一(50分)、设矩阵

$$A=\begin{bmatrix}3 & -1 & 2\\ 8 & -6 & 11\\ 4 & -4 & 7\end{bmatrix}$$

求$A$的Jordan标准型$J$，并求相似变换矩阵$P$，使得$P^{-1}AP=J$.

1. 将$A$的特征矩阵$\lambda I-A$化为Smith标准形:

$$\lambda I-A=\begin{bmatrix}\lambda-3 & 1 & -2\\ -8 & \lambda+6 & -11\\ -4 & 4 & \lambda-7\end{bmatrix}\simeq\begin{bmatrix}0 & 1 & 0\\ -(\lambda-2)(\lambda+5) & \lambda+6 & 2\lambda+1\\ -4(\lambda-2) & 4 & \lambda+1\end{bmatrix}$$

$$\simeq\begin{bmatrix}0 & 1 & 0\\ -(\lambda-2)(\lambda+5) & 0 & 2\lambda+1\\ -4(\lambda-2) & 0 & \lambda+1\end{bmatrix}\simeq\begin{bmatrix}0 & 1 & 0\\ -(\lambda-2)(\lambda-3) & 0 & -1\\ -4(\lambda-2) & 0 & \lambda+1\end{bmatrix}$$

$$\simeq\begin{bmatrix}0 & 1 & 0\\ -(\lambda-2)(\lambda-3) & 0 & -1\\ -(\lambda-2)(\lambda-1)^2 & 0 & 0\end{bmatrix}\simeq\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & (\lambda-2)(\lambda-1)^2\end{bmatrix};$$

2. $\lambda I-A$的不变因子为：$\{1,1,(\lambda-2)(\lambda-1)^2\}$;

3. $\lambda I-A$的初等因子为：$\{\lambda-2,(\lambda-1)^2\}$;

4. 初等因子$\lambda-2$对应的Jordan块为：$\begin{bmatrix}2\end{bmatrix}$,

   初等因子$(\lambda-1)^2$对应的Jordan块为：$\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$;

5. Jordan块构成的$A$的Jordan标准形为：$J=\begin{bmatrix}2 & 0 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$;

6. 将$P$按列分块，记$P=[P_1,P_2,P_3]$，则由$AP=PJ$得：

$$\begin{cases}(A-2I)P_1=\vec{0},\\ (A-I)P_2=\vec{0},\\ (A-I)P_3=P_2.\end{cases}$$

   其中$A-2I=\begin{bmatrix}1 & -1 & 2\\ 8 & -8 & 11\\ 4 & -4 & 5\end{bmatrix}\to\begin{bmatrix}1 & -1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$，可取$P_1=\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}$;

   $A-I=\begin{bmatrix}2 & -1 & 2\\ 8 & -7 & 11\\ 4 & -4 & 6\end{bmatrix}\to\begin{bmatrix}2 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$，可取$P_2=\begin{bmatrix}-1\\ 2\\ 2\end{bmatrix}$;

   $[A-I|P_2]=\begin{bmatrix}2 & -1 & 2 & -1\\ 8 & -7 & 11 & 2\\ 4 & -4 & 6 & 2\end{bmatrix}\to\begin{bmatrix}2 & 0 & 1 & -3\\ 0 & 1 & -1 & -2\\ 0 & 0 & 0 & 0\end{bmatrix}$，可取$P_3=\begin{bmatrix}-3\\ 1\\ 3\end{bmatrix}$.

   综上，可取相似变换矩阵$P=\begin{bmatrix}1 & -1 & -3\\ 1 & 2 & 1\\ 0 & 2 & 3\end{bmatrix}$（不唯一）.

二(50分)、求$A=\begin{bmatrix}1&0\\1&2\\0&1\end{bmatrix}$的奇异值分解.

