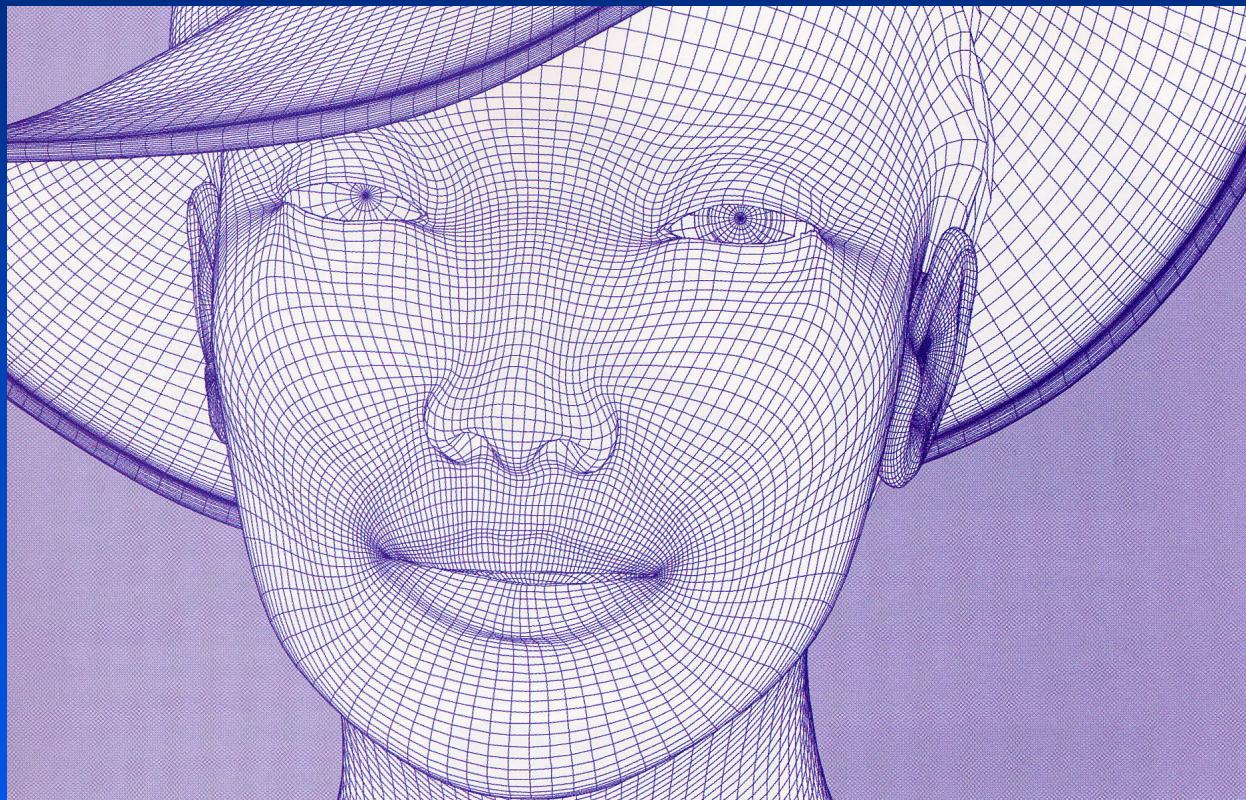


Parametric Curves and Surfaces I



Virtual Celebrity/Marlene Inc.

Higher Degree Curves and Surfaces

- Way to model objects, specify motion curves
- Compact
 - Smooth curve/surface possible with few control points
- Easier to use for modeling
 - User has small set of parameters that control large DOF
- Image rendering more complex
 - Direct or Polygonal approximation
- 3D “rendering” simpler
 - Numerically Controlled (NC) milling of mathematically defined surface

Explicit equation

- $y=f(x)$, $z=g(x)$
- Cannot get multiple y or z for single x
- Not rotationally invariant
- Cannot describe curves with infinite slopes

Implicit equation

- $f(x, y, z)=0$
- Have to solve non-linear equation for each point
- Difficult to move along the curve
- More solution than we want (need to specify additional constraint)
- Difficult to join two curves (such that tangents are same)
- Derivatives not given in terms of direction of motion

Parametric equation

- $x=x(t), y=y(t), z=z(t)$
- As t varies from 0 to 1, function sweeps out a curve (or surface in the case of parametric surfaces in u and v)
- t not plotted

Cubic

- $x(t) = a_x t^3 + b_x t^2 + c_x t + d_x$

$$y(t) = a_y t^3 + b_y t^2 + c_y t + d_y$$

$$0 \leq t \leq 1$$

$$z(t) = a_z t^3 + b_z t^2 + c_z t + d_z$$

- Lower degree poly – too little control (cannot be made to pass through specific point with specified derivatives)
- Higher degree poly – too much complexity (greater the degree, greater the amount of oscillations)
- Need 4 known vector equations to solve for the 4 unknowns
- Depending on the constraints, different curve

Matrix form

$$Q(t) = [x(t) \ y(t) \ z(t)] = \underbrace{[t^3 \ t^2 \ t \ 1]}_T \underbrace{\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \\ d_x & d_y & d_z \end{bmatrix}}_C$$

- Parametric tangent vector

$$\frac{d}{dt} Q(t) = [3a_x t^2 + 2b_x t + c_x \quad 3a_y t^2 + 2b_y t + c_y \quad 3a_z t^2 + 2b_z t + c_z]$$

– Velocity of a point on the curve w.r.t. t

Continuity between two curves

- G^0 geometric continuity – 2 curve segments joined
- G^1 geometric continuity – 2 curve segments joined & direction of tangent same
- C^n parametric continuity – 2 curves joined & n^{th} tangent vector, $\frac{d^n}{dt^n} Q(t)$, same

- In general $C^1 \Rightarrow G^1$
except when $\frac{dQ}{dt} = [0, 0, 0]$
can have tangent vector same, direction of tangent
different
- Cannot determine tangent vector (magnitude) by
looking at resulting curve: t not plotted

Basis functions (Blending functions)

- $Q = T \bullet C$
- Rewrite $C = M \bullet G$
 - M : 4×4 “basis” matrix
 - G : Geometry vector (of vectors)

$$\begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \\ g_{3x} & g_{3y} & g_{3z} \\ g_{4x} & g_{4y} & g_{4z} \end{bmatrix}$$

- $$x(t) = (t^3 m_{11} + t^2 m_{21} + t m_{31} + m_{41}) g_{1x} +$$

$$(t^3 m_{12} + t^2 m_{22} + t m_{32} + m_{42}) g_{2x} +$$

$$(t^3 m_{13} + t^2 m_{23} + t m_{33} + m_{43}) g_{3x} +$$

$$(t^3 m_{14} + t^2 m_{24} + t m_{34} + m_{44}) g_{4x}$$

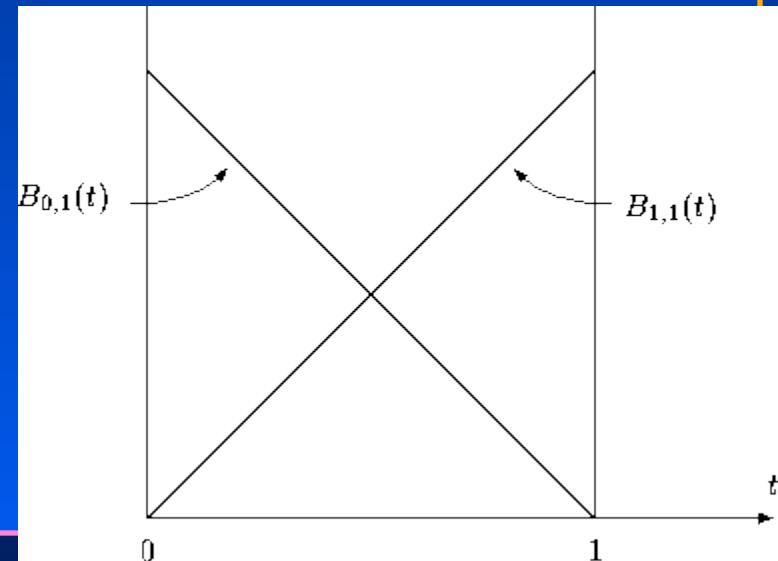
♪ $y(t), z(t)$ similar

- $$x(t) = \sum_{k=0}^3 g_{kx} B_k(t)$$

- Can think of Q as weighted sum of geometry matrix G where weights are the “blending functions” B
- Another way of looking : reconstruction of a function Q by “filtering” 4 samples
 - Basis functions are the filters

