Lecture 5, 9/10/24

ECE 364

Fall, 2024

Vector $x \in \mathbb{R}^n$ ~ n-dim Euclidean space

$$x = (x_1, \ldots, x_n) \equiv [x_1 \; x_2 \; \cdots \; x_n]^T \equiv \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Inner Product

$$x^T y = \sum_{i=1}^{n} x_i y_i$$

Euclidean Norm

$$\|x\| = \sqrt{x^T x} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

$$\angle(x, y) = \cos^{-1}\left(\frac{x^T y}{\|x\| \, \|y\|}\right)$$

standard inner product on $\mathbb{R}^{m \times n}$

$$\langle X, Y \rangle = \operatorname{tr}(X^T Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij} Y_{ij}$$

<u>Gradient</u> (when $f: \mathbb{R}^n \longrightarrow \mathbb{R}$) :

$$\underbrace{\nabla f(x)}_{\parallel} = \underbrace{\left[\frac{\partial f(x)}{\partial x_1} \quad \frac{\partial f(x)}{\partial x_2} \quad \cdots \quad \frac{\partial f(x)}{\partial (x_n)} \right]^T}_{}$$

$D f(x)^T$

$Df(x)$ is $1 \times n$ called derivative

- For $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$Df(x) \in \mathbb{R}^{m \times n}, \quad \text{where}$$

$$Df(x) = \frac{\partial f_i(x)}{\partial x_j}, \quad i=1,\ldots,m; \; j=1,\ldots,n$$

Hessian:

$$\nabla^2 f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots \\ & & \ddots \end{bmatrix}$$

$$[\nabla^2 f(x)]_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Chain Rule for Multivariate Functions

$x = g(\beta)$, $y = h(\beta)$

$z = f(x, y)$

Then

$$\frac{\partial z}{\partial \beta} = \frac{dz}{dx}\,\frac{dx}{d\beta} + \frac{dz}{dy}\cdot\frac{dy}{d\beta}$$

$\underline{x} \in \mathbb{R}^n, \quad \underline{w} \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n \times n}$

- Multivariate function: $f(x) = w^T x$
- Derivative $\dfrac{\partial f}{\partial x} = ?$

e.g. $\begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$w_1 x_1 + w_2 x_2$

$$\underbrace{\frac{\partial f}{\partial x}} = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \partial f / \partial x_n \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = w$$

Multivariate function : $f(x) = x^T A x$

Derivative $\dfrac{\partial f}{\partial x} = ?$

left as HW : $Ax + A^T x = (A + A^T) x$

if A is symmetric : $= 2Ax$