PHIL 220: Introduction to Logic

Week 9 Discussion (10/24/2025)

Notes: If you want to go over the timed test, just let me know.

Today: Review the syntax of quantificational (or predicate) logic.

Vocabulary:

Designators (Constants)

- **What they are:** Expressions that designate or name a *specific individual object*.
- **English Examples:** "Gabriel", "Zhanming", "USC", "42".
- **Formal Symbol:** Lowercase letters like a, b, c, d, e with or without subscripts, called **constants**.

Variables

- **What they are:** Expressions that *do not* designate any specific individual object. They are *placeholders* for individual objects.
- **Formal Symbols:** Lowercase letters like x, y, z with or without subscripts, called **variables**.

Predicates

- **What they are:** Expressions that attribute a *property* or a *relation* to individuals. They say something *about* a designator.
- **English Examples:** "...is a mammal", "...is taller than...", "...is located between...and...".
- **Formal Symbol:** Uppercase letters, called **predicates**.
- **Arity:** The number of arguments (designators or variables) predicates can take.
  - **1-place (monadic):** Px ("x is a philosopher")
  - **2-place (dyadic):** Lxy ("x loves y")
  - **3-place (triadic):** $Bxyz$ ("y is between x and z")

Quantifiers

- **What they are:** Expressions used to show *generality*—to say "how many" things have a property.
- **Universal Quantifier ($\forall$):** "For all...", "Every...". $\forall x Px$ means "For all x, x is P."
- **Existential Quantifier ($\exists$):** "There exists...", "Some...", "At least one...". $\exists x Px$ means "There exists an x such that x is P."
- **Two roles:** 1. Express generality. 2. Bind variables (we'll see this later).

Apart from these new symbols, we keep some old symbols from propositional logic: connectives ($\neg, \wedge, \vee, \rightarrow$) and parentheses.

Nothing else is a primitive symbol of the language.

Exercise 1

Identify the designators, predicates, quantifier phrases in the following sentences:

(a). Every philosopher is a human being.

(b). Nothing has not been discussed by Plato.

(c). Cat, sitting between Alice and Bob, loves Plato so much.

Solution 1

(a) Every philosopher is a human being.

- Designators: none
- Predicates: 'is a human being' (one-place).
- Quantifier phrase: "Every philosopher".

(b) Nothing has not been discussed by Plato.

- Designators: 'Plato'
- Predicates: '...has not been discussed by ...' (2-place).

- Quantifier phrase: 'Nothing'.

(c) Cat, sitting between Alice and Bob, loves Plato so much.

- Designators: 'Cat', 'Alice', 'Bob', 'Plato'.
- Predicates: '…loves…" (2-place); "… is between … and …" (3-place).

Grammar:

Atomic Formulas

- **Definition:** An n-place predicate followed by n argument places filled in by either constant or variable.
- **Examples:** Pa, Qab, Px, $Raxb$.
- Pa says "The individual a has property P."
- Px says "x has property P." (We don't know *what* x is yet!)

Complex Formulas

We build complex formulas from atomic ones in two ways:

1. **With Connectives:** Just like in propositional logic, if ϕ and ψ are formulas, so are: $\neg\phi$, $(\phi \wedge \psi)$, $(\phi \vee \psi)$, $(\phi \rightarrow \psi)$.
   Examples: $\neg Pa$, $(Pa \wedge Qab)$
2. **With Quantifiers:** If ϕ is a formula and v is a *variable*, then the following are also formulas: $\forall v\phi$, $\exists v\phi$.

Nothing else is a formula.

Exercise 2

Using the translation key below, translate the English sentences into *atomic formulas*.

Symbol	Intended Meaning
a	Alice
b	Bob
c	Charlie
H	is happy
L	… loves …
G	… gave … to …

1. Alice is happy.
2. Alice loves Charlie.
3. Charlie loves Alice.
4. Bob gave Alice to Charlie.
5. Alice gave Charlie to Bob.

Solution 2

1. Ha
2. Lac
3. Lca
4. $Gbac$
5. $Gacb$

(Be careful with the order of predicates' arguments!)

Variables: Free vs. Bound

- **Bound Variable:** An occurrence of a variable is **bound** if it falls within the *scope* of a quantifier using that same variable. The quantifier "catches" or "binds" it.
  - In $\forall x Px$, the x in Px is **bound** by the $\forall x$.
  - In $\forall x(Px \rightarrow Qx)$, *all* occurrences of x are **bound**.

- **Free Variable:** An occurrence of a variable is **free** if it is *not* bound by any quantifier.
  - In Px, the x is **free**.
  - In Rxy, both x and y are **free**.
  - In $(\forall xPx \to Qx)$, the first two x's are bound by $\forall$. But the third x (in Qx) is **free**, because the scope of the quantifier only covers Px.

This leads to an important distinction:

- **Open Formula:** A formula with at least one *free* variable.
  - **Examples:** Px, Rxy, $\forall yRxy$ (the x is still free!).
  - An open formula is not a complete thought. It's like "it is tall"—we can't say if it's true or false until we know what "it" (x) refers to.
- **Closed Formula (i.e. Sentence):** A formula with **no** free variables.
  - **Examples:** Pa, $\forall xPx$, $\exists y\forall xRxy$.
  - These are complete thoughts. They make a claim about the world and can be evaluated as True or False (given a model, which we'll learn about later).

Exercise 3

For each formula below

- Identify any **free** variable occurrences (state the variable and where it is).
- Identify any **bound** variable occurrences.
- State whether the formula is an **open formula** or a **sentence**.

1. $Px \land Qay$
2. $\forall x(Px \land Qay)$
3. $\exists zLza$
4. $\forall yPy \land Qy$
5. $\forall x\exists y(Rxy \to Pz)$

6. $\forall x(\exists y Rxy \to Px)$

Solution 3

Formula	**Free vars**	**Bound vars**	**Open/Sentence**
$Px \wedge Qay$	x, y	–	Open
$\forall x\,(Px \wedge Qay)$	y	x	Open
$\exists z\, Lza$	–	z	Sentence
$\forall y\, Py \wedge Qy$	y in Qy	y in Py	Open
$\forall x\, \exists y\,(Rxy \to Pz)$	z	x, y	Open
$\forall x\,(\exists y\, Rxy \to Px)$	–	x, y	Sentence

Scope note. In $\forall y\, Py \wedge Qy$ the quantifier applies only to the immediately following formula Py, so the y in Qy is free unless parentheses extend the scope.