Seat No. ______ Major/Class ______ School ______ Student ID ______ Name ______

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(DO NOT WRITE YOUR ANSWER IN THIS AREA)

WARNING: MISBEHAVIOR AT EXAM TIME WILL LEAD TO SERIOUS CONSEQUENCE.

SCUT Final Exam

Probability and Statistics Exam paper A (2021–2022-2)

Notice:
1. **Make sure that you have filled the form on the left side of seal line.**
2. **Write your answers on the exam answer sheet .**
3. **This is a close-book exam.**
4. **The exam with full score of 100 points lasts 120 minutes.**

Question No.	I	II	III	IV	V	VI	VII	VIII	Sum
Score									

I. (15 points) Two events A and B are such that $\mathrm{P}(A) = 0.5, \mathrm{P}(B) = 0.3$ and $\mathrm{P}(A \cap B) = 0.1$. Calculate

Score

(a) $\mathrm{P}(A \mid B)$; (b) $\mathrm{P}(B \mid A)$; (c) $\mathrm{P}(A \mid A \cup B)$;

(d) $\mathrm{P}(A \mid A \cap B)$; (e) $\mathrm{P}(A \cap B \mid A \cup B)$.

Solution.

Venn diagrams are helpful in understanding some of the events that arise below.

(a) $\mathrm{P}(A \mid B) = \mathrm{P}(A \cap B)/\mathrm{P}(B) = \frac{1}{3}$

(b) $\mathrm{P}(B \mid A) = \mathrm{P}(A \cap B)/\mathrm{P}(A) = \frac{1}{5}$

(c) $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B) = 0.7$, and the event $A \cap (A \cup B) = A$, so

$$\mathrm{P}(A \mid A \cup B) = \mathrm{P}(A)/\mathrm{P}(A \cup B) = \frac{5}{7}$$

(d) $\mathrm{P}(A \mid A \cap B) = \mathrm{P}(A \cap B)/\mathrm{P}(A \cap B) = 1$, since $A \cap (A \cap B) = A \cap B$.

(e) $\mathrm{P}(A \cap B \mid A \cup B) = \mathrm{P}(A \cap B)/\mathrm{P}(A \cup B) = \frac{1}{7}$, since $A \cap B \cap (A \cup B) = A \cap B$.

II. (10 points) A fair coin is tossed repeatedly. Show that, with probability one, a head turns up sooner or later.

Score

Solution.

By the continuity property of P we have that

$$P(\text{ no head ever }) = \lim_{n \to \infty} P(\text{ no head in first } n \text{ tosses }) = \lim_{n \to \infty} 2^{-n} = 0$$

so that $P(\text{ some head turns up }) = 1 - P(\text{ no head ever }) = 1$.

(or $P(\text{ no head ever }) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1$.)

III. (10 points) Of the people passing through an airport metal detector, 0.5% activate it; let $X =$ the number among a randomly selected group of 500 who activate the detector.

Score

(a) What is the (approximate) pmf of X ?

(b) Compute $E(X)$ and $V(X)$.

Solution.

(a) $X \sim \text{Bin}(n=500, p=.005)$. Since n is large and p is small, X can be approximated by a Poisson distribution with $\mu = np = 2.5$. The approximate pmf of X is

$$p(x; 2.5) = \frac{e^{-2.5} 2.5^x}{x!}.$$

(b) $E(X) = V(X) = \mu = 2.5$.

IV. (10 point) Let X denote the time to failure (in years) of a certain hydraulic component. Suppose the pdf of X is $f(x) = 2/(x+1)^3$ for $x > 0$.

Score

(a) Determine the cdf.

(b) If the component has a salvage value equal to $100/(1+x)$ when its time to failure is x, what is the expected salvage value?

Solution.

(a) For $x \le 0, F(x) = 0$; For $x > 0$,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_0^x \frac{2}{(y+1)^3} dy = -\frac{1}{2} \cdot \frac{2}{(y+1)^2}\Big|_0^x = 1 - \frac{1}{(x+1)^2}.$$

(b) The expected salvage value is

$$E\left(\frac{100}{X+1}\right) = \int_0^\infty \frac{100}{x+1} \cdot \frac{2}{(x+1)^3} dx = 200 \int_0^\infty \frac{1}{(x+1)^4} dx = \frac{200}{3}.$$

V. (20 points) Let X and Y be two continuous rv's with joint pdf

Score

$$f(x,y) = \begin{cases} \frac{6}{5}(x^2+y) & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the marginal pdf of X and Y.

(b) Determine whether X and Y are independent rv's and state the reasons.

(c) Compute the covariance between X and Y.

(d) Compute the probability $P(Y \le X^2)$.

Solution.

(a) The marginal pdf of X and Y are

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy = \begin{cases} \frac{6}{5}\int_0^1 (x^2+y)\,dy = \frac{6}{5}\left(x^2 + \frac{1}{2}\right), & 0 \le x \le 1; \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx = \begin{cases} \frac{6}{5}\int_0^1 (x^2+y)\,dx = \frac{6}{5}\left(\frac{1}{3} + y\right), & 0 \le y \le 1; \\ 0, & \text{otherwise} \end{cases}$$

(b) Clearly $f(x,y) \neq f_x(x) \cdot f_y(y)$, so X and Y are not independent.