Linear interpolation

- $$Q(t) = [t \quad 1] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix} = TMG$$
- M is “basis” matrix of coefficients and different for different curves
- G is geometric vector
- TM are the basis functions
 - $1-t$ and t



To Transform Curve

- Just x -form G
- Not for non-affine transform like perspective transform
(ok for rational curves...to come later)

Bézier Curves

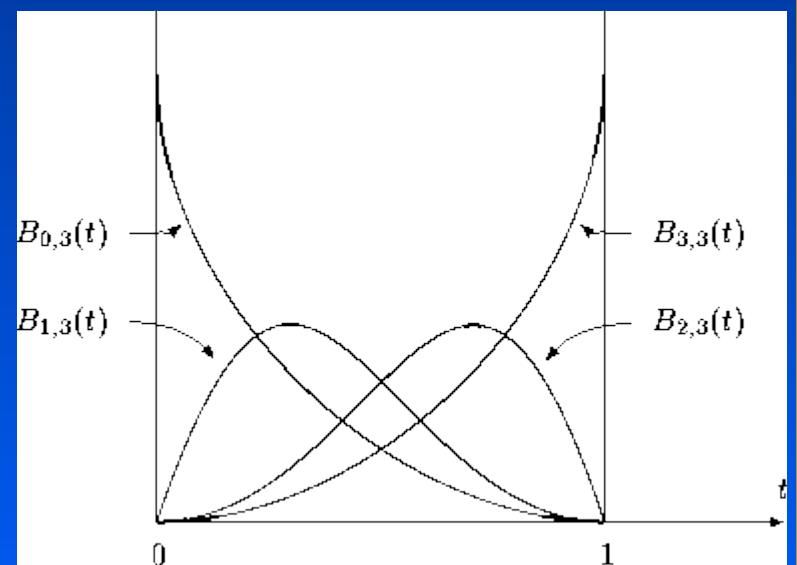
- $$Q(t) = \sum_{k=0}^3 P_k B_k(t)$$
- Where basis functions are given by the Bernstein cubic polynomials

$$B_0(t) = (1-t)^3$$

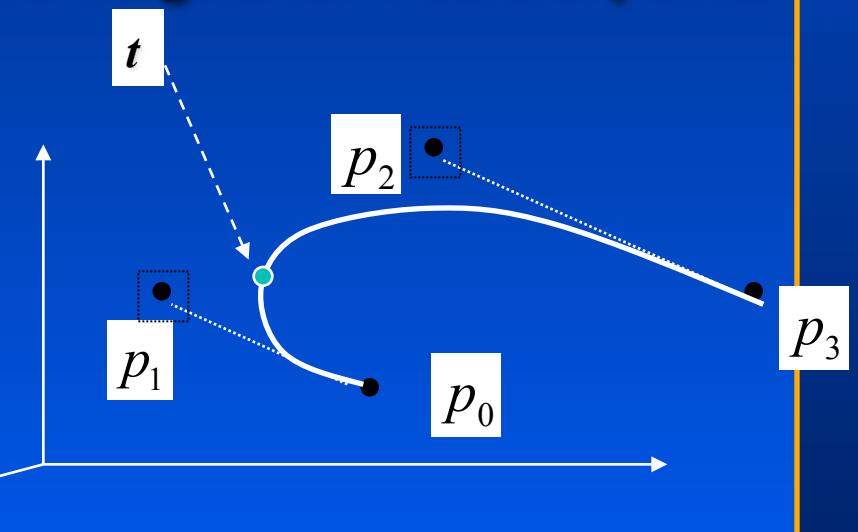
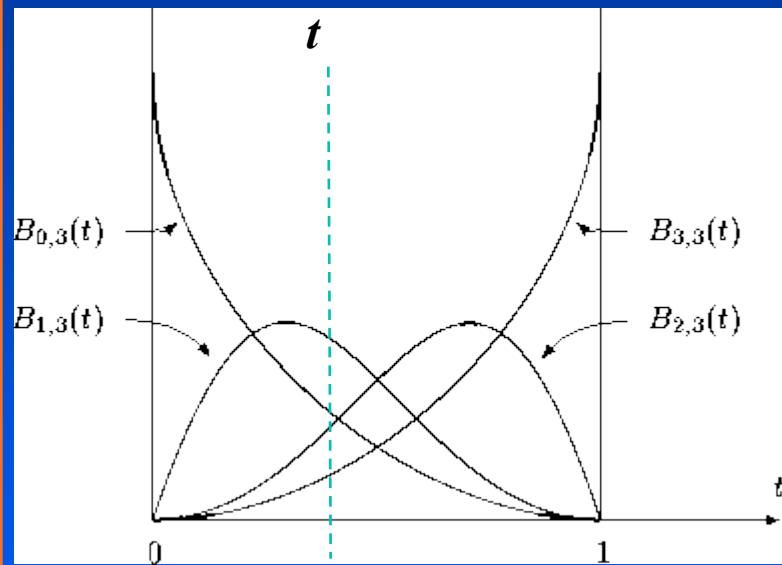
$$B_1(t) = 3t(1-t)^2$$

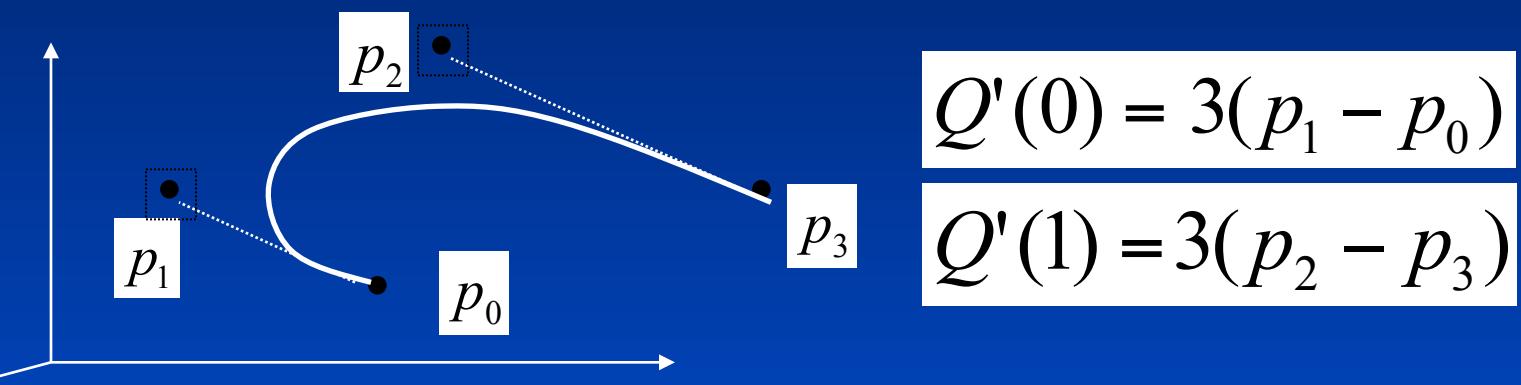
$$B_2(t) = 3t^2(1-t)$$

$$B_3(t) = t^3$$



- At $t=0$, $Q(0) = P_0$
- At $t=1$, $Q(1) = P_3$
- At any t , the 4 blending functions give weights for 4 control points

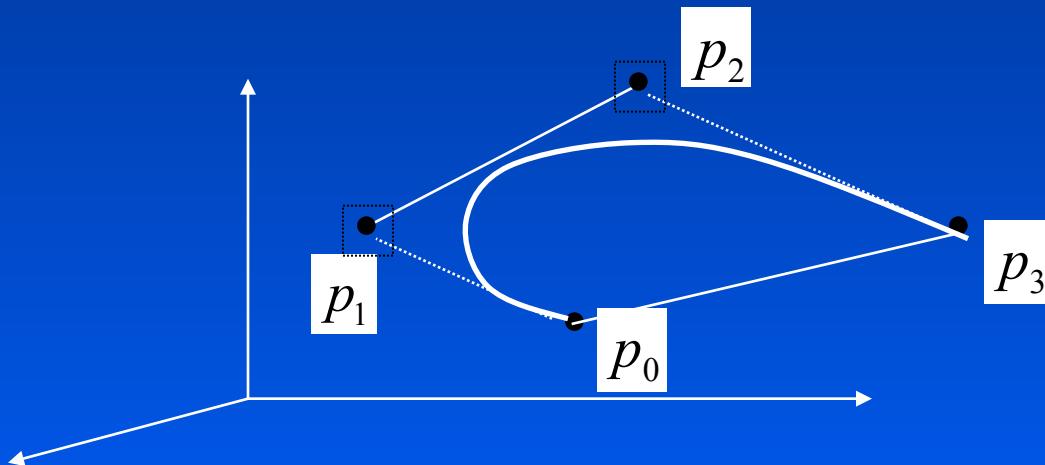




$$\begin{aligned} Q'(0) &= 3(p_1 - p_0) \\ Q'(1) &= 3(p_2 - p_3) \end{aligned}$$

- Basis functions chosen such that p_1-p_0 gives the first point's tangent and p_2-p_3 gives the last point's tangent
- This plus the beginning and end points make up the 4 constraints

