TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

TABLE 2 The Truth Table for the Conjunction of Two Propositions.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

TABLE 3 The Truth Table for the Disjunction of Two Propositions.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

TABLE 4 The Truth Table for the Exclusive Or of Two Propositions.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

TABLE 8 Precedence of Logical Operators.

Operator	*Precedence*
$\neg$	1
$\wedge$	2
$\vee$	3
$\to$	4
$\leftrightarrow$	5

Definition

Another way to create a proposition from propositional function

In English words

all, some, many, none, and few

Types

- Universal quantifier → *For all values…* → **$\forall x\ P(x)$**
- Existential quantifier → *For some values in…* → **$\exists\ x\ P(x)$**
- Unique quantifier → *For one values in…* → **$\exists!\ x\ P(x)$**

TABLE 2 De Morgan's Laws.

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Soal:

Pernyataan:

> "Jika n^2 habis dibagi 3, maka n habis dibagi 9."

Counterexample Proof:

- Ambil $n = 3$
- $n^2 = 9 \to$ habis dibagi 3 ✅
- Tapi $n = 3 \to$ tidak habis dibagi 9 ❌
- Jadi pernyataan **salah**, dan $n = 3$ adalah counterexample ✅

Negation of Quantifiers

Every student in your class has taken a course in calculus

$\forall x P(x)$

Negation → It is not the case that every student in your class has taken a course in calculus

Equivalent to → There is a student in your class who has not taken a course in calculus

$\exists x\ \neg P(x)$

Conclude as $\neg\forall x P(x) \equiv \exists x \neg P(x)$

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \vee q$$
$$p \to q \equiv \neg q \to \neg p$$
$$p \vee q \equiv \neg p \to q$$
$$p \wedge q \equiv \neg(p \to \neg q)$$
$$\neg(p \to q) \equiv p \wedge \neg q$$
$$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$$
$$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$$
$$(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$$
$$(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

De Morgan's Law

¬(p ∧ q) ≡ ¬p ∨ ¬q

¬(p ∨ q) ≡ ¬p ∧ ¬q

p	q	¬p	¬q	p∨q	¬(p∨q)	(¬p)∧(¬q)
0	0	1	1	0	1	1
0	1	1	0	1	0	0
1	0	0	1	1	0	0
1	1	0	0	1	0	0

Logical operators

- NOT (negation) — Symbol ¬
- AND (conjunction) — Symbol ∧
- Inclusive OR (disjunction) — Symbol ∨
- EXclusive OR (XOR) — Symbol ⊕
- Conditional statement (implication) — Symbol →
- Biconditional (iff) — Symbol ↔

Buktikan:

> "Ada tepat satu bilangan bulat x sehingga $2x + 3 = 11$."

Penyelesaian (Unique Proof):

1. **Existence:**
   - Cari x yang memenuhi: $2x + 3 = 11 \implies 2x = 8 \implies x = 4$
   - Jadi ada $x = 4$ yang memenuhi ✅
2. **Uniqueness:**
   - Misal ada y lain yang juga memenuhi: $2y + 3 = 11 \implies 2y = 8 \implies y = 4$
   - Jadi $y = x = 4$
   - Tidak mungkin ada bilangan lain yang memenuhi ✅

Kesimpulan: Ada tepat satu bilangan bulat $x = 4$ ✅

Proof by Contraposition

Making use of the fact that the conditional statement **p → q** is equivalent to its contrapositive, **¬q → ¬p**

e.g. in **p → q**,

We get the contrapositive **¬q → ¬p**

Take **¬q** as premise

we show that **¬p** must follow

Proof of Equivalence

To prove a theorem that is a biconditional statement **p ↔ q**

we show that **p → q** and **q → p** are both true.

