Lecture 6, 9/12/24

ECE 364
Fall 2024

Matrix Operations Summary

- Matrix times a col. vec.

$$\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}\begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \\ \end{bmatrix}$$

- Row vec. times matrix

$$\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \end{bmatrix}$$

- Row vec. times col. vec.

$$\begin{bmatrix} \quad & \quad & \quad \end{bmatrix}\begin{bmatrix} \quad \\ \\ \\ \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Product of $x^T A y$:

$$\begin{bmatrix} \quad & \quad \end{bmatrix} \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \begin{bmatrix} \quad \\ \quad \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

Derivative Summary

- $$\frac{\partial A x}{\partial x} = A^T$$
- $$\frac{\partial x^T A}{\partial x} = A$$
- $$\frac{\partial x^T x}{\partial x} = 2x$$

- $$\frac{\partial x^T A x}{\partial x} = Ax + A^T x$$

$$\frac{\partial x^T A x}{\partial x} = 2Ax \qquad \text{(if } A \text{ is symmetric)}$$

- $$\frac{\partial z}{\partial x} = \frac{\partial y}{\partial x} \frac{\partial z}{\partial y}$$

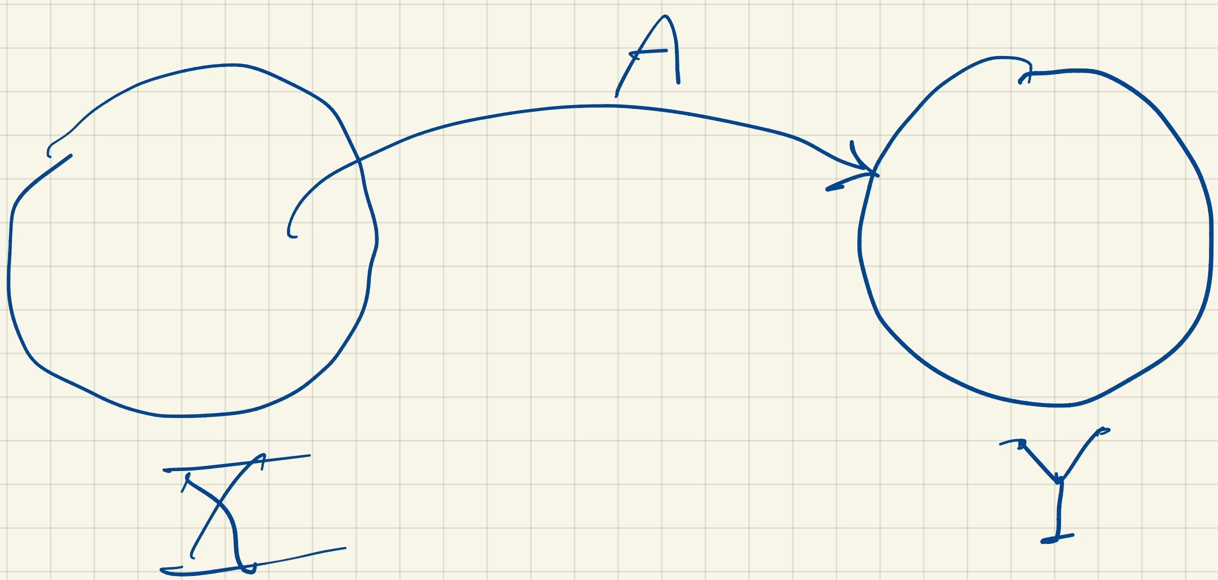
Overview of Linear Systems (operators)

Given $y \in Y$ and a linear operator

$$A : X \longrightarrow Y$$

Finding the inverse:

Find $x \in X$ s.t. $y = Ax$



Prob. specified by:

1. A: operator

2. Object space: X (space where soln. resides)

3. observation: Y " " obs. "

   in general $Y \supset AX$

- An operator $A: X \to Y$ is surjective (onto) if for every $y \in Y$ there is an $x \in X$ s.t. $Ax = y$

