. The free growth phase

We first illustrate the observations in the free growth phase, when the total population of all cities $N(t)$ is less than $N^*$. At this phase, our model reproduces two important phenomena of urbanization systems: Zipf's law for rank size distribution of population across cities, and Clark's law for exponential decay of population density within a city.

City populations typically decay proportionally to the inverse of their rank sizes, which is the well-known Zipf's law. Our model provides an analytic explanation for the Zipf's law in the following way:

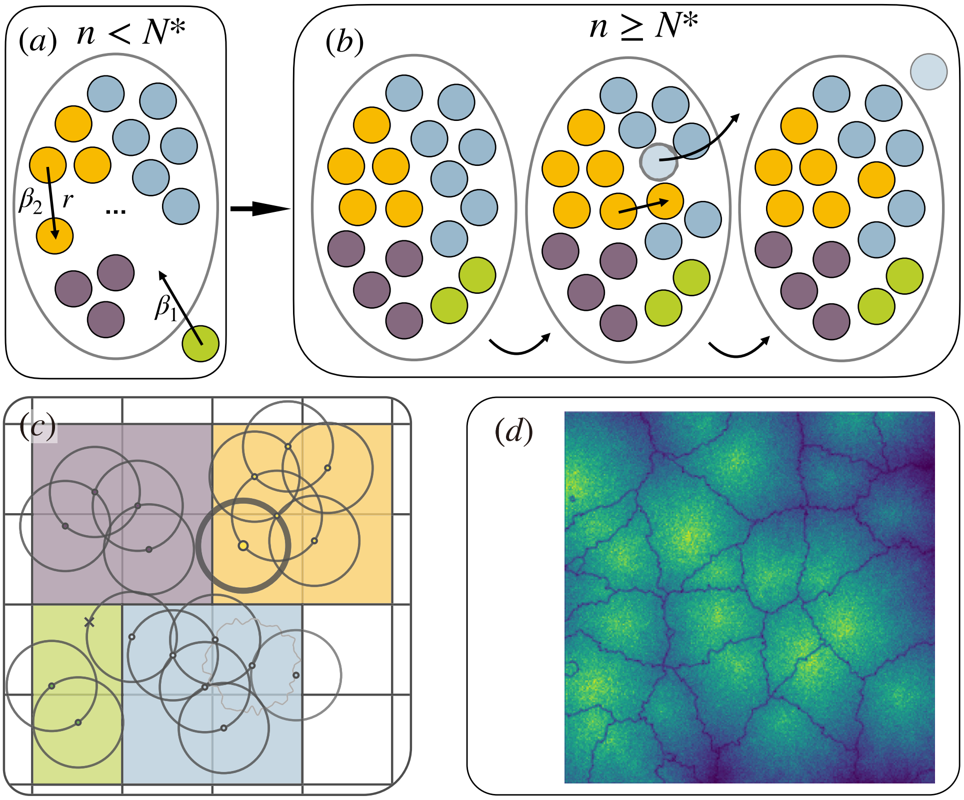


**Fig. 1.** **(a)** Status in the memory kernel at the free growth phase, i.e., the sum of all cities' population is less than $N^*$. Existing citizens introduce new dwellers at the rate $\beta_2$, while each existing city (noted by nodes in different colors) introduces new cities at the rate $\beta_1$. **(b)** When the total number of active citizens reaches $N^*$, the active citizens migrate, rather than grow, toward some existing randomly picked active citizens. **(c)** The spatial aspect of the model is that an offspring citizen's placement is at distance $r$ from an existing active citizen. **(d)** Simulated results for $L$, $r$, $\beta$ equal 256, 0.5, and 4, respectively. We choose $\beta = 4$ to avoid confusion of too many cities shown. This is equivalent to a quarter of a $2L \times 2L$ simulation with $\beta = 1$.

