Chinese Name________ Student ID □□□□□□□□□□□□ School________ Major________ Sit NO.________

(DONOY WRITE YOUR ANSWER IN THIS AREA)

Sale Line

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

2020-2021-2《Calculus II》 Exam Paper A

Notice: **1. Make sure that you have filled the form on the left side of seal line.**
2. Write your answers on the exam paper.
3. This is a close-book exam.
4. The exam with full score of 100 points lasts 120 minutes.

Question No.	1-5	6-12	13-22	Sum
Score				

一. **Answer the questions.** $(3'\times 6=18')$

1. L is the boundary of circle $x^2+y^2=9$ with counterclockwise direction, then curve integral $\oint\limits_L (2xy-2y)dx+(x^2-4x)dy=$ ______.

-18π

2. $L: x^2+y^2=a^2$ $\oint\limits_L \sqrt{x^2+y^2}\,ds=$ ________.

$\oint\limits_L \sqrt{x^2+y^2}\,ds=\oint\limits_L |a|\,ds=2\pi a^2$

3. Let $D: x^2+y^2\le 2x$, write the $\iint\limits_D f(x,y)dxdy$ in polar coordinate,

then $\iint\limits_D f(x,y)dxdy=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\theta\int_0^{2\cos\theta} f(r\cos\theta, r\sin\theta)rdr$

4. If $\vec{a}=\{-1,2,2\},\vec{b}=\{2,-1,2\}$, then $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=$ ______ $\{12,12,-6\}$ ______.

5. The equation of plane through the three points $P_1(1,-2,3), P_2(4,1,-2), P_1(-2,-3,0)$ is

________________________. $7x-12y-3z=22.$

6. The divergence of $\vec{A}=e^{xy}\vec{i}+\cos(xy)\vec{j}+xz^2\vec{k}=$ ____________.

The divergence of $\vec{A}=e^{xy}\vec{i}+\cos(xy)\vec{j}+xz^2\vec{k}$ $ye^{xy}-x\sin(xy)+2xz$.

二、Finish the following questions. (7-16: $7' \times 10 = 70'$; 17-18: $6' \times 2 = 12'$)

7. Please show that $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ is conditionally convergent.

Solu:

By the Alternating Series Test,

$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ converges.

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$.

$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges. So, it is conditionally convergent.

8. Expand the function $f(x) = \arctan\frac{1+x}{1-x}$ into power series of x.

Solu:

$$f(x) = \arctan\frac{1+x}{1-x} \Rightarrow f(0) = \arctan 1 = \frac{\pi}{4}$$

$$f'(x) = \frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\cdot\frac{1\cdot(1-x)-(1+x)\cdot(-1)}{(1-x)^2}$$

$$= \frac{1}{1+x^2}(-1 < x < 1)$$

$$\text{by}\int_0^x f'(t)\,dt = f(t)\big|_0^x = f(x) - f(0) = f(x) - \frac{\pi}{4}.$$

$$\Rightarrow f(x) = \frac{\pi}{4} + \int_0^x f'(t)\,dt$$

$$= \frac{\pi}{4} + \int_0^x \frac{1}{1+t^2}\,dt$$

$$= \frac{\pi}{4} + \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt$$

$$= \frac{\pi}{4} + \sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n}\,dt$$

$$= \frac{\pi}{4} + \sum_{n=0}^{\infty}(-1)^n\frac{1}{2n+1}x^{2n+1}.(-1 \le x < 1)$$

9. Line integral $\int_L xy^2dx + y\varphi(x)dy$ is independent of path, and $\varphi(x)$ is derivative, $\varphi(0)=0$, find $\int_{(0,0)}^{(1,1)} xy^2dx + y\varphi(x)dy$.