- An operator $A : X \to Y$ is injective (one-to-one) if $Ax_1 = Ax_2$ implies

$$x_1 = x_2$$

- An operator which is both surjective & injective is called bijective

Square matrix $A_{n \times n}$

- A is singular if $\det(A) = 0$, else non-sing
- If $\det(A) \neq 0$, A^{-1} exists and

$$= \frac{1}{\det(A)} \operatorname{adj}(A)$$

Eigenvalue and Eigenvectors

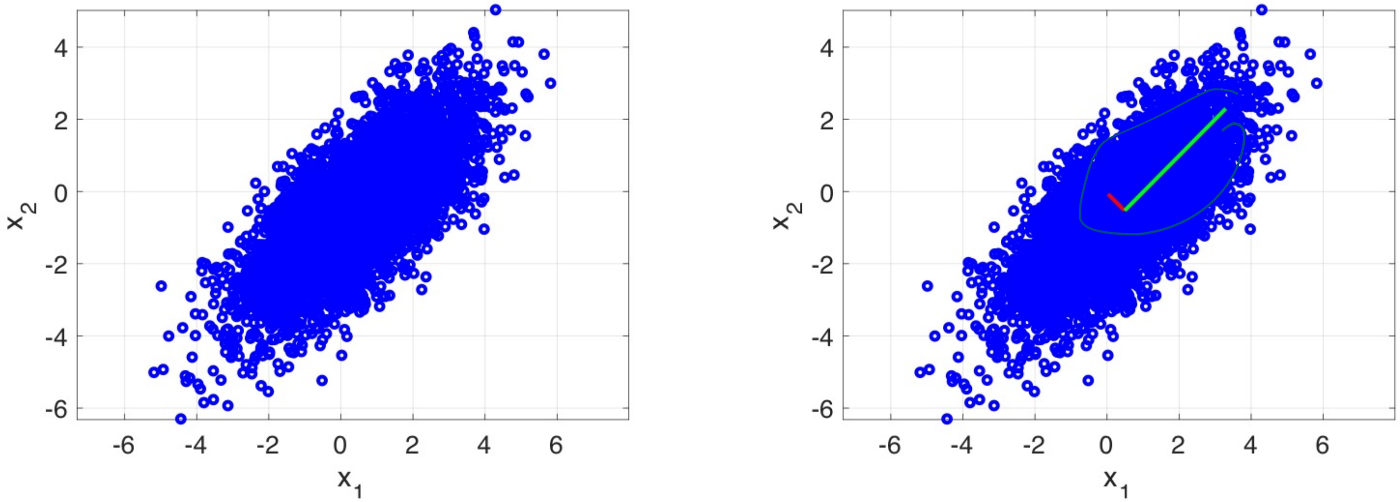
$A_{n \times n}$ has eigenvector $u \neq 0$ if

$$A \underline{u} = \lambda \underline{u}$$

$$\Longrightarrow (A - \lambda I) u = 0$$

$$\Longrightarrow \det(A - \lambda I) = 0 \longleftarrow \text{characteristic polyn.}$$

Example:



What if we want to find the direction with second, third largest variance that's orthogonal to the first, first and second?

Eigendecomposition of Symmetric Matrices

Let A be a symmetric $n \times n$ matrix, i.e., $A^T = A$

1) <u>Result</u> all eigenvalues of A are real

2) <u>Result</u> Eigenvectors corresponding to distinct e-values are orthogonal

<u>Conclusion</u> For real symm. A, eigenvalues are real, and we can find orthonormal eigenvectors, $u_1, \dots, u_n$.