- Curve contained in “convex hull” formed by control points
 - Basis functions sum to unity for all t
 - Useful for defining bounding box



Matrix form

$$Q(t) = \sum_{k=0}^3 P_k B_k(t)$$

$$Q(t) = P_0(1-t)^3 + P_1 3t(1-t)^2 + P_2 3t^2(1-t) + P_3(t)^3$$

$$Q(t) = T M G$$

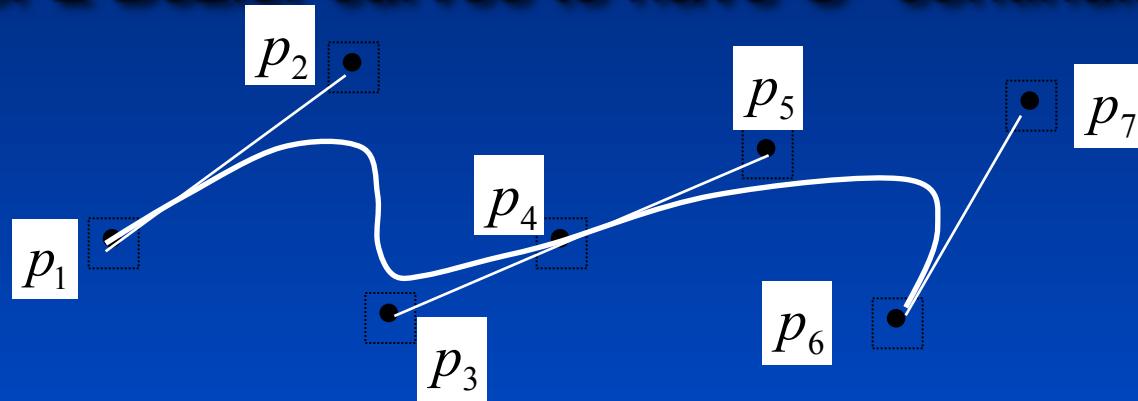
$$T = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Combine Bezier curves

- For 2 Bézier curves to have C^1 continuity



- $p_3 - p_4 - p_5$ collinear and equally spaced
- If not equally spaced, G^1 continuous

Problems with Bezier curves

- Global control
 - All weighting function non-zero everywhere except at the ends (for one curve)
 - Means that small changes in one control point changes the whole curve
- Continuity between curves explicitly controlled by the linearity of the control points

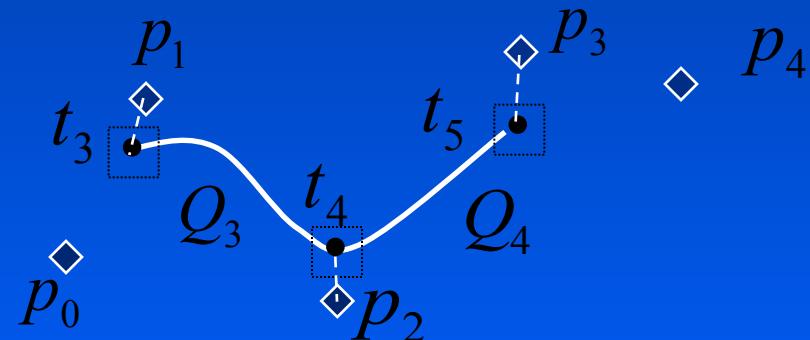
Splines

- Natural spline
 - Global control
 - At least C^2 continuous
 - Flexible strip & “ducks” or “whales”



Frets.com

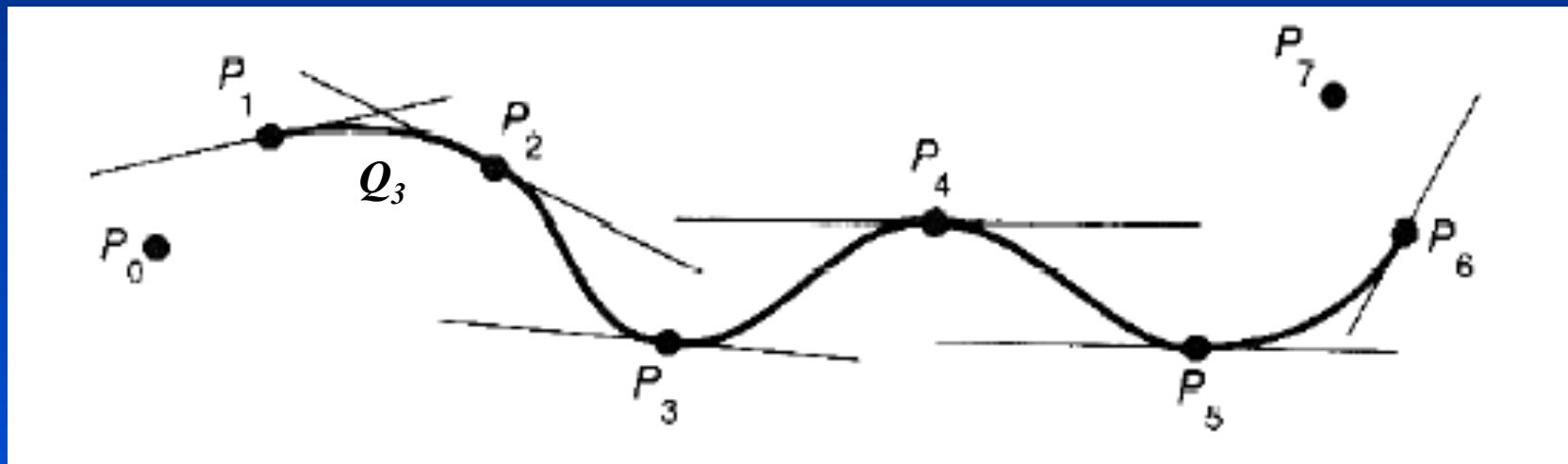
- Knot: where two curves join
- Interpolating
 - Control point = knot point
- Approximating
 - Control point \neq knot point



control pts : $p_0 \dots p_m$
 curves : $Q_3 \dots Q_m$
 Parameter for Q_i :
 $t_i \leq t \leq t_{i+1}$

Cardinal splines

- Interpolating
- C^1 continuity



- Assume tangent at p_j given by $\overline{p_{j-1} p_{j+1}}$

$$Q(t) = At^3 + Bt^2 + Ct + D$$

or

$$Q(t) = [t^3 \ t^2 \ t \ 1] [A \ B \ C \ D]^T$$

$$Q(0) = p_{i-2} = D$$

$$Q(1) = p_{i-1} = A + B + C + D$$

- Parametric derivative

$$Q'(t) = [3t^2 \ 2t \ 1 \ 0] [A \ B \ C \ D]^T$$

- At t=0, beginning tangent and at t=1, end tangent

$$Q'(0) = a(p_{i-1} - p_{i-3}) \equiv S_0 = C$$

$$Q'(1) = a(p_i - p_{i-2}) \equiv S_1 = 3A + 2B + C$$

$$M_1$$

- 4 constraints from previous two slides give

$$\begin{bmatrix} p_{i-2} \\ p_{i-1} \\ S_0 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix}$$

- Definitions of S_0 and S_1 give

$$\begin{bmatrix} p_{i-2} \\ p_{i-1} \\ S_0 \\ S_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -a & 0 & a & 0 \\ 0 & -a & 0 & a \end{bmatrix} \begin{bmatrix} p_{i-3} \\ p_{i-2} \\ p_{i-1} \\ p_i \end{bmatrix}$$

$$M_2$$

- Equating the two equations

$$\begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix} = M_1^{-1} M_2 \begin{bmatrix} p_{i-3} \\ p_{i-2} \\ p_{i-1} \\ p_i \end{bmatrix}$$