Firstly, we try to prove that city $i$'s population $N_i(t)$ has a geometric distribution of $P(N_i(t) = n) = e^{-\beta_1 t}(1 - \exp(-\beta_1 t))^{n-1}$ through a similar proof in [23]. The geometric distribution is a single-parameter probability distribution thus to determine it needs only one parameter, the mean. So to prove the claim, we first determine its mean, then determine its form of geometric distribution.

To prove the expected population of a city initiated $t$ ago (denoted as $Z(t)$), we note that each of the $Z(t)$ citizens introduce new comers at the same rate $\beta_2$, we note that its growth rate $\mathrm{d}Z(t)/\mathrm{d}t = \beta_2 Z(t)$, which implies the expected population of a city at time $t$ $Z(t) = e^{\beta_2 t}$. To validate the geometric distribution form of $P(N_i(t) = n)$, we use Kolmogorov's Forward Equation, that the probability distribution of registering $j$ people in $i$ cities at time $t$ is given by the differential equation

$$p'_t(1, j) = -\beta_2 j p_t(1, j) + \beta_2(j-1)p_t(1, j-1). \tag{2}$$

, where $p_t(1, 0) = 0$. When $j = 1$, we have

$$p'_t(1, 1) = -\beta_2 p_t(1, 1),$$

which leads to $p_t(1, 1) = e^{-\beta_2}$; When $j > 1$, we assume the $P(Z_t)$ follows the geometric form above, and differentiate $P(Z_t = j)$ by time $t$,

$$\begin{aligned}
\frac{dP(Z_t = j)}{dt} &= -\beta_2 e^{-\beta_2 t}(1 - e^{-\beta_2 t})^{j-1} + e^{-\beta_2 t}(j-1)(1 - e^{-\beta_2 t})^{j-2}\beta_2 e^{-\beta_2 t} \\
&= -\beta_2 e^{-\beta_2 t}(1 - e^{-\beta_2 t})^{j-1} + e^{-\beta_2 t}(j-1)(1 - e^{-\beta_2 t})^{j-2}[-(1 - e^{-\beta_2 t})\beta_2 + \beta_2] \\
&= -\beta_2 e^{-\beta_2 t}(1 - e^{-\beta_2 t})^{j-1} - \beta_2 e^{-\beta_2 t}(j-1)(1 - e^{-\beta_2 t})^{j-1} + \beta_2 e^{-\beta_2 t}(j-1)(1 - e^{-\beta_2 t})^{j-2} \\
&= -\beta_2 j e^{-\beta_2 t}(1 - e^{-\beta_2 t})^{j-1} + \beta_2 e^{-\beta_2 t}(j-1)(1 - e^{-\beta_2 t})^{j-2}
\end{aligned} \tag{3}$$

which agrees with Eq. (2) when comparing each of the terms. Thus we complete the proof of the probability distribution of a city's population. The probability distribution of the region's population is easy to find according to the given result.

