1. Write the following numbers in exponential forms

(1) $(1+\sqrt{3}i)^{2023}$ (2) $(1+\sqrt{3}i)^{\frac{1}{2023}}$ (3) $\frac{1+i}{1+\sqrt{3}i}$.

(1) $(1+\sqrt{3}i)^{2023} = (2e^{i\frac{\pi}{3}})^{2023} = 2^{2023}e^{i\frac{2023\pi}{3}} = 2^{2023}e^{\frac{\pi}{3}i}$

(2) $(1+\sqrt{3}i)^{\frac{1}{2023}} = (2e^{i(\frac{\pi}{3}+2k\pi)})^{\frac{1}{2023}} = 2^{\frac{1}{2023}}e^{i(\frac{\pi}{6069}+\frac{2k\pi}{2023})}, k\in\mathbb{Z}$

(3). $\frac{1+i}{1+\sqrt{3}i} = \frac{\sqrt{2}e^{\frac{\pi}{4}i}}{2e^{\frac{\pi}{3}i}} = \frac{\sqrt{2}}{2}e^{-\frac{\pi}{12}i}$.

2. Are the following functions analytic? why?

(1) $f(z)=(\bar{z})^2$ (2) $f(z)=e^{x+2iy}$

(1). No. Let $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$.

Then $f(z)=(\bar{z})^2=(x-iy)^2=x^2-y^2-2xyi$.

$\Rightarrow \begin{cases} u(x,y)=x^2-y^2 \\ v(x,y)=-2xy \end{cases} \Rightarrow \begin{matrix} u_x'=2x & v_x'=-2y \\ u_y'=-2y & v_y'=-2x. \end{matrix}$

If the C-R equations are to hold at a point (x,y).

it follows that $\begin{cases} 2x=-2x \\ -2y=2y \end{cases} \Rightarrow \cancel{x=0,\ \forall y\in\mathbb{R}}\ x=y=0$

By the definition of analytic, $f(z)$ is not analytic anywhere.

(2) No. Let $f(z)=u(x,y)+iv(x,y)$.

Then $f(z)=e^{x+2iy} = e^x\cos 2y + ie^x\sin 2y$.

$\Rightarrow u(x,y)=e^x\cos 2y$, $v(x,y)=e^x\sin 2y$.

If the C-R equations are hold at a point (x,y).

it follows that $\begin{cases} u_x'=e^x\cos 2y = 2e^x\cos 2y = v_y' \\ u_y'=-2e^x\sin 2y = -e^x\sin 2y = -v_x' \end{cases} \Rightarrow \begin{cases} \cos 2y=0 \\ \sin 2y=0. \end{cases}$

but it's not going to hold.

Thus, $f(z)$ is not analytic anywhere.

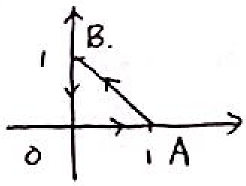
3. Evaluate the following integrals

(1) $\int_C x\,dz$ (2) $\int_C dz$.

where C is the boundary of the triangle

$\{(x,y)\in\mathbb{R}^2;\ 0<x<1,\ 0<y<1-x\}$ with the counterclockwise orientation

(1).



$$\int_C x\,dz = \int_{OA} x\,dz + \int_{AB} x\,dz + \int_{BO} x\,dz.$$

$$= \int_0^1 x\,dx + \int_1^0 x\cdot(1-i)\,dx. + \int_{BO} 0\,dz$$

$$= \frac{1}{2}x^2\Big|_0^1 + \frac{1}{2}(1-i)x^2\Big|_1^0$$

$$= \frac{1}{2}i.$$

(2). Let $f(z)=u(x,y)+iv(x,y)=1$. Then $u(x,y)=1$, $v(x,y)=0$.

and $u_x' = 0 = v_y'$, $u_y' = 0 = -v_x'$ satisfy the C-R equations.

Then $f(z)=1$ is analytic everywhere

Thus $\int_C dz = 0$.

4. Evaluate the following integrals with positive orientations.

(1) $\int_{|z|=2} \frac{\sin z}{z-1}\,dz$, (2) $\int_{|z|=2} \frac{\sin z}{(z-1)^{2022}}\,dz$.

(1). $\int_{|z|=2} \frac{\sin z}{z-1}\,dz = 2\pi i\cdot \sin 1$

(2) $\cancel{\oint_{|z|}}$ $\int_{|z|=2} \frac{\sin z}{(z-1)^{2022}}\,dz = \frac{2\pi i}{2021!}\cdot (\sin z)^{(2021)}\Big|_{z=1}$

$$= \frac{2\pi i}{2021!}\cdot \cos 1.$$

Remark. $(\sin z)^{(n)}$.

$n=1$, $(\sin z)' = \cos z$.

$n=2$, $(\sin z)'' = (\cos z)' = -\sin z$

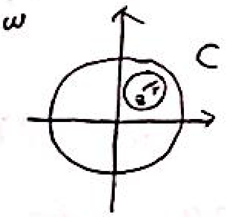
$n=3$, $(\sin z)^{(3)} = (-\sin z)' = -\cos z$

$n=4$, $(\sin z)^{(4)} = (-\cos z)' = \sin z$.

5. For any z with $|z|<2$, let $f(z)=\int_{|w|=2} \frac{e^{w^2}}{w-z}\,dw$.
Where $|z|=2$ is equipped with the positive orientation.
(1). Evaluation $f(0)$. (2) Is $f(z)$ analytic ? why?

(1). $f(0)=\int_{|w|=2} \frac{e^{w^2}}{w}\,dw = 2\pi i \cdot e^{0^2} = 2\pi i$

(2) Since $|z|<2$. then exist enough small $r>0$, s.t.
$C_0: |w-z|=r$ is in the interior of $C: |w|=2$.



Then $f(z)=\int_{|w|=2} \frac{e^{w^2}}{w-z}\,dw$

$=\int_{|w-z|=2} \frac{e^{w^2}}{w-z}\,dw$

$= 2\pi i \cdot e^{z^2}$

Let $\cancel{f(z)=}$ $z=x+iy$, $f(z)=u(x,y)+iv(x,y)$.

Then $f(z)=2\pi i\cdot e^{(x+iy)^2} = 2\pi i \cdot e^{x^2-y^2+2xyi}$

$= 2\pi i \cdot e^{x^2-y^2}\cdot(\cos 2xy + i\sin 2xy)$.

$\Rightarrow u(x,y) = -2\pi e^{x^2-y^2}\cdot \sin(2xy)$.

$v(x,y) = 2\pi e^{x^2-y^2}\cdot\cos(2xy)$.

$\Rightarrow$ $\cancel{u_x' = -8\pi xy e^{x^2-y^2}\cos(2xy)}$.

$\cancel{u_y' =}$ $u_x' = -2\pi\,[2xe^{x^2-y^2}\sin(2xy)+2ye^{x^2-y^2}\cos(2xy)] = v_y'$

$u_y' = -2\pi\,[2xe^{x^2-y^2}\cos(2xy) - 2y\,e^{x^2-y^2}\sin(xy)] = -v_x'$

satisfy the C-R equations.

Thus, $f(z)$ is analytic everywhere.