An: $P = xy^2$, $Q = y\varphi(x)$, $\frac{\partial P}{\partial y} = 2xy = \frac{\partial Q}{\partial x} = y\varphi'(x) \Rightarrow \varphi'(x) = 2x \Rightarrow \varphi(x) = x^2 + C$

By $\varphi(0)=0$ obtain $C=0$, so $\varphi(x)=x^2$

$$\int_{(0,0)}^{(1,1)} xy^2dx + y\varphi(x)dy = \int_{(0,0)}^{(1,1)} xy^2dx + yx^2dy = \int_0^1 ydy = \frac{1}{2}$$

10. Find $\int_{\frac{1}{4}}^{\frac{1}{2}} dx \int_{\frac{1}{2}}^{\sqrt{x}} e^{\frac{x}{y}} dy + \int_{\frac{1}{2}}^{1} dx \int_{x}^{\sqrt{x}} e^{\frac{x}{y}} dy$.

Solu: $I = \iint_D e^{\frac{y}{x}} dxdy = \int_{\frac{1}{2}}^{1} dy \int_{y^2}^{y} e^{\frac{x}{y}} dx = \frac{3}{8}e - \frac{1}{2}e^{\frac{1}{2}}$

11. Find $\iiint_\Omega (x^2+y^2+z^2)dv$, Ω is bounded by $x^2+y^2+z^2=1$.

Solu: $I = \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 r^2 \cdot r^2 \sin\varphi dr = \frac{4\pi}{5}$

12. Find $\iiint_\Omega zdv$, and Ω is bounded by $x^2+y^2=1$ and $z=0$, $z=1$.

Solu: Method A: Cylindrical, $I = \int_0^{2\pi} d\theta \int_0^1 rdr \int_0^1 zdz = \frac{\pi}{2}$

Method B: Cutting plane method: $I = \int_0^1 \pi zdz = \frac{\pi}{2}$

13. Let $f(u,v)$ is differentiable, $z = z(x,y)$ is determined by $(x+1)z - y^2 = x^2 f(x-z, y)$ find $dz|_{(0,1)}$.

Ans:

$x=0, y=1 \Rightarrow z=1$

Partial derivatives of x and y on both sides of the equation

$$z+(x+1)\frac{\partial z}{\partial x}=2xf(x-z,y)+x^2 f_1'\cdot(1-\frac{\partial z}{\partial x})$$

$$(x+1)\frac{\partial z}{\partial y}-2y=x^2[f_1'\cdot(-\frac{\partial z}{\partial y})+f_2']$$

when $x=0, y=1, z=1$

$$\Rightarrow \frac{\partial z}{\partial x}\Big|_{(0,1)}=-1, \frac{\partial z}{\partial y}\Big|_{(0,1)}=2$$

$$dz\big|_{(0,1)}=-dx+2dy$$

14. $\sum$ is the surface $z=x^2+y^2 (z\le 1)$ with upper side, please calculate

$$I=\iint_{\Sigma}(x-1)^3dydz+(y-1)^3dzdx+(z-1)dxdy.$$

Ans :

Let $\sum_1$ is the lower side of surface $x^2+y^2\le 1, z=1$

Let the area enclosed by $\sum_1$ and $\sum$ is Ω.

$$\oiint_{\Sigma_1+\Sigma}(x-1)^3dydz+(y-1)^3dzdx+(z-1)dxdy=-\iiint_{\Omega}[3(x-1)^2+3(y-1)^2+1]dzdxdy$$

$$z=1, dz=0\Rightarrow \iint_{\Sigma_1}(x-1)^3dydz+(y-1)^3dzdx+(z-1)dxdy=0$$

$$\iiint_{\Omega}xdxdydz=\iiint_{\Omega}ydxdydz=0$$

$$I=-\iiint_{\Omega}(3x^2+3y^2+7)dxdydz=-\int_0^{2\pi}d\theta\int_0^1 dr\int_{r^2}^1(3r^2+7)rdz=-4\pi$$

15. Find the minimum distance between the original point and the surface $z^2=x^2y+4$..

Solu:

$z^2=x^2y+4.$

$d^2=x^2+y^2+z^2=x^2+y^2+x^2y+4,$

$f_x=0, f_y=0\Rightarrow 2x+2xy=0, 2y+x^2=0,$

$\Rightarrow 2x-x^3=0\Rightarrow x=0$ or $x=\pm\sqrt{2}\Rightarrow y=0, y=-1.$

critical points: $(0,0)\left(\sqrt{2},-1\right)\left(-\sqrt{2},-1\right).$

$D(x,y)=f_{xx}f_{yy}-f_{xy}^{\ 2}=(2+2y)2-4x^2.$

$D(\pm\sqrt{2},-1)=-8<0,$ neither $\left(\sqrt{2},-1\right)$ nor $\left(-\sqrt{2},-1\right)$ yiled extremum.