- Multiply out $M_1^{-1}M_2$
 - Call this M

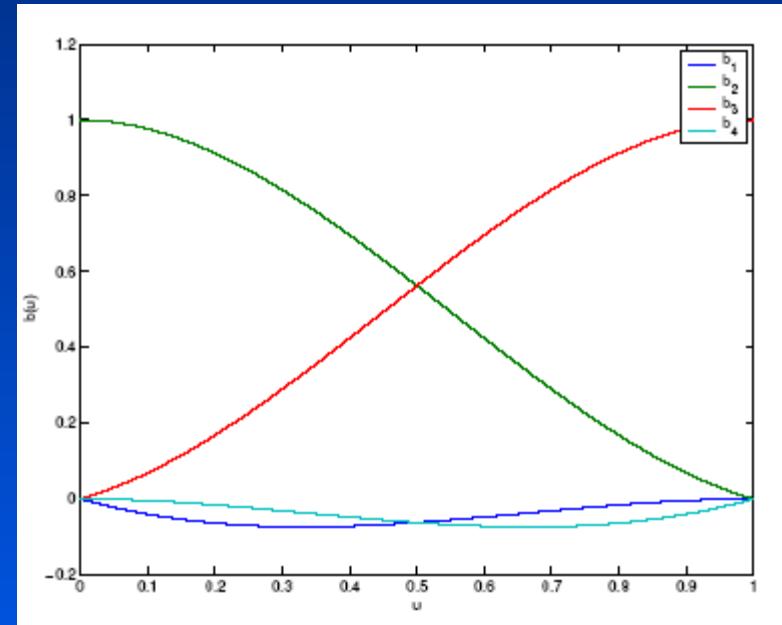
$$M_1^{-1}M_2 = \begin{bmatrix} -a & 2-a & a-2 & a \\ 2a & a-3 & 3-2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \equiv M$$

Matrix form of Cardinal Spline

$$Q(t) = [t^3 \ t^2 \ t \ 1] [A \ B \ C \ D]^T$$

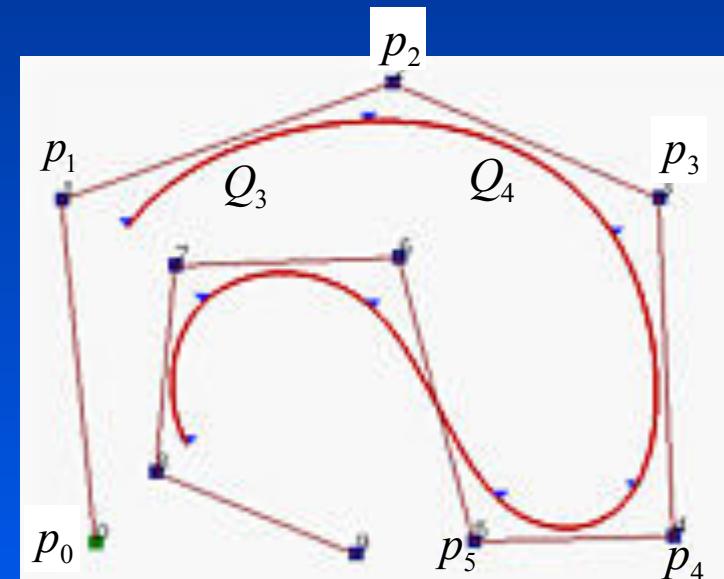
$$Q(t) = [t^3 \ t^2 \ t \ 1] M \begin{bmatrix} p_{i-3} \\ p_{i-2} \\ p_{i-1} \\ p_i \end{bmatrix}$$
$$M = \begin{bmatrix} -a & 2-a & a-2 & a \\ 2a & a-3 & 3-2a & -a \\ -a & 0 & a & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

- Blending functions for Catmull-Rom spline



Uniform (Nonrational) B-splines (“basis spline”)

- Approximating
- C^0, C^1, C^2 continuous



$$Q_i(t) = \sum_{k=0}^3 p_{i-3+k} B_{i-3+k}(t)$$

- p : control points (geometric vector components)
- B : basis functions (or blending functions)
- i : segment number
- k : local control point index
- t : 0 to 1

- For parameter range $t_i \leq t \leq t_{i+1}$

$$B_i = \frac{1}{6} t^3$$

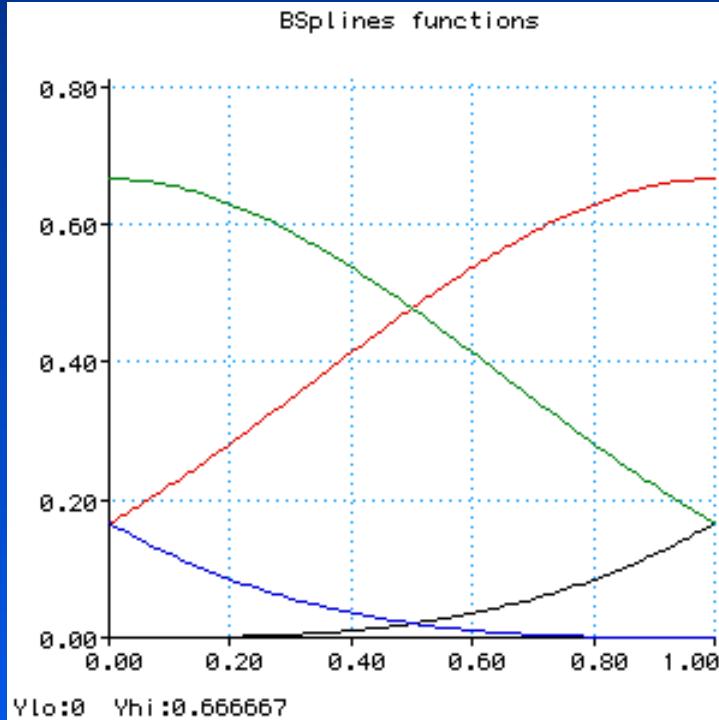
$$B_{i-1} = \frac{1}{6} (-3t^3 + 3t^2 + 3t + 1)$$

$$B_{i-2} = \frac{1}{6} (3t^3 - 6t^2 + 4)$$

$$B_{i-3} = \frac{1}{6} (1 - t)^3$$

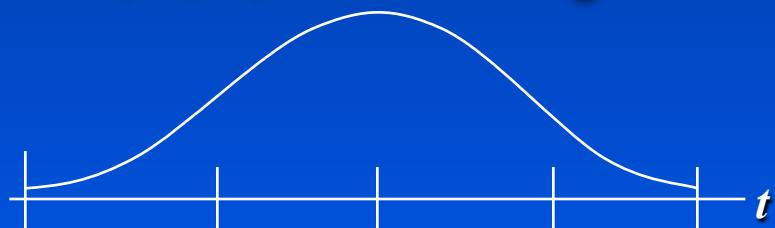
- Basis functions in range

$$t_i \leq t \leq t_{i+1}$$



Define as one B-spline curve

- i : non-local control point number
- t : global parameter (not just 0 to 1)
- Basis function in entire range given by shifting right given equations by 0, 1, 2, 3 units to right