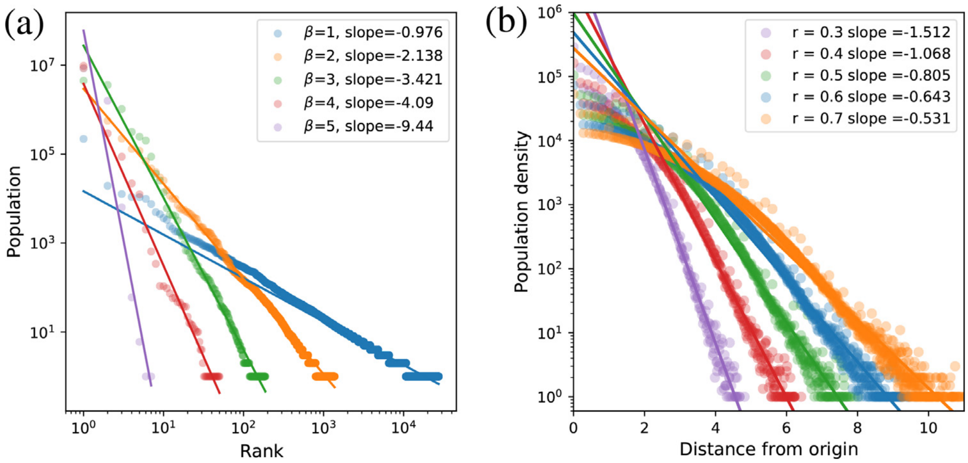


**Fig. 2.** **(a)** The distribution of population among cities. In the simulation we take $N^* = 10^5$ and alternate $\beta$s. The realistic Zipf's coefficient is reproduced when $\beta \approx 1$. The theoretical predictions of the slopes are $-\beta$, and are well approximated when $\beta$s are small. Larger $\beta$ reduces the chance of the emergence of latter city. Thus the spatial aspects of the SYM strengthen inequality. This result confirms that Zipf's law is valid for growing urban systems where all cities share the same rate of growth. From the other master equation, we analyze that this observation vanishes if the total growing force is finite. **(b)** The population distribution as a function of distance from a district's center. The vertical axis is logarithmic processed, which represents the exponential decay of population distribution. Regardless of the finite-sample effect, we fit the middle part of spatial population density to the exponential distribution with a slope of $-1.076$.

The probability distribution of registering $j$ people in $i$ cities at time $t$, is

$$P_t(i,j) = \begin{pmatrix} j-1 \\ i-1 \end{pmatrix} \left(e^{-\beta t}\right)^i \left(1 - e^{-\beta t}\right)^{j-i}.$$

Recall that the number of cities grows exponentially at rate $\beta_1$, if we randomly pick an existing city, its existence time is exponential with parameter $\beta_1$. Thus, the distribution of a city's population is

$$f(n) = \frac{\Gamma(1+1/\beta)\Gamma(n)}{\beta\Gamma(n+1+1/\beta)} \approx Cn^{-1-1/\beta}, \text{ as } n \to \infty, \tag{4}$$

where $\Gamma(\cdot)$ is the *gamma* function and $C$ is a normalizing constant. This equation implies a Zipfian relationship with $n(\text{rank}) \sim \text{rank}^{-\beta}$. Noticing that $\beta$ takes its value from all positive real numbers in our model, we can derive arbitrary scaling behaviors by altering $\beta$. According to some empirical research [9], the power law dependence of population frequency is $2.03 \pm 0.05$ for the entire world, indicating that the average relative emerging rate of cities is around 1, meaning that the relative growth rate of the number of cities and active citizens are identical, if rescaled by the current number of cities and active citizens, respectively.

Zipf's law is a useful tool for analyzing different geographical and urban systems, although the Zipfian exponent can vary depending on the system being studied. In our model, we use the tunable parameter $\beta$ to examine the variation of the Zipfian exponent, which reflects the intensity of regional population concentrations in large cities. Fig. 2 supports our analytic results during the free growth phase. Interestingly, when $\beta$ is large ($> 2$), the simulated Zipfian exponents are notably larger than their theoretical predictions. This occurs due to a spatial effect, where the competition for space favors small cities because the expected population density at boundary cells is higher than in larger cities. However, in systems where $\beta$ is large for cities at the same rank, the probability of a successful new city emergence decreases due to the relatively large area of existing cities.

Next, we examine the intra-city population density in our Spatial Yule Model (SYM). The SYM assumes that the intra-city population density evolves as a two-dimensional diffusion from the location of the first active citizen [24]. Therefore, we can focus on the density dynamics in the radial direction. To derive the spatial distribution of the population within a species, we first use the mean-field approach. We assume that a node is placed on a broad area, and the second node is placed at a distance $r$ to the right of the first with a probability of $1/2$. Along this axis, the $n$th node is placed at $k$ from the right end with $C_n^k/2^n$. Using the Stirling formula, we can approximate the density at distance $k$ from the center as:

$$\rho(k) \approx \frac{n^{n+1/2}}{\sqrt{2\pi}k^{k+1/2}(n-k)^{n-k+1/2}} \sim e^{-k}. \tag{5}$$

