Stanford CME 241 (Winter 2021) - Assignment 6

Assignments:

1. Assume the Utility function is $U(x) = x - \frac{\alpha x^2}{2}$. Assuming $x \sim \mathcal{N}(\mu, \sigma^2)$, calculate:

   - Expected Utility $\mathbb{E}[U(x)]$
   - Certainty-Equivalent Value x_{CE}
   - Absolute Risk-Premium π_A

   Assume you have a million dollars to invest for a year and you are allowed to invest z dollars in a risky asset whose annual return on investment is $\mathcal{N}(\mu, \sigma^2)$ and the remaining (a million minus z dollars) would need to be invested in a riskless asset with fixed annual return on investment of r. You are not allowed to adjust the quantities invested in the risky and riskless assets after your initial investment decision at time $t = 0$ (static asset allocation problem). If your risk-aversion is based on this Utility function, how much would you invest in the risky asset? In other words, what is the optimal value for z, given your level of risk-aversion (determined by a fixed value of α)?

   Plot how the optimal value of z varies with α.

2. Repeat the calculations for the *Portfolio application of CRRA* (that we covered in class) with a Utility function of $U(x) = \log(x)$ (instead of $U(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$).

3. Assume you are playing a casino game where at every turn, if you bet a quantity x, you will be returned $x \cdot (1+\alpha)$ with probability p and returned $x \cdot (1-\beta)$ with probability $q = 1-p$ for $\alpha, \beta \in \mathbb{R}^+$ (i.e., the return on bet is α with probability p and $-\beta$ with probability $q = 1-p$) . The problem is to identify a betting strategy that will maximize one's expected wealth over the long run. The optimal solution to this problem is known as the Kelly criterion, which involves betting a constant fraction of one's wealth at each turn (let us denote this optimal fraction as f^*).

   It is known that the Kelly criterion (formula for f^*) is equivalent to maximizing the Expected Utility of Wealth after a single bet, with the Utility function defined as: $U(W) = \log(W)$. Denote your wealth before placing the single bet as W_0. Let f be the fraction (to be solved for) of W_0 that you will bet. Therefore, your bet is $f \cdot W_0$.

   (1) - Write down the two outcomes for wealth W at the end of your single bet of $f \cdot W_0$.
   (2) - Write down the two outcomes for log (Utility) of W.
   (3) - Write down $\mathbb{E}[\log(W)]$.
   (4) - Take the derivative of $\mathbb{E}[\log(W)]$ with respect to f.
   (5) - Set this derivative to 0 to solve for f^*. Verify that this is indeed a maxima by evaluating the second derivative at f^*. This formula for f^* is known as the Kelly Criterion.
   (6) - Convince yourself that this formula for f^* makes intuitive sense (in terms of it's dependency on α, β and p).

$$f = \frac{p\alpha - \beta + \beta p}{\beta\alpha} = \frac{p\alpha - \beta q}{\alpha\beta}$$

(1)

$$\begin{tikzcd}[column sep=large]
 & fW_0(1+\alpha) + (1-f)W_0 \\
W_0 \arrow[ur, "p"] \arrow[dr, "1-p"'] & \\
 & fW_0(1-\beta) + (1-f)W_0
\end{tikzcd}$$

(2) $\log(\qquad\qquad)$

$\log(\qquad\qquad)$

(3) $p\log[fW_0(1+\alpha) + (1-f)W_0] + (1-p)\log[fW_0(1-\beta) + (1-f)W_0]$

(4)
$$\frac{d}{df} = p\frac{W_0(1+\alpha) - W_0}{fW_0(1+\alpha) + (1-f)W_0} + (1-p)\frac{W_0(1-\beta) - W_0}{fW_0(1-\beta) + (1-f)W_0}$$

(5)
$$\frac{d}{df} = \frac{pW_0\alpha(-f\beta W_0 + W_0) + (1-p)(-W_0\beta)(f\alpha W_0 + W_0)}{[fW_0(1+\alpha) + (1-f)W_0][fW_0(1-\beta) + (1-f)W_0]}$$

$$p\alpha(-f\beta+1) - \beta(1-p)(f\alpha+1) = 0$$

$$\cancel{-p\alpha f\beta} + p\alpha - \beta f\alpha + \cancel{\beta p f\alpha} - \beta + \beta p = 0$$

$$f = \frac{p\alpha - \beta + \beta p}{\beta\alpha}$$

$$\frac{\partial^2}{\partial f^2} = \frac{\partial}{\partial f} \frac{(p\alpha - \beta f\alpha - \beta + \beta p)}{[f\alpha+1][-f\beta+1]}$$

$$= \frac{-\alpha\beta(f\alpha+1)(1-f\beta) - (p\alpha-\beta f\alpha-\beta+\beta p)(-2\alpha\beta f-\beta+\alpha)}{(f\alpha+1)^2(-f\beta+1)^2}$$

$$= -\alpha\beta(f\alpha+1-f^2\alpha\beta-f\beta) - (-2p\alpha^2\beta f + \cancel{2}\alpha^2\beta^2 f^2 + 2\alpha\beta^2 f - 2\alpha\beta^2 f p - \cancel{p\alpha\beta} + \cancel{\beta^2 f\alpha} + \beta^2 - \beta^2 p + p\alpha^2 - \cancel{\beta f\alpha^2} - \cancel{\beta\alpha} + \cancel{\beta\alpha p})$$