$$Q(t) = \sum_{i=0}^m p_i B_i(t)$$

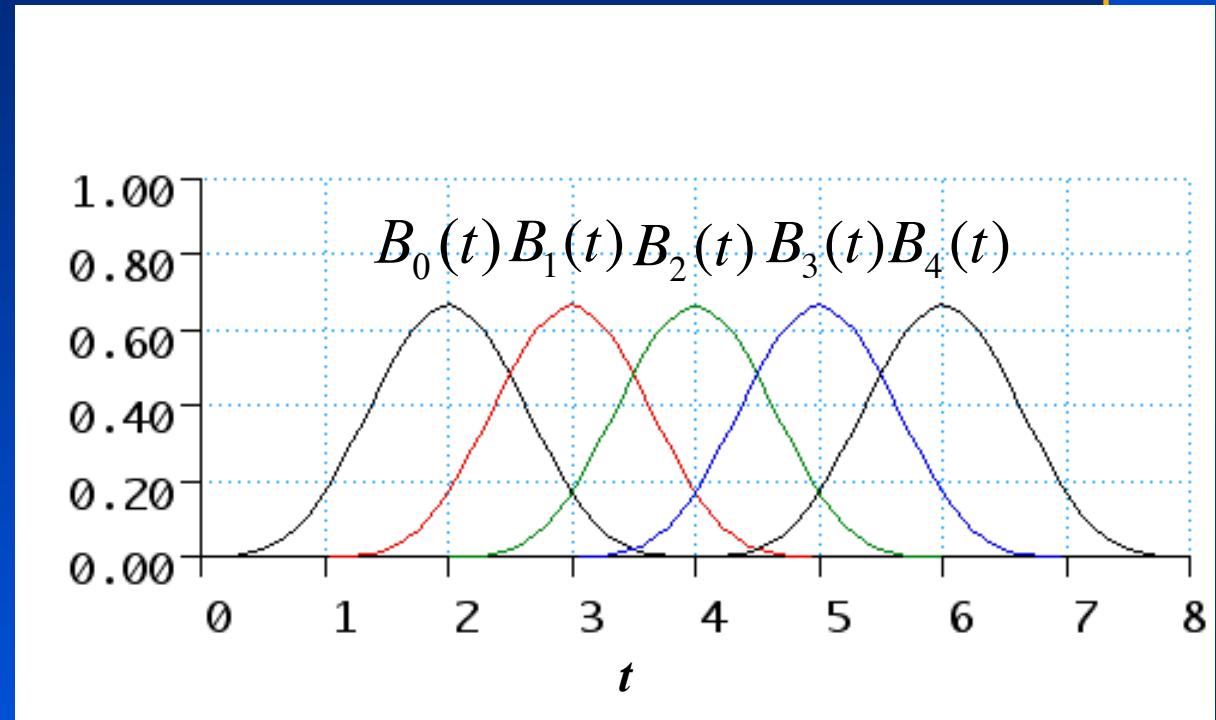
$$B_i = \frac{1}{6} t^3$$

$$B_{i-1} = \frac{1}{6} (-3t^3 + 3t^2 + 3t + 1)$$

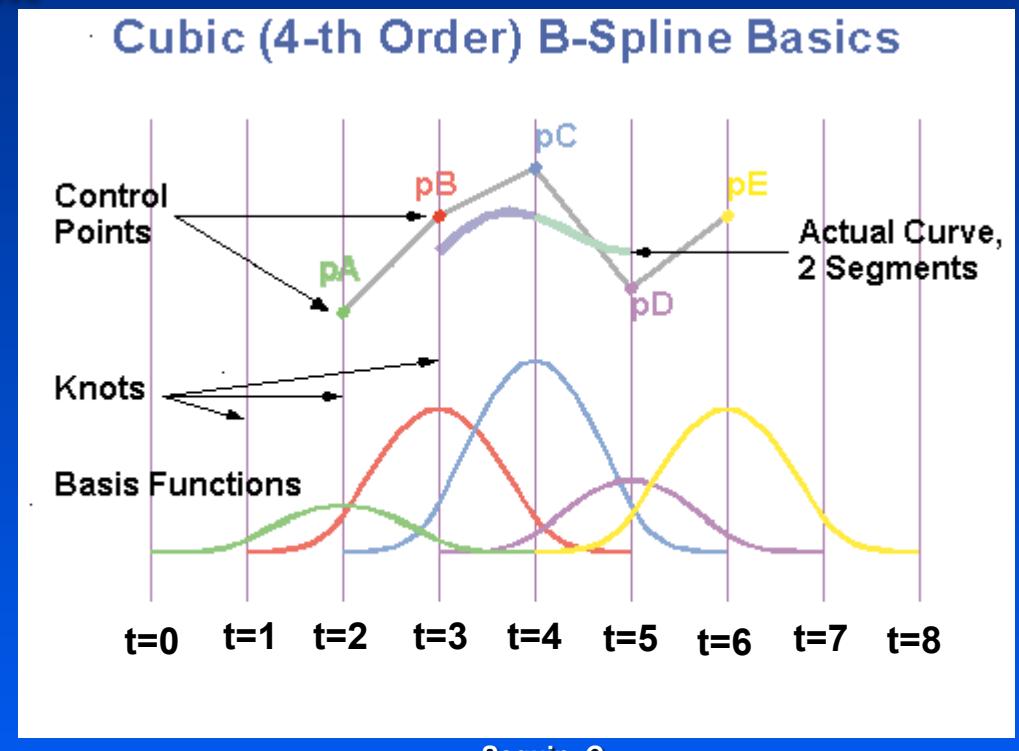
$$B_{i-2} = \frac{1}{6} (3t^3 - 6t^2 + 4)$$

$$B_{i-3} = \frac{1}{6} (1 - t)^3$$

- For the useful range, basis functions sum to 1
 - $t \geq 3$ for curve
- For example on right, two curves between $t=3$ and 5

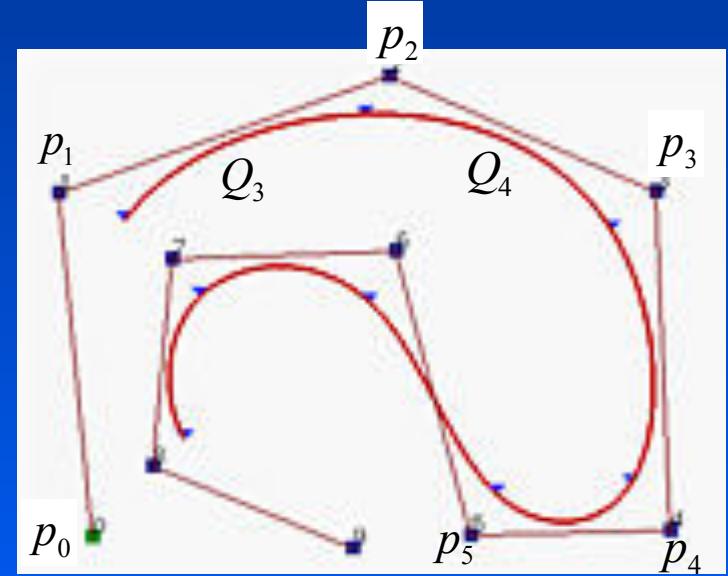


- Curve segments end/start at **equal** intervals with respect to global parameter t : “uniform”
 - $[0, 1, 2, \dots m]$
 - Called knot vector
- On right, 9 knots
 - Basis functions shown weighted by value of control points



Sequin, C.

- For the B-spline on the right:
knot vector is [0,1,2,3,4,5,6,7,8,9,10,11,12,13] with useful
knot values at [3,4,5,6,7,8,9,10]
where basis function sum to 1