$D(0,0)=4>0,\ f_{xx}(0,0)=2>0.$

So, $(0,0)$ yields the minmum distance . $d^2=4. d_{\min}=2.$

16. Let $z = f(u, x, y), u = xe^y$, f has the second-order continuous partial derivative, find $\dfrac{\partial^2 z}{\partial x \partial y}$.

Ans:

$$\frac{\partial z}{\partial x} = f_1' \cdot \frac{\partial u}{\partial x} + f_2' = f_1' \cdot e^y + f_2'$$

$$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = (f_{11}'' \cdot \frac{\partial u}{\partial y} + f_{13}'')e^y + f_1' \cdot e^y + f_{21}'' \cdot \frac{\partial u}{\partial y} + f_{23}''$$

$$= f_{11}'' \cdot xe^{2y} + f_{13}'' \cdot e^y + f_1' \cdot e^y + f_{21}'' \cdot xe^y + f_{23}''$$

$$= e^y f_1' + xe^{2y} f_{11}'' + e^y f_{13}'' + xe^y f_{21}'' \cdot + f_{23}''.$$

17. Find $\oint_L \dfrac{(x-1)dy - ydx}{(x-1)^2 + y^2}$, L represents a simple closed curve, including point $(1, 0)$ with counterclockwise direction.

An: Take a circle $C:(x-1)^2 + y^2 = r^2$ inside of L with counterclockwise direction.

Parametric equations of C: $\begin{cases} x = 1 + r\cos\theta \\ y = r\sin\theta \end{cases} \quad 0 \le \theta \le 2\pi$

By Green formula

$$\oint_L \frac{(x-1)dy - ydx}{(x-1)^2 + y^2} = \oint_C \frac{(x-1)dy - ydx}{(x-1)^2 + y^2} = \int_0^{2\pi} d\theta = 2\pi$$

18. Find convergent region and sum function of power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n+2}}{(n+1)(2n+1)}$.

Ans :

$$\because \lim_{n\to\infty}\frac{\left|\dfrac{x^{2n+4}}{(n+2)(2n+3)}\right|}{\left|\dfrac{x^{2n+2}}{(n+1)(2n+1)}\right|}=x^2$$

$\Rightarrow$ when $|x|<1$, absolute convergence;when $|x|>1$, divergence

when $x=\pm 1$, series $\displaystyle\sum_{n=0}^{\infty}\frac{1}{(n+1)(2n+1)}$ is convergence.

$\Rightarrow$ Convergence region is $[-1,1]$.

let $\displaystyle f(x)=\sum_{n=0}^{\infty}\frac{x^{2n+2}}{(n+1)(2n+1)}, x\in[-1,1]$

$$f'(x)=2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1},\ f''(x)=2\sum_{n=0}^{\infty}x^{2n}=\frac{2}{1-x^2}, x\in(-1,1),$$

$\because f'(0)=0, f(0)=0$

$$\Rightarrow f'(x)=\int_0^x f''(t)dt=\int_0^x\frac{2}{1-t^2}dt=,$$

$$f(x)=\int_0^x f'(t)dt=\int_0^x[\ln(1+t)-\ln(1-t)]dt$$

$$=(1+x)\ln(1+x)+(1-x)\ln(1-x), x\in(-1,1)$$

$$f(1)=\lim_{x\to 1^-}f(x)=2\ln 2,\ f(-1)=\lim_{x\to -1^+}f(x)=2\ln 2$$