最优化第四次作业

郭英明

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1.1

1.To prove self-concordant function preserves properties under positive scaling $\alpha \geq 0$, and sum

Let

$$h(x) = \alpha f(x) \quad (\alpha \geq 0)$$

We know f is self-concordant, then

$$|f^{'''}(x)| \leq 2f^{''}(x)^{\frac{3}{2}}$$

if h is self-concordant, then

$$|h^{'''}(x)| \leq 2h^{''}(x)^{\frac{3}{2}}$$

$$\alpha|f^{'''}(x)| \leq 2\alpha^{\frac{3}{2}}f^{''}(x)^{\frac{3}{2}}$$

if $\alpha > 0$,

$$|f^{'''}(x)| \leq 2\alpha^{\frac{1}{2}}f^{''}(x)^{\frac{3}{2}}$$

if $\alpha \geq 1$, it satisfies
if $f(x_i)$ $(i \in [1, n])$ is self-concordant,then

$$|f^{'''}(x_i)| \leq 2f^{''}(x_i)^{\frac{3}{2}} \quad i \in [1, n]$$

$$\sum_{i=1}^{n} |f^{'''}(x_i)| \leq 2\sum_{i=1}^{n} f^{''}(x_i)^{\frac{3}{2}}, \quad i \in (1, n)$$

for $h(x) = \sum_{i=1}^{n} f(x_i)$ if $h(x)$ is self-concordant, then

$$|h^{'''}(x)| \leq 2h^{''}(x)^{\frac{3}{2}}$$

$$|\sum_{i=1}^{n} f^{'''}(x_i)| \leq 2\sum_{i=1}^{n} f^{''}(x_i)^{\frac{3}{2}}, \quad i \in [1, n]$$

Due to

$$|\sum_{i=1}^{n} f^{'''}(x_i)| \leq \sum_{i=1}^{n} |f^{'''}(x_i)|, \quad i \in [1, n]$$

It satisfies

1.2

2.To prove self-concordant function preserves properties under composition with affine function

Let

$$h(x) = f(ax + b)$$

we have

$$h^{''}(x) = a^2 f^{''}(ax + b)$$

$$h^{'''}(x) = a^3 f^{'''}(ax + b)$$

if h is self-concordant, then

$$|h^{'''}(x)| \leq 2h^{''}(x)^{\frac{3}{2}}$$

$$|a^3 f^{'''}(ax + b)| \leq 2(a^2 f^{''}(ax + b))^{\frac{3}{2}}$$

which (after dividing by a^3) is the self-concordant inequality for f

1.3

3.To prove if g is convex with $dom\ g = R_{++}$ and $|g^{'''}(x)| \leq 3g^{''}/x$, then $f(x) = -log(-g(x)) - log(x)$ is self-concordant

$$f^{''}(x) = \frac{x^2(g^{'}(x)^2 - g(x)g^{''}(x)) + g(x)^2}{x^2 g(x)^2}$$

$$f^{'''}(x) = \frac{3g(x)g^{'}(x)g^{''}(x) - g(x)^2 g^{'''}(x) - 2g^{'}(x)^3}{g(x)^3} - \frac{2}{x^3}$$

According to $g^{'''}(x) \leq 3\frac{g^{''}(x)}{x}$

$$|f^{'''}(x)| \leq \frac{3g''(x)}{-xg(x)} + 2(\frac{|g'(x)|}{-g(x)})^3 + \frac{3g''(x)|g'(x)|}{g(x)^2} + \frac{2}{x^3}$$

if $f(x)$ is self-concordant, then it needs to satisfies

$$f^{'''}(x) \leq 2f^{''}(x)^{3/2}$$

So we can transfer the original problem to

$$\frac{\frac{3g''(x)}{-xg(x)} + 2(\frac{|g'(x)|}{-g(x)})^3 + \frac{3g''(x)|g'(x)|}{g(x)^2} + \frac{2}{x^3}}{2((\frac{g'(x)}{g(x)})^2 - \frac{g''(x)}{g(x)} + \frac{1}{x^2})^{\frac{3}{2}}} \leq 1$$