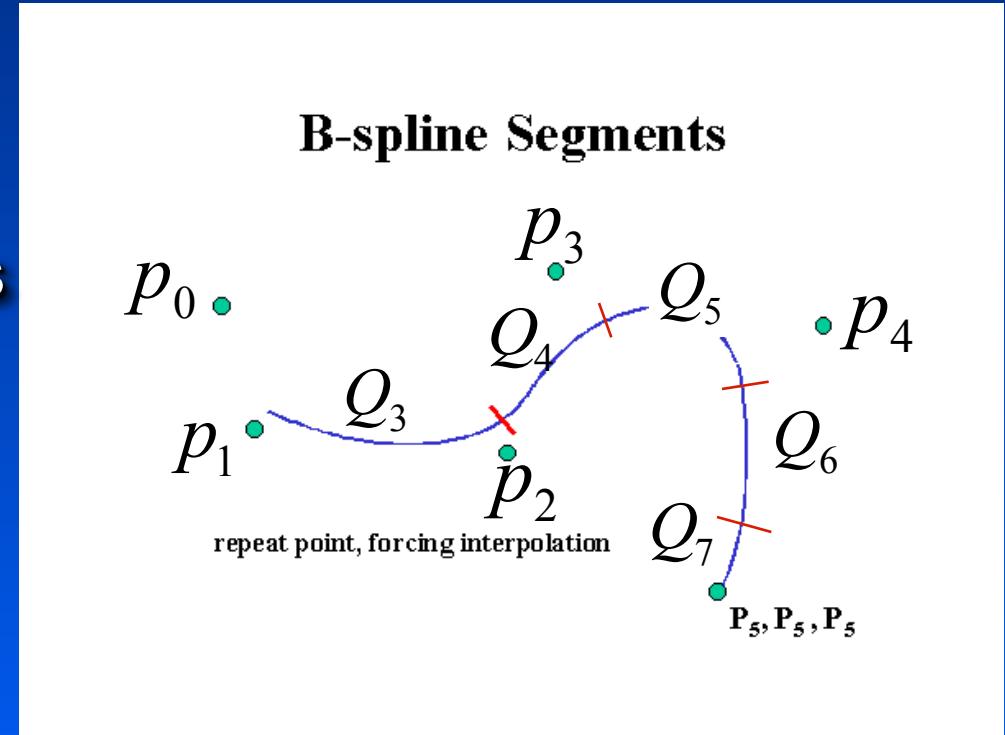
Matrix form

$$Q_i(t) = T_i M G_i$$

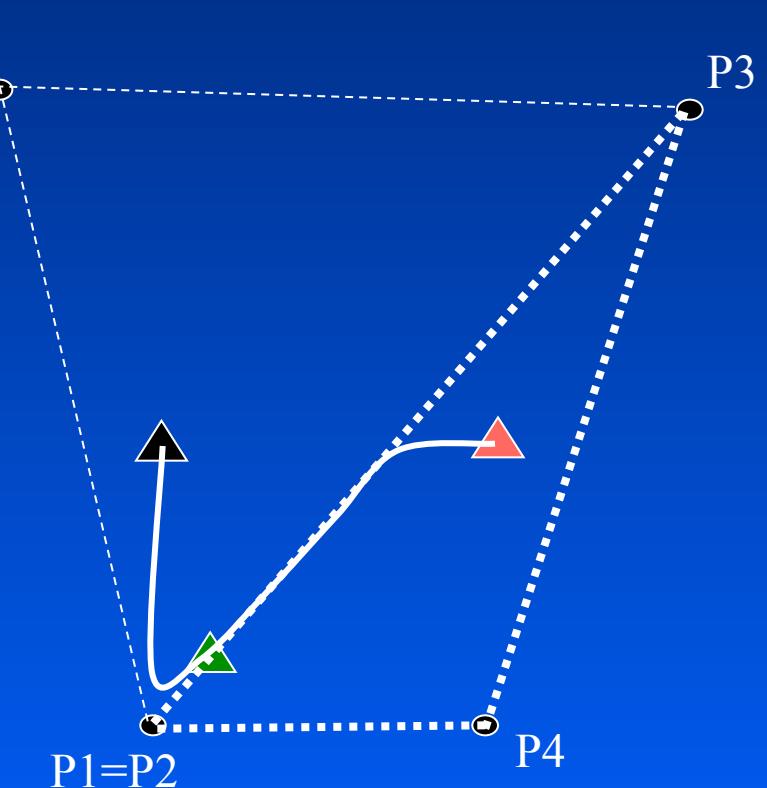
$$M = \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix}$$

Multiplicity of Control Points♪

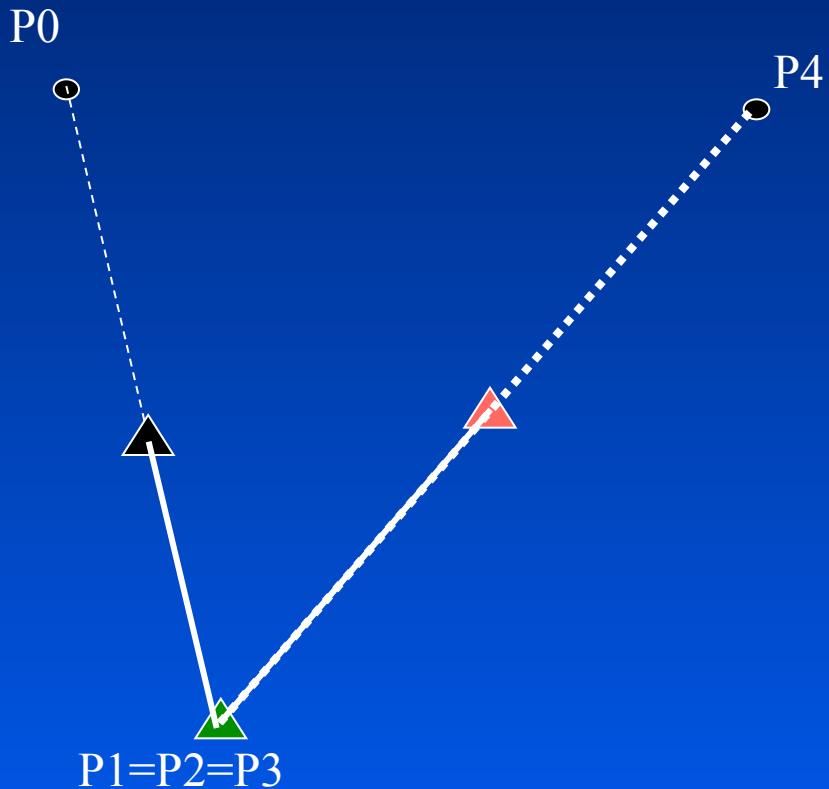
- Example: Last control point used three times
- Q5 uses p_5 once
- Q6 uses p_5 twice
- Q7 uses p_5 three times

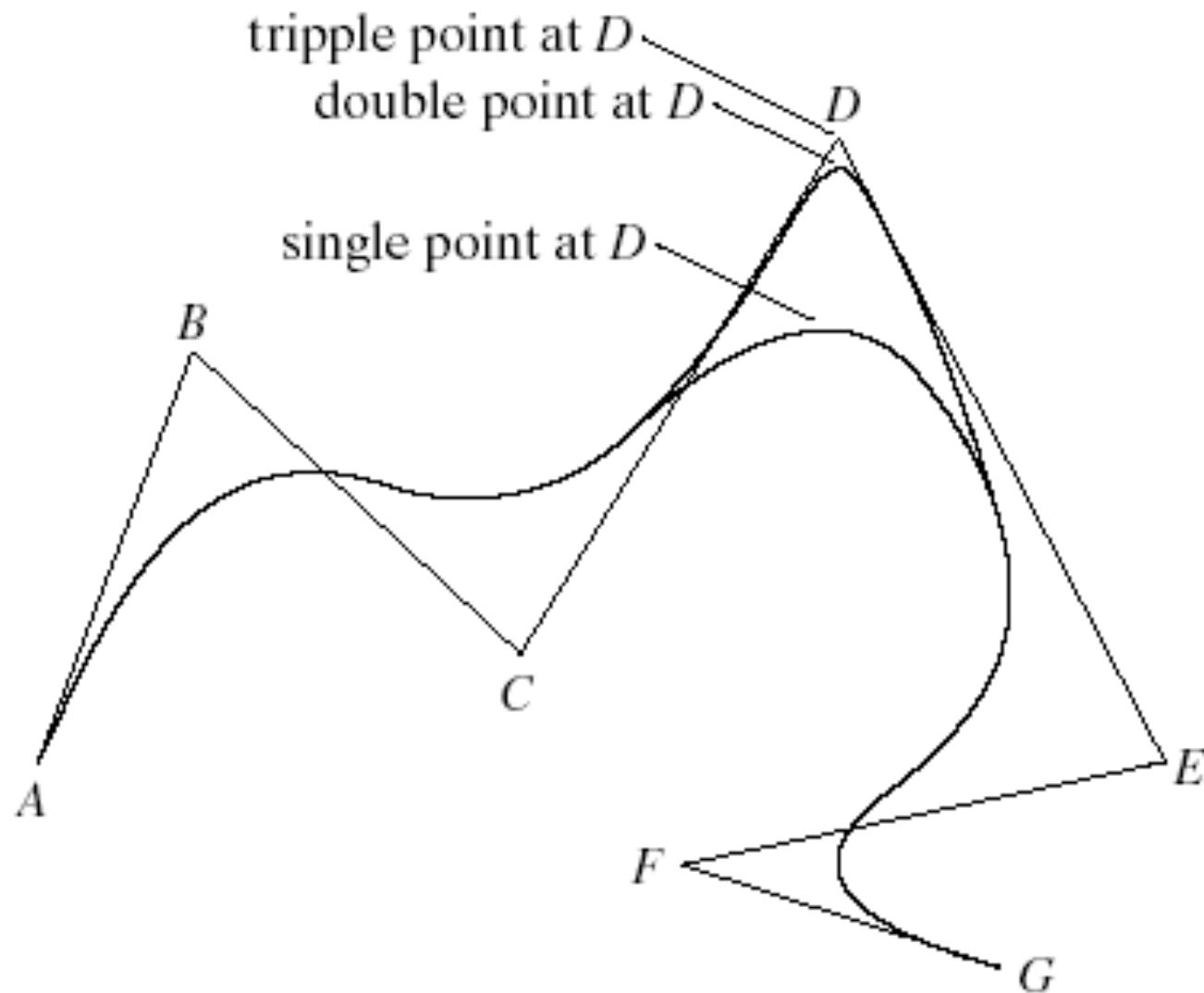


- Example: Interior control point used twice♪
- Continuity C^2G^1 :
 - Continuity within still C^2
 - Continuity across knot G^1