Let $\rho(d)$ denote the population density of the active citizens at distance $d$ from a city's center, and let $t_n$ denote the time for the $n$th citizen to be generated. We can use the equation

$$\rho_{t_{n+1}}(d) = (\rho_{t_n}(d-r) + \rho_{t_n}(d+r))/2 \tag{6}$$

to model the density evolution. Neglecting spatial constraints and rescaling time as $\tau_n = t_n \cdot (n_c(t)\beta_1 + n_c(t)\beta_2)/T$, where $n_c(t)$ is the number of cities at time $t$ and $T$ is a sufficiently large time, we obtain an exponential decay of density:

$$\rho(d) \sim e^{-\alpha d}, \tag{7}$$

known as Clark's law [25]. The density of population turns out to follow an exponential distribution. This suggests that the local properties of the spatial Yule model can be viewed as a discrete version of a maximum entropy system.

The above analysis suggests that the population density of a city follows an exponential decay with distance from the city center, which can be expressed by Clark's law. This law has implications for the competition among cities, particularly at urban intersections, which can influence the local Zipfian coefficients. In particular, the population density at the edges of a city is important because it determines the competitive advantages for space, which also influences the local Zipfian coefficients. In particular, the population density at the edges of a city is important because it determines the competitive advantages of space.

To estimate the population density at the edges of a city, we recall Clark's law which states that the population density is just a function of a city's age and the distance from an urban center. The population within an edge cell of city $j$ is estimated by $e^{(T-T_j)} \int_d^{d+1} \rho(r)\, \mathrm{d}r/(2\pi d)$, where $T_j$ is the emerging time of city $j$. We also have the waiting time $T_{n+1} - T_n \sim 1/n$, and the total population approximation $e^{\beta_1+\beta_2}$, by combining which we derive the density of edging cells if time and the urban radius are given. We denote that the distance between the first and the furthest node of a city as $O$ and $F$, respectively, and the moment that $F$ lands its first offspring as $t+\tau$, where $t$ is the moment that $F$ is landed. We investigate the radius of the city, which is defined by the distance between $O$ and $F$. The radius of a city at time $t$, $R_t$, changes to $R_{t+\tau}$ when the offspring is given birth. The expectation of $R_{t+\tau}$ goes as

$$\begin{aligned} ER_{t+\tau} &= \int_0^{\frac{\pi}{2}} \sqrt{(R_t + r\cos\theta)^2 + (r\sin\theta)^2} d\theta \\ &= \sqrt{2R_t r} \int_0^{\frac{\pi}{2}} \sqrt{\frac{R_t}{2r} + \frac{r}{2R_t} + \cos\theta d\theta} \\ &= 2(R_t + r)\mathbb{E}\left(\pi/4 | \frac{4R_t r}{(R_t + r)^2}\right) \end{aligned} \tag{8}$$

where $\mathbb{E}$ is the elliptic function. We find that it decreases as $r$ increases. So that the smaller cities have more active edging cells.

Since a large urban center is more attractive, the population at the edge of large cities is actually smaller than that of minor cities. We validate our prediction with simulations, shown in Fig. 2. In addition, a larger $r$ will flatten the difference of spatial competitiveness among cities with different sizes, as the spatial distribution of new emerging citizens is less skewed, so a larger proportion of citizens lives near city boundaries. In reality, metropolis areas over the world have very different densities. In SYM, $r$ corresponds to the spatial sprawl of each city for the studied urban system. It can also be taken to study the proportion of a city's area in the studied region. On the other hand, it also controls the radial limit of cities if $N^*$ is relatively small, which we will cover in the following discussion of the economic constraint phase.

### *3.2. The economic constraint phase*

The economic constraint phase we describe in this section reflects the dynamics of highly urbanized regions or countries where urban expansion is no longer driven by rural migration but by migration from other cities. This transition can occur when the non-urban habitable space becomes scarce or when the urbanization rate exceeds a certain threshold. To account for these scenarios in our model, we define a limit on the total population of cities, denoted by $N^*$, which serves as a generalization of many real-world regulations aimed at promoting sustainable development and limiting resource depletion, but objectively limits the region to be further urbanized.

