1. Write the following number in exponential forms.

(1) $(1+\sqrt{3}i)^{2023}$

$= [2e^{i\frac{\pi}{3}}]^{2023}$

$= 2^{2023} e^{i\frac{\pi}{3}(2022+1)}$

$= 2^{2023} e^{i\frac{\pi}{3}}$.

(2) $(1+\sqrt{3}i)^{\frac{1}{2023}}$

$= [2e^{i\frac{\pi}{3}}]^{\frac{1}{2023}}$

$= 2^{\frac{1}{2023}} e^{i(\frac{\pi}{3}+2k\pi)\cdot\frac{1}{2023}}$

$= 2^{\frac{1}{2023}} e^{i\frac{\pi+6k\pi}{6069}}$.

(3) $\frac{1+i}{1+\sqrt{3}i}$

$= \frac{\sqrt{2}e^{i\frac{\pi}{4}}}{2e^{i\frac{\pi}{3}}}$

$= \frac{\sqrt{2}}{2}e^{i(\frac{\pi}{4}-\frac{\pi}{3})} = \frac{\sqrt{2}}{2}e^{-i\frac{\pi}{12}}$.

2. Are the following functions analytic? Why?

(1) $f(z) = (\bar{z})^2$

let $z = x+iy$.

then $\bar{z} = x-iy$.

$f(z) = (\bar{z})^2 = (x-iy)^2$

$= x^2 - 2xyi - y^2$

let $u(x,y) = x^2-y^2$

$v(x,y) = -2xy$.

then

$$\begin{cases} u_x = 2x \neq -2x = v_x & (\text{if } x \neq 0) \\ u_y = -2y \neq 2y = -v_y. & (\text{if } y \neq 0) \end{cases}$$

Since $f(z)$ donot satisifies Cauchy-Rieman equations in any area D

So $f(z)$ is not analytic.

(2) $f(z) = e^{x+2yi}$

$= e^x e^{2yi}$

$= e^x(\cos 2y + i\sin 2y)$

$= e^x\cos 2y + ie^x\sin 2y$

let $u(x,y) = e^x\cos 2y$

$v(x,y) = e^x\sin 2y$

then

$$\begin{cases} u_x = e^x\cos 2y \neq 2e^x\cos 2y & (\text{if } y \neq \frac{\pi}{4}+k\pi) \\ u_y = -2e^x\sin 2y \neq e^x\sin 2y & (\text{if } y \neq \frac{\pi}{2}+k\pi) \end{cases}$$

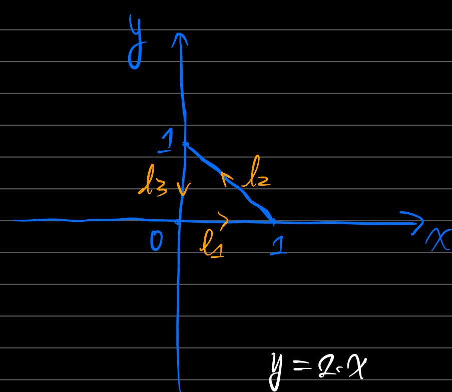
Since $f(z)$ donot satisifies Cauchy-Riemann equations in any area

So $f(z)$ is not analytic.

3. Evaluate the following integrals.

(1) $\int_C x\,dz$ (2) $\int_C dz$.

where C is the boundary of the trangle $\{(x,y) \in \mathbb{R}^2 : 0 < x < 1,\ 0 < y < 1-x\}$ with the counterclockwise orientation. (as the gram show)



$y = 1-x$

$x = 1-y$

$dx = -dy$.

(1) $\int_C x\,dz$

$= \int_C x\,d(x+iy)$

$= \int_C x\,dx + i\int_C x\,dy$

$= \int_{l_1} x\,dx + \int_{l_2} x\,dx + \int_{l_3} x\,dx + i\left(\int_{l_1} x\,dy + \int_{l_2} x\,dy + \int_{l_3} x\,dy\right)$

$= \int_0^1 x\,dx + \int_0^1 (1-y)(-dy) + 0 + i\left(0 + \int_1^0 x(-dx) + \int_1^0 0 \cdot dy\right)$

$= \frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}(y-1)^2\Big|_0^1 + i\left(\frac{1}{2}x^2\Big|_0^1\right)$

$= \frac{1}{2} - \frac{1}{2} + + i\frac{1}{2} = \frac{1}{2}i.$

(2) $\int_C dz$

Since $f(z) = 1$ is analytic

$\int_C dz = 0.$

4. Evaluate the following integrals with positive orientations.

(1). $\int_{|z|=2} \frac{\sin z}{z-1}\,dz$

the only sigularity of $\frac{\sin z}{z-1}$ is $z=1$.

and it is contain in $|z| \le 2$.

By Cauchy integral formula

$$\int_{|z|=2} \frac{\sin z}{z-1}\,dz = 2\pi i (\sin z)\Big|_{z=1} = i2\pi \sin 1.$$

(2) $\int_{|z|=2} \frac{\sin z}{(z-1)^{2022}}\,dz$

for the same reason with (1)

$$\int_{|z|=2} \frac{\sin z}{(z-1)^{2022}}\,dz = \frac{2\pi i}{(2021)!}(\sin z)^{(2021)}\Big|_{z=1}$$

$$= \frac{2\pi i}{(2021)!} \sin\left(z + \frac{2021\pi}{2}\right)\Big|_{z=1}$$

$$= \frac{2\pi i}{(2021)!}(\cos z)\Big|_{z=1}$$

$$= i\frac{2\pi}{(2021)!}\cos 1$$

5. For any z with $|z|<2$, let

$$f(z) = \int_{|w|=2} \frac{e^{w^2}}{w-z}\,dw.$$

Where $|z|=2$ is equipped with the positive orientation.

(1) Evaluation $f(0)$

$$f(0) = \int_{|w|=2} \frac{e^{w^2}}{w}\,dw$$

By Cauchy intergral formular.

$f(0) = 2\pi i\, e^{w^2}\big|_{w=0} = 2\pi i$.

(2) for $\forall z \in \{z \in \mathbb{C} \mid |z| < 2\}$,

the function $\frac{e^{w^2}}{w-z}$ has

only sigularity $w = z$ at $|w| \le 2$.

By Cauchy integral formular.

$f(z) = \int_{|w|=2} \frac{e^{w^2}}{w-z}\, dw = 2\pi i\, e^{w^2}\big|_{w=z} = 2\pi i\, e^{z^2}$

Since e^{z^2} is analytic at $|z|<2$.

So $f(z) = 2\pi i\, e^{z^2}$ is also analytic at $|z|<2$. #.