$$-\cancel{\alpha^2\beta f} - \cancel{\alpha\beta} + \cancel{f^2\alpha^2\beta^2} + \cancel{f\beta^2\alpha}$$

$$= 2p\alpha^2\beta f - f^2\alpha^2\beta^2 - 2\alpha\beta^2 f + 2\alpha\beta^2 f p - \beta^2 + \beta^2 p - p\alpha^2$$

$$f\alpha\beta(2\alpha p - f\alpha\beta - 2\beta + 2\beta p) - \beta^2 + \beta^2 p - p\alpha^2$$

$$(p\alpha - \beta + \beta p)(\cancel{2}\alpha p - \cancel{p\alpha} + \cancel{\beta} - \cancel{\beta p} - \cancel{2}\beta + \cancel{2}\beta p) - \beta^2 + \beta^2 p - p\alpha^2$$

$$f = \frac{p\alpha - \beta + \beta p}{\beta\alpha} = \frac{p\alpha - \beta q}{\alpha\beta}$$

$$(p\alpha - \beta + \beta p)(\alpha p - \beta + \beta p) - \beta^2 + \beta^2 p - p\alpha^2$$

$$p^2\alpha^2 + \cancel{\beta^2} + \beta^2 p^2 - 2p\alpha\beta - \cancel{2}\beta^2 p + 2p^2\alpha\beta$$

$$-\cancel{\beta^2} + \cancel{\beta^2} p - p\alpha^2$$

$$p^2\alpha^2 + \beta^2 p^2 - 2p\alpha\beta - \beta^2 p + 2p^2\alpha\beta - p\alpha^2$$

$$p^2(\alpha^2 + \beta^2 + 2\alpha\beta) - p(\alpha^2 + \beta^2 + 2\alpha\beta)$$

$$\underbrace{(p^2 - p)(\alpha + \beta)^2 < 0}$$

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   Assume you have a million dollars to invest for a year and you are allowed to invest z dollars in a risky asset whose annual return on investment is $\mathcal{N}(\mu, \sigma^2)$ and the remaining (a million minus z dollars) would need to be invested in a riskless asset with fixed annual return on investment of r. You are not allowed to adjust the quantities invested in the risky and riskless assets after your initial investment decision at time $t = 0$ (static asset allocation problem). If your risk-aversion is based on this Utility function, how much would you invest in the risky asset? In other words, what is the optimal value for z, given your level of risk-aversion (determined by a fixed value of α)?

   Plot how the optimal value of z varies with α.

$$E[U(x)] = \int p(x)U(x) = \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \left(x - \frac{\alpha x^2}{2}\right) dx$$

$$= E(X) - \frac{\alpha}{2}E(X^2) = \mu - \frac{\alpha}{2}(\mu^2 + \sigma^2)$$

$$U(X_{CE}) = E[U(x)] \qquad X_{CE} - \frac{\alpha X_{CE}^2}{2} = \mu - \frac{\alpha}{2}(\mu^2+\sigma^2)$$

$$\frac{\alpha}{2}X_{CE}^2 - X_{CE} + \mu - \frac{\alpha}{2}(\mu^2+\sigma^2) = 0$$

$$X_{CE} = \frac{1 \pm \sqrt{1 - 2\alpha\left[\mu - \frac{\alpha}{2}(\mu^2+\sigma^2)\right]}}{\alpha}$$

$$\pi_A = E(x) - X_{CE}$$

$$= \int \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} x\, dx - X_{CE} = \mu - X_{CE}$$

$$E[U(X)] = E[U(zX + (1-z)r)]$$

$$= E\left[zX + (1-z)r - \frac{\alpha}{2}(zX + (1-z)r)^2\right]$$

$$= z\mu + (1-z)r - \frac{\alpha}{2} E\left[z^2X^2 + (1-z)^2r^2 + 2z(1-z)Xr\right]$$

$$= z\mu + (1-z)r - \frac{\alpha}{2}\Big[z^2(\mu^2+\sigma^2) + (1-z)^2 r^2 + 2z(1-z)\mu r \Big]$$

$$\frac{\partial}{\partial z} = \mu - r - \frac{\alpha}{2}\Big[(\mu^2+\sigma^2)2z - 2(1-z)r^2 + 2(1-2z)\mu r \Big]$$

$$= 0$$

$$\mu - r - \frac{\alpha}{2}\Big[(2\mu^2+2\sigma^2+2r^2-4\mu r)z - 2r^2 + 2\mu r \Big] = 0$$

$$\frac{\mu - r + \alpha r^2 - \alpha\mu r}{\alpha(\mu^2+\sigma^2+r^2-2\mu r)} = z$$

2. Repeat the calculations for the *Portfolio application of CRRA* (that we covered in class) with a Utility function of $U(x) = \log(x)$ (instead of $U(x) = \frac{x^{1-\gamma}-1}{1-\gamma}$).

$$R(x) = \frac{-U''(x) \cdot x}{U'(x)}$$

$$= \frac{\frac{1}{x^2} \cdot x}{\frac{1}{x}} = 1$$