- Example: Interior control point used three times♪
- Continuity C^2G^0
 - Continuity within still C^2
 - Continuity across knot G^0



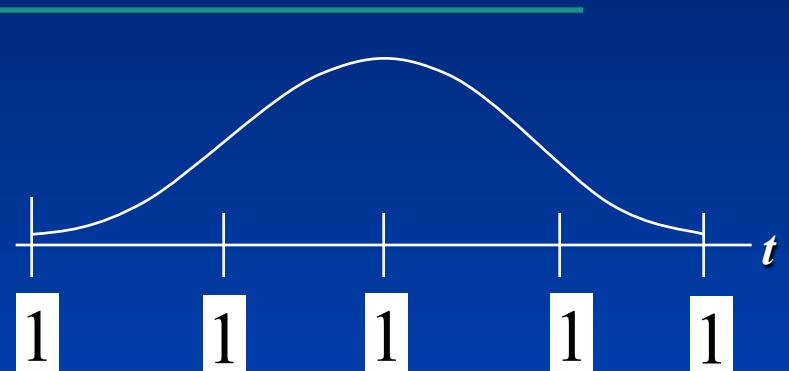


Nonuniform (Nonrational) B-Spline

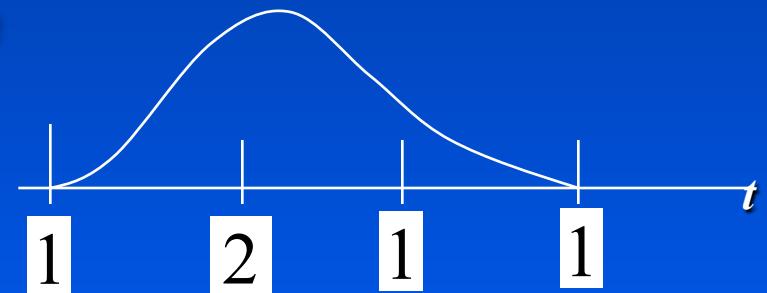
- Parameter interval between knot values need not be uniform
 - Most common is reducing intervals between successive knots to 0 (multiplicity of knots)
 - When $t_i = t_{i+1}$ multiple knot and Q_i becomes a point
- Basis functions can vary from curve segment to curve segment
- Continuity can be reduced from C^2 to C^1 to C^0 to none
- C^0 : curve **interpolates** control point w/o straight line segment as in uniform B-spline w/ multiplicity of control points
- Possible to add additional knots and control points

Effect of knot multiplicity On single basis function

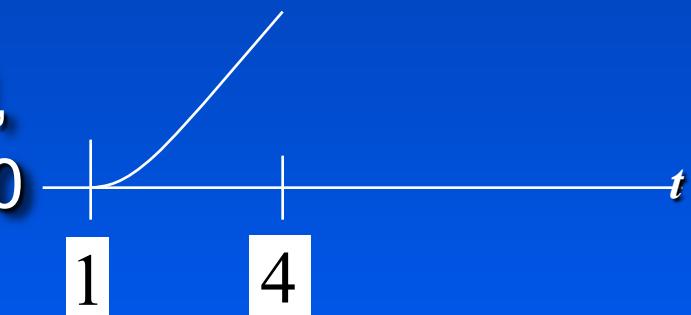
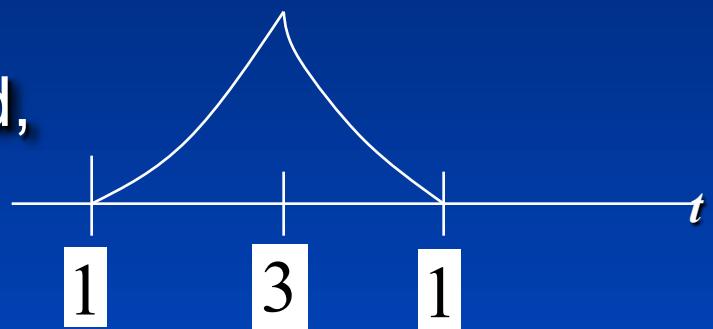
- [0,1,2,3,4]
 - Uniform interval between knot values
 - Right figure shows basis function with multiplicity of one for each knot (usual B-spline as before)



- [0,1,1,2,3]
 - Interval between second and third knot values is 0
 - B_{i-1} (second segment) shrinks to 0
 - Eliminate second derivative continuity, first derivative continuity remain



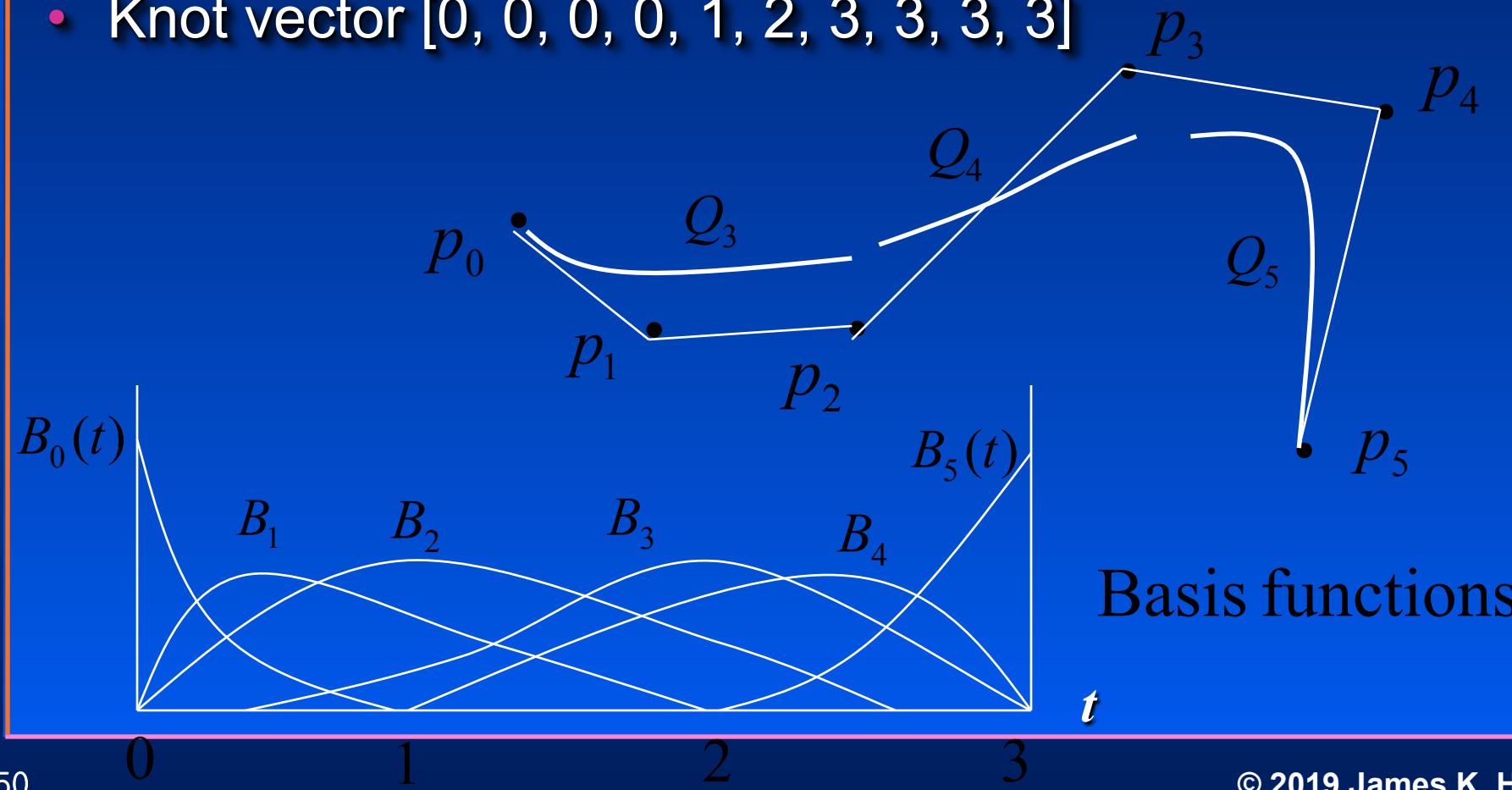
- $[0,1,1,1,2]\text{♪}$
 - Interval between second, third, fourth knot values is 0
 - Only positional continuity
- $[0,1,1,1,1]\text{♪}$
 - Interval between second, third, fourth, and fifth knot values is 0
 - No positional continuity