We supposing

$$a = \frac{(-g''(x)/g(x))^{\frac{1}{2}}}{((\frac{g'(x)}{g(x)})^2 - \frac{g''(x)}{g(x)} + \frac{1}{x^2})^{\frac{1}{2}}}$$

$$b = \frac{-\frac{|g'(x)|}{g(x)}}{((\frac{g'(x)}{g(x)})^2 - \frac{g''(x)}{g(x)} + \frac{1}{x^2})^{\frac{1}{2}}}$$

$$c = \frac{\frac{1}{x}}{((\frac{g'(x)}{g(x)})^2 - \frac{g''(x)}{g(x)} + \frac{1}{x^2})^{\frac{1}{2}}}$$

$$\{a, b, c > 0 \mid a^2 + b^2 + c^2 = 1\}$$

Then we just need to prove

$$\frac{3}{2}a^2c + b^3 + \frac{3}{2}a^2b + c^3 \leq 1$$

$$\frac{3}{2}a^2c + b^3 + \frac{3}{2}a^2b + c^3 = \frac{1}{2}(b+c)(3 - (b+c)^2) \leq 1$$

So $f(x)$ is self-concordant

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2.1

1.To prove $-\sum_{i=1}^{m} log(b_i - a_i^T x)$ on $\{x \mid a_i^T x < b_i,\ i = 1, 2, \cdots, m\}$ **is self-concordant**

let

$$g(x) = -log(b - a^T x)$$

$$g^{''}(x) = \frac{(a^T)^2}{(b - a^T x)^2}$$

$$g^{'''} = \frac{2(a^T)^3}{(b - a^T x)^3}$$

$$\frac{g^{'''}(x)}{2g^{''}(x)^{3/2}} = 1$$

$g(x)$ is self-concordant

According to function preserves self-concordant under sum

$f(x) = \sum_{i=1}^{m} g(x)$ is self-concordant

2.2

2.To prove $f(X) = -log\ det(X)\ on\ S_{++}^n$ is self-concordant

$$f^{''}(X) = -(X^{-2})^T$$

$$f^{'''}(X) = -2(X^{-3})^T$$

$$\frac{f^{'''}(X)}{2f^{''}(X)^{3/2}} = 1$$

$f(X)$ is self-concordant

2.3

3.To prove $f(x,y) = -log(y^2 - x^T x)\ on\ \{(x,y) \mid ||x||_2 < y\}$ is self-concordant

Limit the function to lines:$x = \hat{x} + tv,\ \ y = \hat{y} + tw$

Then we have

$$f(\hat{x} + tv, \hat{y} + tw) = -log(\hat{y} + tw) - log(\hat{y} + tw - \frac{(\hat{x})^T\hat{x} + 2t(\hat{x})^T v + t^2 v^T v}{\hat{y} + tw})$$

when $w = 0$

$$f(t) = -log((\hat{y})^2 - (\hat{x} + tv)^2) \quad domf = \{t \big| (\hat{y})^2 - (\hat{x} + tv) > 0\}$$

$$f(t)=-log(w^2(t-a)(t-b))=-log(w^2)-log(t-a)-log(t-b) \quad (a,b\,are\,the\,solutions)$$

As $-log\ x$ is self-concordant and affine principles, $f(x,y)$ is self-concordant
when $w\neq 0$, let $t=\frac{y-\hat{y}}{w}$

$$f=-log(\alpha+\beta y-\frac{\gamma}{y})-log(y)$$

$$\alpha=2\frac{\hat{y}v^Tv}{w^2}-2\frac{(\hat{x})^Tv}{w}$$

$$\beta=1-\frac{v^Tv}{w}$$

$$\gamma=(\hat{x})^T(\hat{x})-2\frac{\hat{y}(\hat{x})^Tv}{w}+\frac{(\hat{y})^2v^Tv}{w^2}$$

let $g(y)=-\alpha-\beta y+\frac{\gamma}{y}$,
then

$$f(\hat{x}+tv,\hat{y}+tw)=-log(-g(y))-log(y)$$

$$g''(y)=\frac{2\gamma}{y^3}$$

$$g'''(y)=-\frac{6\gamma}{y^4}$$

$$|g'''(y)|\leq\frac{3g''(y)}{y}\ and\ g(x)\ is\ convex$$

According to problem 3, $f(x,y)$ is self-concordant

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3.1

1.To rewrite $B^+=B+\frac{(y-Bs)(y-Bs)^T}{(y-Bs)^Ts}$ **as** $C^+=C+\frac{(s-Cy)(s-Cy)^T}{(s-Cy)^Ty}$
According to Sherman-Morrison formula:

$$(A+uvT)^{-1}=A^{-1}-\frac{A^{-1}uv^TA^{-1}}{1+v^TA^{-1}u}$$

$$C^+=C-\frac{C(y-Bs)(y-Bs)^TC}{1+(y-Bs)^TC(y-Bs)}$$

As $BC = I,\ (y - Bs)^T s = 1$

$$C^+ = C - \frac{(s - Cy)(s - Cy)^T}{(y - Bs)^T s + (y - Bs)^T (Cy - s)}$$

$$C^+ = C + \frac{(s - Cy)(s - Cy)^T}{(s - Cy)^T y}$$

3.2

2.To rewrite $B^+ = B - \frac{Bss^T B}{s^T Bs} + \frac{yy^T}{y^T s}$ **as** $C^+ = (I - \frac{sy^T}{y^T s})C(I - \frac{ys^T}{y^T s}) + \frac{ss^T}{y^T s}$
According to Sherman-Morrison formula:

$$(A + \frac{uu^t}{t})^{-1} = A^{-1} - \frac{A^{-1}(uu^T)A^{-1}}{t + u^T A^{-1} u}$$

$$(B + \frac{yy^T}{y^T s} - \frac{Bss^T B^T}{s^T Bs})^{-1} = (B + \frac{yy^T}{y^T s})^{-1} + \frac{(B + \frac{yy^T}{y^T s})^{-1} Bss^T B (B + \frac{yy^T}{y^T s})^{-1}}{s^T Bs - s^T B (B + \frac{yy^T}{y^T s})^{-1} Bs}$$

$$C^+ = (B + \frac{yy^T}{y^T s})^{-1} + (B + \frac{yy^T}{y^T s})^{-1} \frac{Bss^T B}{s^T Bs - s^T B (B + \frac{yy^T}{y^T s})^{-1} Bs} (B + \frac{yy^T}{y^T s})^{-1}$$

$$C^+ = C + \frac{ss^T y^T s}{(s^T y)^2} + \frac{ss^T yCy^T}{(s^T y)^2} - \frac{Cys^T}{s^T y} - \frac{sy^T C}{y^T s}$$

$$C^+ = C(I - \frac{sy^T}{y^T s}) - \frac{sy^T C}{s^T y}(I - \frac{sy^T}{s^T y}) + \frac{ss^T}{s^T y} = (I - \frac{sy^T}{y^T s})C(I - \frac{ys^T}{y^T s}) + \frac{ss^T}{y^T s}$$

3.3

3.We know $C^+ = C - \frac{Cyy^T C}{y^T Cy} + \frac{ss^T}{y^T s}$**, to prove** $B^+ = (I - \frac{ys^T}{y^T s})B(I - \frac{sy^T}{y^T s}) + \frac{yy^T}{y^T s}$

$$(C^+)^{-1} = (C + \frac{ss^T}{y^T s})^{-1} + \frac{(C + \frac{ss^T}{y^T s})^{-1} Cyy^T C (C + \frac{ss^T}{y^T s})^{-1}}{y^T Cy + y^T C (C + (\frac{ss^T}{y^T s})^{-1} Cy}$$

$$(C^+)^{-1} = C^{-1} + \frac{yy^T ys}{(s^T)^2} + \frac{yy^T SC^{-1} S^T}{(s^T y)^2} - \frac{C^{-1} sy^T}{y^T s} - \frac{ys^T C^{-1}}{s^T y}$$

$$(C^+)^{-1} = C^{-1}(I - \frac{sy^T}{y^T s}) - \frac{ys^T C^{-1}}{s^T y}(I - \frac{ys^T}{s^T y}) + \frac{yy^T}{s^T y}$$

$$B^+ = (I - \frac{ys^T}{y^T S})B(I - \frac{sy^T}{y^T s}) + \frac{yy^T}{y^T s}$$