For example, the Greater London region introduced the Congestion Charge to reduce traffic congestion and curb excessive business growth, while zoning laws and territorial development strategies are employed in the US, Hong Kong, and other areas to restrict clustering development [26]. These examples illustrate the need for internal regulations in megacity systems to prevent unchecked urban expansion and the resulting depletion of resources and increased living costs.

Our model aims to explain the phenomenon of *urban shrinkage*, which refers to a significant population decline in a city. We examine this phenomenon in the context of the economic constraint phase of the SYM. Specifically, we propose the following logic to explain population shrinkage: when the total urbanized population of a region is limited by $N^*$, a city's population can decrease if its attractiveness to migrants from other cities is lower than the attractiveness of those other cities.

To quantitatively analyze the effect of migration on urban shrinkage, we consider the rate at which migrants from other cities resettle in city $i$ as $\beta_2(N^* - n_i)n_i/N^{*2}$, where $(N^- n_i)/N^*$ is the probability that a migrant is not a citizen of city $i$, and $n_i/N^*$ is the probability of the migrant resettling in city $i$. Similarly, the rate at which individuals move from city $i$ to other cities is $((N^* - n_i)\beta_2 + n_c\beta_1)n_i/N^*$, which can lead to the formation of a new urban cluster or the growth

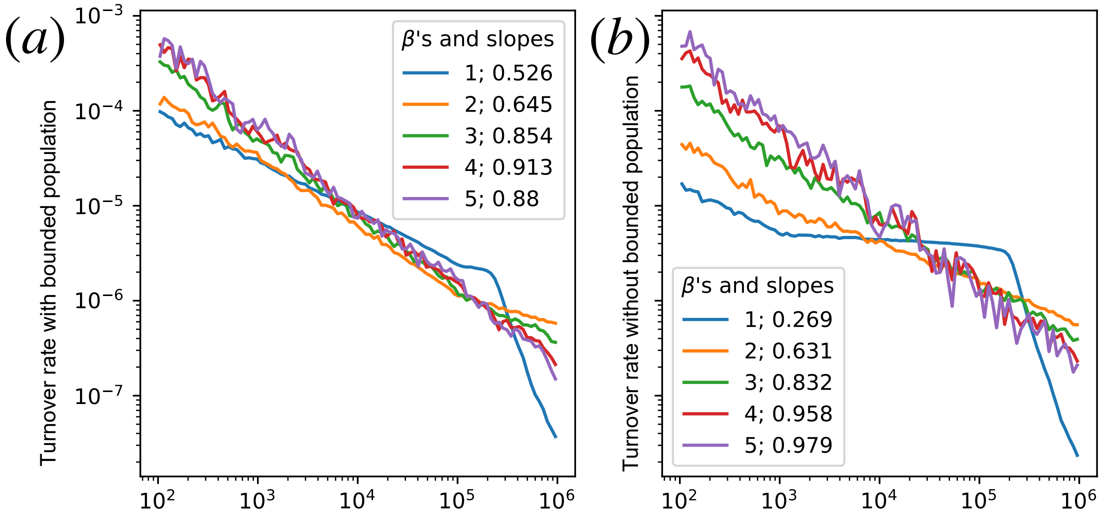


**Fig. 3.** The change rate statistics with $(a)$ and without $(B)$ a constraint in the region's total population. The $x$-axis is the number of the simulation steps, and the $y$-axis is the percentage of trials whose second largest city at the time exceeds the size of the largest at the $x$'th step. The intercepts are the constraint that keeps the turnover of the ranks of cities' population more often. With the same $\beta$'s, the turnover rate of an SYM with a constraint in population decays slower. These results validate our prediction that with a finite resource, advantages are more likely to be kept.

of an existing city. The equilibrium condition is when these two rates are equal, which gives the critical size of the urban cluster as:

$$n_i^* = N^* + (n_c/\beta_1 + 2)(N^* - 1)/N^*,$$

where $n_c$ is the number of existing cities and $\beta_1$ is the rate of new urban cluster formation. When $n_i < n_i^*$, the city will gradually shrink.