Based on the tautology **(p ↔ q) ↔ (p → q) ∧ (q → p)**

Proofs

Vacuous Proof

p → **q** is true if **p** is false

p	q	p → q
0	0	1
0	1	1
~~1~~	~~0~~	~~0~~
~~1~~	~~1~~	~~1~~

if n > 1 then n^2 > n, for n = 0

domain: all integers

p : 0 > 1 (FALSE)

q : 0^2 > 0 (FALSE)

hence p → q TRUE

Trivial Proof

p → **q** is true if **q** is true

p	q	p → q
~~0~~	~~0~~	~~1~~
0	1	1
~~1~~	~~0~~	~~0~~
1	1	1

if a ≥ b then $a^n \geq b^n$ a, b are positive integers and n = 0

domain: positive integers

p : a ≥ b

q : $a^0 \geq b^0$ (TRUE)

hence, the theorem is true

TABLE 1 Rules of Inference.

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \land (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \land (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \land (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \lor q$ $\neg p$ $\therefore q$	$((p \lor q) \land \neg p) \to q$	Disjunctive syllogism
p $\therefore p \lor q$	$p \to (p \lor q)$	Addition
$p \land q$ $\therefore p$	$(p \land q) \to p$	Simplification
p q $\therefore p \land q$	$((p) \land (q)) \to (p \land q)$	Conjunction
$p \lor q$ $\neg p \lor r$ $\therefore q \lor r$	$((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$	Resolution

Rule of Inference	*Name*
$\forall x P(x)$ $\therefore P(c)$	Universal instantiation
$P(c)$ for an arbitrary c $\therefore \forall x P(x)$	Universal generalization
$\exists x P(x)$ $\therefore P(c)$ for some element c	Existential instantiation
$P(c)$ for some element c $\therefore \exists x P(x)$	Existential generalization

Soal:

Buktikan bahwa untuk setiap bilangan bulat n dengan $1 \le n \le 4$, $n^2 \le 16$.

Penyelesaian (exhaustive proof):

Periksa semua nilai $n = 1, 2, 3, 4$:

1. $n = 1$: $1^2 = 1 \le 16$ ✅
2. $n = 2$: $2^2 = 4 \le 16$ ✅
3. $n = 3$: $3^2 = 9 \le 16$ ✅
4. $n = 4$: $4^2 = 16 \le 16$ ✅

STEP 1	
Kalimat	**Simbol**
It rains	R
It is foggy	F
The sailing race will be held	S
The lifesaving demonstration will go on	L
The trophy will be awarded	T

STEP 2	
Kalimat	**Simbol**
"If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on"	$(\neg R \lor \neg F) \to (S \land L)$
"If the sailing race is held, then the trophy will be awarded"	$S \to T$
"The trophy was not awarded"	$\neg T$
"It rained"	R

STEP 3		
No	**Langkah**	**Alasan**
1	$\neg T$	Premis
2	$S \to T$	Premis
3	$\neg S$	Modus Tollens (1) dan (2)
4	$(\neg R \lor \neg F) \to (S \land L)$	Premis
5	$\neg(S \land L) \to \neg(\neg R \lor \neg F)$	Contrapositive
6	$\neg S \lor \neg L \to (R \land F)$	De Morgan's Law
7	$\neg S \lor \neg L$	Addition Rule dari $\neg S$ (3)
8	$(R \land F)$	Modus Ponens (6) dan (7)
9	R	Simplification

> "Jika n adalah bilangan bulat, maka $n^2 + n$ adalah genap."

Penyelesaian (Proof by Cases):

1. **Kasus 1:** n genap, misal $n = 2k$
   - $n^2 + n = (2k)^2 + 2k = 4k^2 + 2k = 2(2k^2 + k)$ ✅ genap
2. **Kasus 2:** n ganjil, misal $n = 2k + 1$
   - $n^2 + n = (2k + 1)^2 + (2k + 1) = 4k^2 + 4k + 1 + 2k + 1 = 4k^2 + 6k + 2 = 2(2k^2 + 3k + 1)$ ✅ genap

Karena kedua kasus menunjukkan hasil genap, pernyataan **benar untuk semua bilangan bulat n**.