(c) Using the definition,

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x,y)dxdy = \frac{7}{20};$$
$$E(X) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xf(x,y)dxdy = \frac{3}{5};$$
$$E(Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} yf(x,y)dxdy = \frac{3}{5};$$
$$\text{Cov}(X,Y) = E(XY) - E(X)E(Y) = \frac{7}{20} - \frac{9}{25} = -0.01.$$

(d) The probability is

$$P(Y \leq X^2) = \iint_{y \leq x^2} f(x,y)dxdy = \int_0^1 \left[\int_0^{x^2} \frac{6}{5}(x^2+y)dy \right] dx = \frac{9}{25}.$$

VI. (10 points) Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from the pdf

Score

$$f(x;\alpha) = \begin{cases} \frac{1}{2}e^{-\frac{x-\alpha}{2}}, & x \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

(a) Use the method of moments to find an estimator for α.

(b) Find the maximum likelihood estimator for α.

Solution.

(a) $E(X) = \int_{-\infty}^{+\infty} xf(x,\alpha)dx = \int_{\alpha}^{+\infty} x \cdot \frac{1}{2}e^{-\frac{x-\alpha}{2}}dx = 2+\alpha$, then $\alpha = E(X) - 2$.

So the moment estimator of α is $\hat{\alpha}_M = \overline{X} - 2$.

(b) The likelihood function is $L(\alpha) = L(x_1, \cdots, x_n; \alpha) = f(x_1, \alpha) \cdots f(x_n, \alpha)$

$$= \begin{cases} \left(\frac{1}{2}\right)^n e^{-\sum_{i=1}^n \frac{x_i - \alpha}{2}}, & x_i \geqslant \alpha \quad (i = 1, \cdots, n) \\ 0, & \text{otherwise.} \end{cases}$$

The ln of the likelihood fucntion is

$$\ln L(\alpha) = -n\ln 2 - \sum_{i=1}^n \left(\frac{x_i - \alpha}{2}\right) = -n\ln 2 - \frac{1}{2}\sum_{i=1}^n x_i + \frac{n}{2}\alpha \quad (x_i \geqslant \alpha, i = 1, \cdots n).$$

We have

$$\frac{d\ln L(x_1, \cdots, x_n; \alpha)}{d\alpha} = \frac{n}{2} > 0 \quad (x_i \geqslant \alpha, i = 1, \cdots n)$$

So as long as $\min(x_i) \geqslant \alpha$, $L(\alpha)$ increase as α increase. However, as soon as $\min(x_i) < \alpha$, the likelihood function drops to 0. Thus, the mle of α is $\hat{\alpha}_L = \min(X_i)$.

VII. (10 points) A sample of 100 pH-values of a chemical compound was detected. Suppose that the population of pH is a random variable with expected value μ and standard deviation 0.1 and that the sample mean pH is 8.55.

Score

(a) Calculate a confidence interval of μ at the 90% confidence level.

(b) What sample size would be required for the width of a 90% confidence interval to be at most 0.02?

(Using the Central Limit Theorem. $z_{0.025} = 1.96,\ z_{0.05} = 1.65,\ z_{0.1} = 1.28$.)

Solution.

According to the CLT, $\bar{X}$ has approximately a normal distribution $N\left(\mu, \frac{0.01}{n}\right)$.

(a) With $n = 100, \overline{x} = 8.55, \sigma = 0.1$ and $z_{\alpha/2} = z_{0.05} = 1.65$, the 90% confidence interval for μ is

$$\overline{x} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = 8.55 \pm 1.65\frac{0.1}{10} = 8.55 \pm 0.0165 = (8.5335, 8.5665).$$

(b) The width of 90% CI is $w = 2z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$. So

$$n = \left(\frac{2z_{\alpha/2}\sigma}{w}\right)^2 = \left(\frac{2(1.65)(0.1)}{0.02}\right)^2 = (16.5)^2 = 272.25,$$

So, a sample of size at least 273 is required.

VIII. (15 points) In a test of time perception for $n = 16$ smokers after a 12-hour smoking abstinence, each of them was asked to estimate how much time had elapsed during a 45-second period. The average estimating time is 50 seconds. Suppose the population of estimating times of all smokers is known to be normal with $\sigma = 9$ seconds.

Score

(a) Does this data suggest that the population mean more than 45 seconds? State and test the appropriate hypotheses using $\alpha = 0.05$. ($z_{0.025} = 1.96,\ z_{0.05} = 1.65,\ z_{0.1} = 1.28$.)

(b) If a level 0.05 test is used, what is $\beta(51)$, the probability of making a type II error when $\mu = 51$ seconds? ($\Phi(1.017) = 0.84, \Phi(0.707) = 0.76$.)

(c) What value of n is necessary to ensure that $\beta(51) = 0.10$ when $\alpha = 0.05$?

Solution.

$H_0 :\ \mu = 45$ vs $H_a :\ \mu > 45$.

The sample mean is $\bar{x} = 50$.

(a) $z = \frac{\overline{x}-\mu_0}{\sigma/\sqrt{n}} = \frac{50-45}{9/\sqrt{16}} = 2.22 \geq z_\alpha = z_{0.05} = 1.65$. Reject H_0.

(b) The probability of making a type II error when $\mu = 51$ is

$$\beta(51) = \Phi\left(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) = \Phi\left(1.65 + \frac{45-51}{9/\sqrt{16}}\right)$$
$$= \Phi(-1.017) = 1 - 0.84 = 0.16.$$

(c) Since $z_{0.1} = 1.28$, the requirement that the level 0.05 test also have $\beta(51) = 0.1$ necessitates

$$\Phi\left(1.65 + \frac{45-51}{9/\sqrt{n}}\right) = 0.1, \quad 1.65 + \frac{45-51}{9/\sqrt{n}} = -1.28$$

$$n = \left[\frac{9(1.65+1.28)}{45-51}\right]^2 = 19.32.$$

So $n = 20$ should be used.