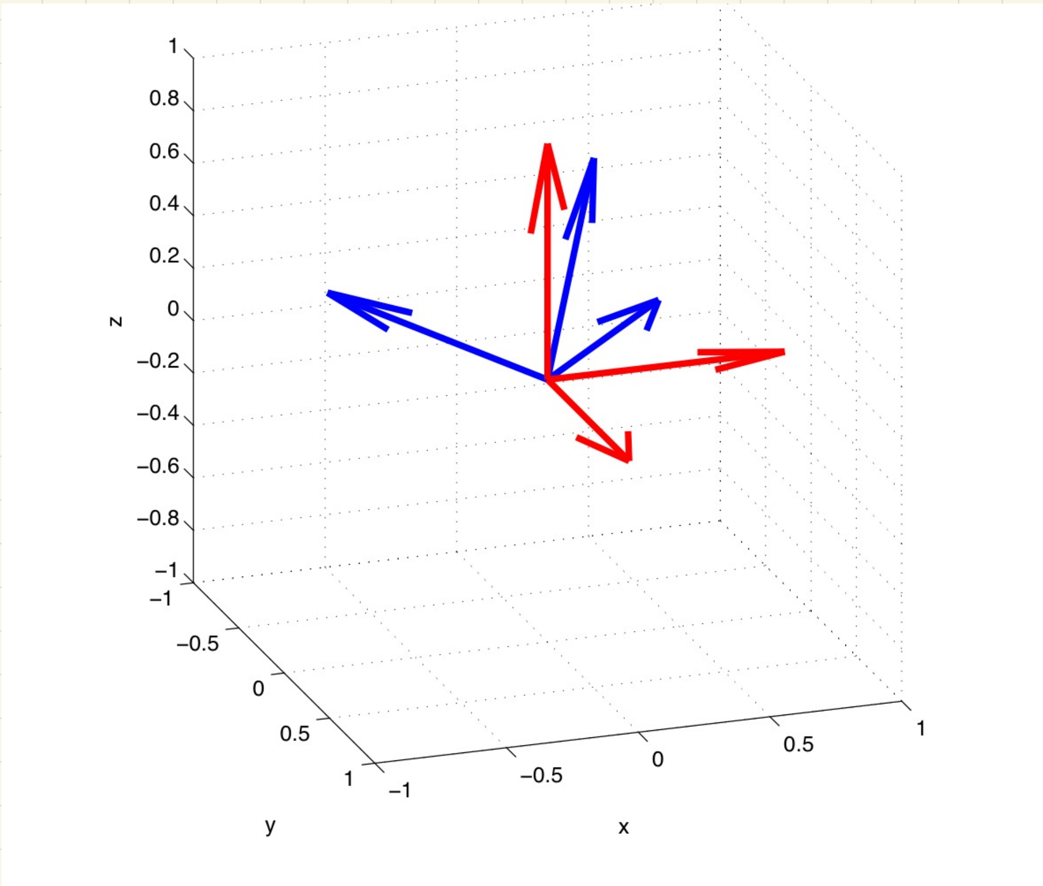
$$u_i^T u_j = \begin{cases} 1, & \text{if } i = j. \\ 0, & \text{if } i \neq j \end{cases}$$

Ex.

$$A = \begin{bmatrix} 1.4698 & -0.5223 & -0.1634 \\ -0.5223 & 1.4848 & -0.1297 \\ -0.1634 & -0.1297 & 0.0453 \end{bmatrix}.$$

$$U = [u_1 \mid u_2 \mid u_3] =$$

$$\begin{bmatrix} -0.7036 & -0.6926 & 0.1588 \\ 0.7105 & -0.6894 & 0.1412 \\ 0.0117 & 0.2122 & 0.9772 \end{bmatrix}$$



Eigen-structure of A.
Red vectors: standard basis in $\mathbb{R}^3$
Blue vectors: orthonormal vectors u_1, u_2, u_3 representing the Eigenbasis for A.

Diagonalization of Real Symm. Matrices

$$A = \sum_{i=1}^{n} \lambda_i \underline{u}_i \underline{u}_i^T = U \Lambda U^T$$

$A u_i = \lambda_i u_i \; , \; i = 1, \ldots, n$

where

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & 0 \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix} \qquad U = [u_1 \; u_2 \; \cdots \; u_n]$$

$\Lambda = \text{diag.} (\lambda_1, \ldots, \lambda_n)$

Note: $U^T U = I \implies U^T = U^{-1}$

since $u_i^T u_j = 0, \; \forall i \neq j \; ; \; u_i^T u = 1, \; \forall i$

$$A = U \Lambda U^T = \sum_{i=1}^{n} \lambda_i \underline{u}_i \underline{u}_i^T$$

then $$A^{-1} = (U^T)^{-1} \Lambda^{-1} U^{-1} = U \Lambda^{-1} U^T$$

$$= \sum_{i=1}^{n} \frac{1}{\lambda_i} \underline{u}_i \underline{u}_i^T$$