STEP 1	
Kalimat	**Simbol**
x is a student in this class	C(x)
x has read the book	B(x)
x passed the first exam	P(x)

STEP 2	
Kalimat	**Simbol**
"A student in this class has not read the book"	$\exists x(C(x) \land \neg B(x))$
"Everyone in this class passed the first exam"	$\forall x(C(x) \to P(x))$
"Someone who passed the first exam has not read the book"	$\exists x(P(x) \land \neg B(x))$

STEP 3		
No	**Langkah**	**Alasan**
1	$\exists x(C(x) \land \neg B(x))$	Premis
2	$C(a) \land \neg B(a)$	Existensial instantiation dari (1)
3	$C(a)$	Simplification dari (2)
4	$\forall x(C(x) \to P(x))$	Premis
5	$C(a) \to P(a)$	Universal instantiation dari (4)
6	$P(a)$	Modus Ponens dengan (5) dengan (3)
7	$\neg B(a)$	Simplification of $C(a) \land \neg B(a)$ / (2)
8	$P(a) \land \neg B(a)$	Conjuction between (6) and (7)
9	$\exists x(P(x) \land \neg B(x))$	Existensial Generalization

Contoh Soal 3 (lebih abstrak)

Soal:

Buktikan:

> "Untuk setiap bilangan bulat positif n, ada bilangan bulat positif m sehingga $m > n$."

Penyelesaian (Construction Proof):

- Cara konstruksi: ambil $m = n + 1$
- Jelas $m > n$ dan m positif ✅

Kita **menunjukkan secara langsung cara membuat** m yang memenuhi, jadi bukti selesai.

Contoh:

> Jika $3n + 2$ genap, maka n genap.

Bukti kontradiksi:

- Asumsikan $3n + 2$ genap **(P)**
  dan n ganjil **(¬Q)**
- Dari n ganjil, kita dapat $3n + 2$ ganjil.
  Tapi itu **berlawanan** dengan $3n + 2$ genap.
- 🔥 Kontradiksi → asumsi salah → maka n genap.

Soal:

Buktikan:

> "Ada bilangan prima yang lebih besar dari 1000."

Penyelesaian (Non-Construction Proof):

- Kita tahu bilangan prima **tak terbatas** (Teorema Euclid).
- Maka pasti ada bilangan prima yang lebih besar dari 1000. ✅
- Kita **tidak menulis bilangan primanya**, cukup argumen eksistensi.

Soal:

Buktikan:

> "Jika x ganjil, maka x^2 ganjil."

Penyelesaian (Forward Reasoning):

1. Diketahui: x ganjil → $x = 2k + 1$
2. Hitung x^2:

$$x^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$$

3. Karena $x^2 = 2(\text{bilangan bulat}) + 1$, maka x^2 ganjil ✅

Penalaran maju ini **langsung dari fakta ke kesimpulan** tanpa mundur dari tujuan.

Buktikan:

> "Jika $x + 1$ ganjil, maka x genap."

Penyelesaian (Backward Reasoning):

1. Tujuan: Buktikan x genap → $x = 2k$
2. Diketahui: $x + 1$ ganjil → $x + 1 = 2m + 1$
3. Maka $x = (2m + 1) - 1 = 2m$ → genap ✅

Teks soal. Use a proof by contraposition to show: if $x + y \ge 2$ (real x, y) then $x \ge 1$ or $y \ge 1$.

q

p

$p \to q$

$\neg q \to \neg p$

$p: x + y \ge 2 \qquad q: x \ge 1 \lor y \ge 1$

$\neg p: x + y < 2 \qquad \neg q: x < 1 \land y < 1$

$\neg q \to \neg p$

$x < 1 \land y < 1 \to x + y < 2$ ✓benar