1. $A^TA=\begin{bmatrix}1&1&0\\0&2&1\end{bmatrix}\begin{bmatrix}1&0\\1&2\\0&1\end{bmatrix}=\begin{bmatrix}2&2\\2&5\end{bmatrix}$，$|\lambda I-A^TA|=\begin{vmatrix}\lambda-2&-2\\-2&\lambda-5\end{vmatrix}=(\lambda-6)(\lambda-1)$，$A^TA$的特征值为$\{6,1\}$，$A$的奇异值为$\{\sqrt{6},1\}$.

2. 依次计算$A^TA$的对应于特征值$\{6,1\}$的单位特征向量:

   $6I-A^TA=\begin{bmatrix}4&-2\\-2&1\end{bmatrix}\to\begin{bmatrix}2&-1\\0&0\end{bmatrix}$，单位特征向量可取为$\begin{bmatrix}\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}$;

   $I-A^TA=\begin{bmatrix}-1&-2\\-2&-4\end{bmatrix}\to\begin{bmatrix}1&2\\0&0\end{bmatrix}$，单位特征向量可取为$\begin{bmatrix}-\frac{2}{\sqrt{5}}\\\frac{1}{\sqrt{5}}\end{bmatrix}$.

3. 令$V=\begin{bmatrix}\frac{1}{\sqrt{5}}&-\frac{2}{\sqrt{5}}\\\frac{2}{\sqrt{5}}&\frac{1}{\sqrt{5}}\end{bmatrix}$,则$AV=\begin{bmatrix}\frac{1}{\sqrt{5}}&-\frac{2}{\sqrt{5}}\\\sqrt{5}&0\\\frac{2}{\sqrt{5}}&\frac{1}{\sqrt{5}}\end{bmatrix}$,记$\alpha_1=\begin{bmatrix}\frac{1}{\sqrt{5}}\\\sqrt{5}\\\frac{2}{\sqrt{5}}\end{bmatrix}$,$\alpha_2=\begin{bmatrix}-\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}$.

4. 将$\alpha_1,\alpha_2$依次单位化得：$\beta_1=\frac{1}{\sqrt{6}}\alpha_1=\begin{bmatrix}\frac{1}{\sqrt{30}}\\\frac{\sqrt{5}}{\sqrt{6}}\\\frac{2}{\sqrt{30}}\end{bmatrix}$,$\beta_2=\alpha_2=\begin{bmatrix}-\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}$.

5. 取$\beta_3$使得$\{\beta_1,\beta_2,\beta_3\}$构成$\mathbb{R}^3$的一组单位正交基底，即求解齐次线性方程组 $\begin{bmatrix}\beta_1^T\\\beta_2^T\end{bmatrix}\beta_3=\begin{bmatrix}0\\0\end{bmatrix}$，可取$\beta_3=\begin{bmatrix}\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\end{bmatrix}$.

6. 令$U=\begin{bmatrix}\frac{1}{\sqrt{30}}&-\frac{2}{\sqrt{5}}&\frac{1}{\sqrt{6}}\\\frac{\sqrt{5}}{\sqrt{6}}&0&-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{30}}&\frac{1}{\sqrt{5}}&\frac{2}{\sqrt{6}}\end{bmatrix}$，则$AV=U\begin{bmatrix}\sqrt{6}&0\\0&1\\0&0\end{bmatrix}$，$A=U\begin{bmatrix}\sqrt{6}&0\\0&1\\0&0\end{bmatrix}V^T$.

7. 代入即得$A$的奇异值分解为：

$$A=\begin{bmatrix}\frac{1}{\sqrt{30}}&-\frac{2}{\sqrt{5}}&\frac{1}{\sqrt{6}}\\\frac{\sqrt{5}}{\sqrt{6}}&0&-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{30}}&\frac{1}{\sqrt{5}}&\frac{2}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}\sqrt{6}&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{5}}&\frac{2}{\sqrt{5}}\\-\frac{2}{\sqrt{5}}&\frac{1}{\sqrt{5}}\end{bmatrix}.$$