The expression shows qualitatively that the presence of more existing cities and the formation of new urban clusters lead to the fading out of smaller cities, and this is supported by empirical evidence from urban systems worldwide. For instance, Tianjin, which was once the third-largest city in China, has seen a decline in its population due to the emergence of numerous high-tech subcenters in Beijing. As a result, talented individuals have migrated to Beijing for better career opportunities, hindering Tianjin's development in high-tech industries and other emerging sectors.

Similarly, the analysis can be moved to the analysis of populations of the grid cells, that when the exploration radius $r$ is small, whether a grid cell's population will remain depends on a critical threshold $\rho^*$, which can be an explanation of the *urban sprawl* phenomenon.

On the mean-field approximations, we have derived the critical population of maintaining an active city. Here, we further discuss the competition of the urban cluster in a probabilistic way by introducing the turnover rate, defined as the average frequency that the second largest urban cluster's population surpasses the largest urban cluster. We conduct numerical experiments to test the relationship between reinvestment rate $\beta$ and the turnover rate, and whether the population bound $N^*$ plays a role in probabilistic allometric growth (Fig. 3). A baseline result is when $N^* \to \infty$, regardless of the reinvestment rate $\beta$, the turnover rate decays in the shape of a power law with the total population in the urban system (Fig. 3's left panel), meaning that the ranking order of cities gets harder to change over time as the cities gain accumulative advantages. Next, we set $N^* = 10^6$, which would increase the turnover rate because the largest city are more likely to lose their citizens (with the probability $n_1/N^*$ which decays with $N^*$) when the small cities are picked to reproduce in the economic constraint phase. We see a significant divergence of the turnover rate's reliance on the reinvestment rate $\beta$ (Fig. 3's right panel). Generally, the turnover rate decays slower if the system has constraints. This result also indicates that an arrest of growth (small $N^*$) harms the potential of economies of scale (large $n_1$).

## 4. Discussion

A two-phased spatial growth model is proposed to simulate and analytically derive the properties of the regional urbanization process. First, the model reproduced Zipf's law for urban cluster distribution and Clark's hypothesis of the urban population's exponential decay from each urban center. Further, we introduced the parameter for economic constraint N* to emphasize the latter stage of urbanization: the limits of the governmental redistribution that leads to higher urban efficiency and presents some perspectives of the model to explain urban shrinkage and inter-city competition.

Although the work briefly introduces our intuition that the constraints in resources quantitatively explain the post-urbanization phenomena (e.g., urban shrinkage, and the deviations of cluster size distribution), more directions are necessary to characterize it, including: 1. Analytical solutions on more aspects of the model; 2. Further numerical analysis based on current urban agglomerations; 3. Reconsider the 'soft' approaches of a city's occupying spaces.

Preferential attachment has been extensively studied in economic geography, but empirical evidence often contradicts the hypothesis that urban growth trajectories have shifted from crowding into cities to intercity population flows. Additionally, the size distributions of cities often significantly deviate from Zipf's law [15,27,28], and various models have been proposed to explain these deviations based on factors such as city fitness or value [29,30]. However, these factors only determine new cities' success, and the presented stochastic model provides a preliminary attempt to link several post-urbanization phenomena with intercity competition. With more detailed inputs, such as considering the fitness of the city at the early stage of development, this model could serve as a reliable benchmark for quantitatively tracing urban clusters' size redistribution.

## CRediT authorship contribution statement

**Gezhi Xiu:** Conceptualization, Methodology, Formal analysis, Experiment design, Validation, Writing – original draft, Writing – review & editing. **Jianying Wang:** Experiment design, Validation, Writing – original draft, Writing – review & editing, Visualization. **Yu Liu:** Writing – original draft, Writing – review & editing, Supervision, Funding acquisition.

## Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: Yu Liu reports financial support was provided by National Natural Science Foundation of China no. 41830645.

## Data availability

No data was used for the research described in the article.

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