Project euler

1 2 3 5 8 13 21 34 55

odd even odd odd even odd odd even

every third value is even

fib(2), fib(5), fib(8), ..., fib(n×3 − 1) $\forall n \in \mathbb{N}$

max: fib(n×3 − 1) < 4 000 000

Binet's formula (wiki): $F_n = \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$ where $\varphi = \frac{1+\sqrt{5}}{2}$

$F_0 = 0 \quad F_1 = F_2 = 1$

fib(2) = F_2

$\Rightarrow$ fib(n×3−1) $= F_{n\times3} = \frac{\varphi^{n\times3} - (-\varphi)^{-(n\times3)}}{\sqrt{5}}$

sol'n $= \sum_{i=1}^{maxi} F_{i\times3}$

$F_{n\times3} = \frac{\varphi^{n\times3} - (-\varphi)^{-n\times3}}{\sqrt{5}} < 4000000$

$\Rightarrow \varphi^{n\times3} - (-\varphi)^{-n\times3} < \sqrt{5}\cdot 4mil$ $\qquad -\varphi\cdot\varphi = -(\varphi^2)$

$\Rightarrow \varphi^{n\times3} - \frac{1}{(-\varphi)^{n\times3}} < \sqrt{5}\cdot 4mil$

$\Downarrow$

$\Rightarrow \frac{-\varphi^{(n\times3)2} - 1}{\varphi^{n\times3}} < \sqrt{5}\cdot 4mil$ $\qquad \frac{\varphi^{n\times3}\cdot(-\varphi)^{n\times3} - 1}{(-\varphi)^{n\times3}} < \sqrt{5}\cdot 4mil$

$\Rightarrow \frac{\varphi^{n\times6} + 1}{\varphi^{n\times3}} < \sqrt{5}\, 4mil$ $\qquad \frac{(-(\varphi^2))^{n\times3} - 1}{(-1)^{n\times3}} < \varphi^{n\times3}\sqrt{5}\cdot 4mil$

$\Rightarrow$ $\qquad \frac{(-1)^{n\times3}(\varphi^2)^{n\times3}}{(-1)^{n\times3}} - \frac{1}{(-1)^{n\times3}} < \varphi^{n\times3}\cdot\sqrt{5}\cdot 4mil$

$$F_n = \lfloor \frac{\varphi^n}{\sqrt{5}} + \frac{1}{2} \rfloor \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$$n(F) = \lfloor \log_\varphi \left(F \cdot \sqrt{5} + \frac{1}{2}\right) \rfloor$$

$$\Rightarrow n(4{,}000{,}000) = \lfloor \log_\varphi \left(4\,000\,000 \cdot \sqrt{5} + \frac{1}{2}\right) \rfloor$$

$$\cancel{\log\varphi} \quad \log_\varphi X = \frac{\ln(X)}{\ln(\varphi)} = \frac{\log(X)}{\log(\varphi)}$$

$$= 33.26$$

~~(+1 for us)~~

$\cancel{\nexists}$

$F_{33} = 3.5$ mil

$F_{34} = 5.7$ mil

$F_0 = 0$

$F_1 = 1$

$F_2 = \cancel{1}$

F_3

$F_{n \times 3} = F_{33} \qquad n = 11$

$\mathrm{Fib}(n \times 3 - 1) \quad \forall n \in \{1, \dots, 11\}$

$$\underset{F_3}{\underset{\parallel}{fib(2)}} + \underset{F_6}{\underset{\parallel}{fib(5)}} + \dots + \underset{F_{33}}{\underset{\parallel}{fib(32)}}$$

$$F_n = \lfloor \frac{\varphi^n}{\sqrt{5}} + \frac{1}{2} \rfloor \neq \frac{\varphi^n - (-\varphi)^{-n}}{\sqrt{5}}$$

$$\Sigma = \frac{1}{\sqrt{5}} \left(\varphi^3 + \varphi^6 + \dots + \varphi^{33} - \cancel{\frac{1}{(-\varphi)^3 + (-\varphi)^6 + (-\varphi)^9 + \dots + (-\varphi)^{33}}} \right.$$

$$(\varphi^3) + (\varphi^3)^2 + (\varphi^3)^3 + \dots + (\varphi^3)^{11} - \underbrace{\frac{1}{(-\varphi)^3} - \frac{1}{(-\varphi)^6} - \frac{1}{(-\varphi)^9} - \dots - \frac{1}{(-\varphi)^{33}}}$$

$$\frac{1}{\varphi^3} - \frac{1}{\varphi^6} + \frac{1}{\varphi^9} - \frac{1}{\varphi^{12}} + \dots + \frac{1}{\varphi^{33}}$$

$$+ (\varphi^{-3}) - (\varphi^{-3})^2 + (\varphi^{-3})^3 - (\varphi^{-3})^4 + \dots + (\varphi^{-3})^{11}$$

$$Sum = 1 + x + x^2 + x^3 + \dots + x^n$$

$$Sum \times x + 1 = x^{n+1} + \dots + 1 = x^{n+1} + Sum$$

$$Sum \cdot (x-1) + 1 = x^{n+1}$$

$$Sum = \frac{x^{n+1} - 1}{x - 1}$$

$$\sum_{i=0}^{n} x^i = \frac{x^{n+1}-1}{x-1} \Rightarrow \sum_{i=0}^{n} x^i = \sum_{i=1}^{n} x^i + 1$$

$$\Rightarrow \sum_{i=1}^{n} x^i = \frac{x^{n+1}-1}{x-1} - 1$$

$$\sum_{i=1}^{11} (\varphi^3)^i = \frac{(\varphi^3)^{12} - 1}{\varphi^3 - 1} - 1 = 10316618.19$$

$$+(\varphi^{-3}) - (\varphi^{-3})^2 + (\varphi^{-3})^3 \cancel{+} \dots$$

$$= x - x^2 + x^3 - x^4 + \dots + x^n$$

$$\cancel{(-1)^{i+1} x^i}$$

$$\{(\varphi^{-3}) + (\varphi^{-3})^3$$

$$- [(\varphi^{-3})^2 + (\varphi^{-3})^4 + \dots] + (\varphi^{-3})^{10}]$$

$$- [(\varphi^{-6}) + (\varphi^{-6})^2 + \dots + (\varphi^{-6})^5$$

$$= \cancel{\frac{1}{\varphi^3}(} \quad = \varphi^3 [(\varphi^{-3})^2 + (\varphi^{-3})^4 + \dots]$$

$$\cancel{\varphi^3}$$

$$= \varphi^3 [(\varphi^{-6}) + (\varphi^{-6})^2 + (\varphi^{-6})^3 + \dots]$$

$$(\varphi^{-3}) + (\varphi^{-3})^3 + \dots (\varphi^{-3})^{11}$$

1	3	...	11
2	4	....	12
1	2	...	6

$$\Phi = \varphi^{-6}$$

$$\varphi^3 [\Phi + \Phi^2 + \dots + \Phi^6] - [\Phi + \dots + \Phi^5]$$

$$\varphi^3 \cdot \left(\frac{\Phi^7 - 1}{\Phi - 1} - 1 \right) - \left(\frac{\Phi^6 - 1}{\Phi - 1} - 1 \right) = 0.19$$

$$= \boxed{4613732} \checkmark$$