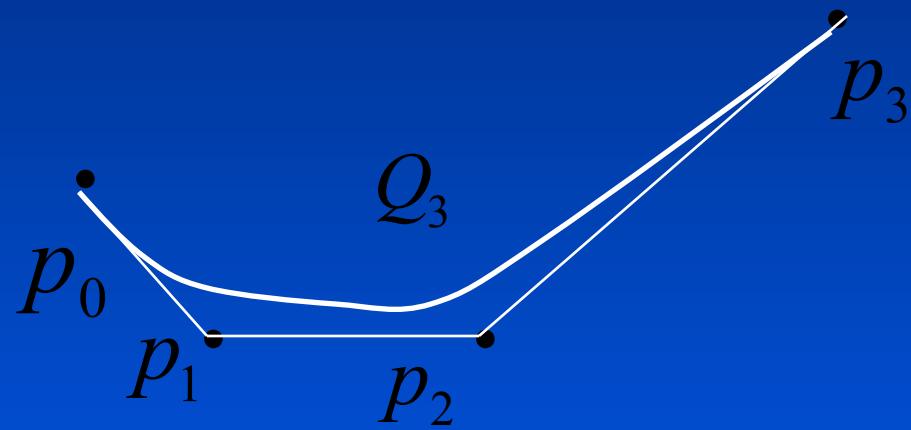
Knot multiplicity	Continuity Conditions	Continuity
1	Positional Tangential Curvature	C^2
2	Positional Tangential	C^1
3	Positional	C^0
4	None	None

Example of knot multiplicity on ends

- Knot vector $[0, 0, 0, 0, 1, 2, 3, 3, 3, 3]$

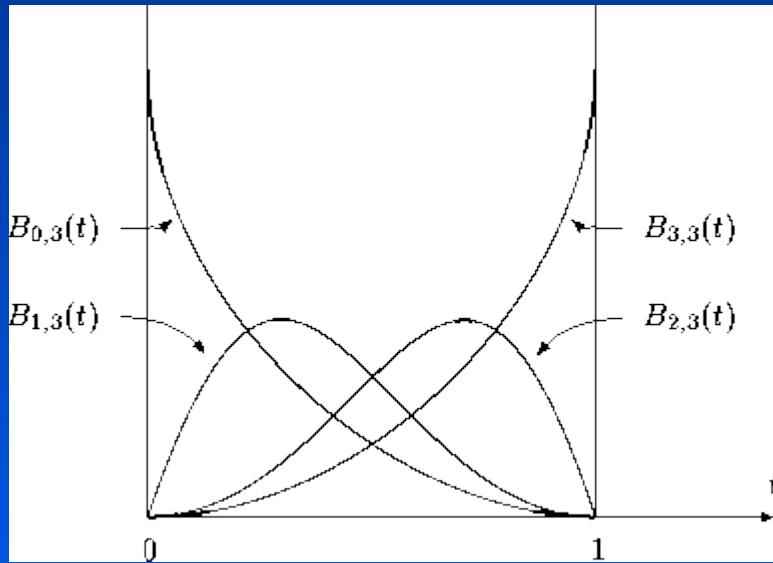


- Knot vector $[0, 0, 0, 0, 1, 1, 1, 1]$



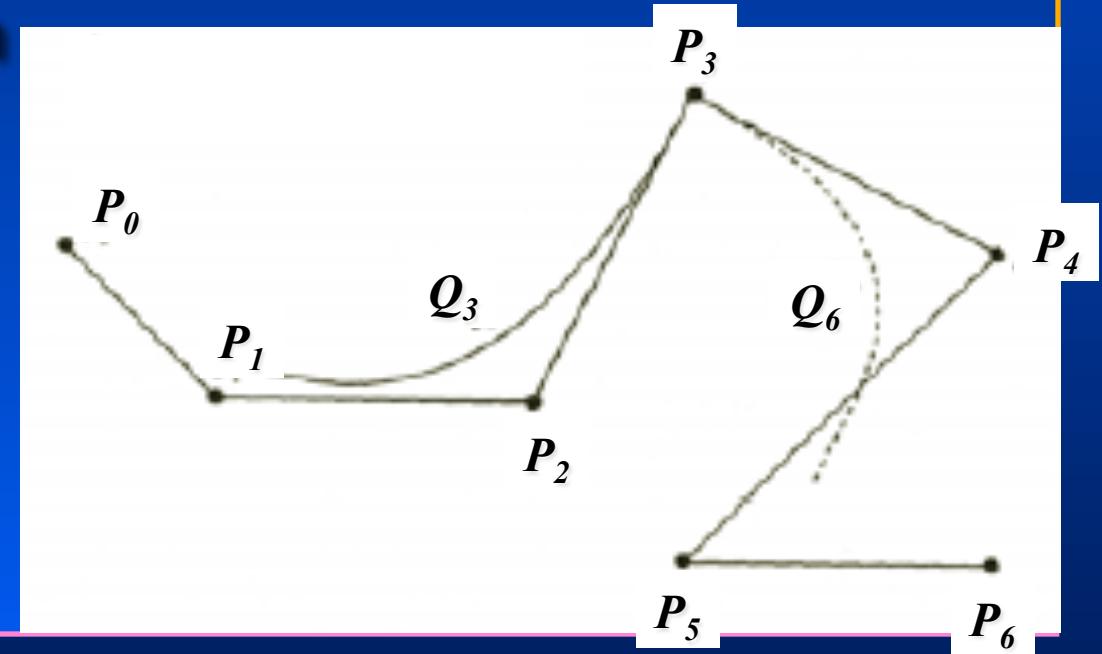
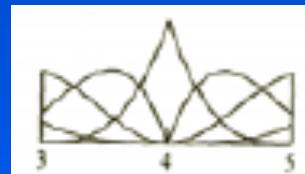
– equal to Bézier curve

- Basis functions are Bezier basis functions
- Bezier a subset of B-splines

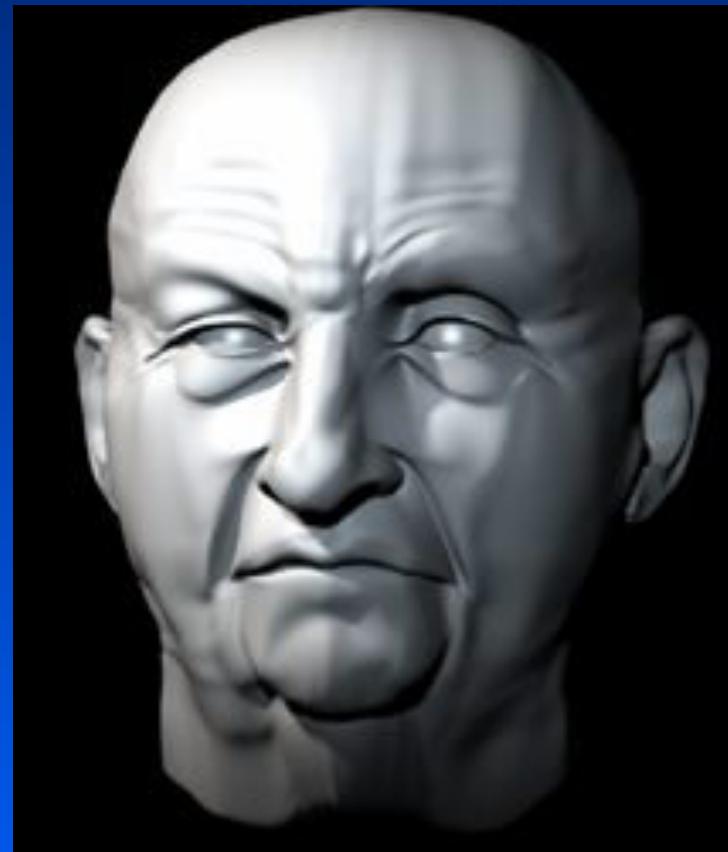


Example of knot multiplicity in interior knot

- Triple knot vector [0, 1, 2, 3, 4, 4, 4, 5, 6, 7, 8]
- Q_4 and Q_5 shrink to zero
- C^0 continuity between Q_3 and Q_6



Next: Parametric surfaces



Turbo Squid