Name: ____________ Student ID: ____________ Major: ______

Question 1 (ca. 9 marks)

Consider the function f defined by

$$f(x,y) = \frac{x^3+y^3}{xy} = \frac{x^2}{y} + \frac{y^2}{x} \qquad \text{for } x>0,\ y>0.$$

a) Which obvious symmetry property does f have? What can you conclude from this about the contours of f?

b) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{(x,y)\to(1,0)} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibility $+\infty$), or show that the limit does not exist.

c) Show that there is no point (x,y) in the domain of f for which $\nabla f(x,y) = 0$. What does this imply for the contours of f?

d) Make an accurate figure of the 2-contour of f (unit length at least 5 cm!). The drawing should indicate the intersection point(s) with the line $y = x$, points with horizontal or vertical tangents, and the shape of the curve near $(0,0)$. *Hints:* $\sqrt[3]{2} \approx 1.26$, $\sqrt[3]{4} \approx 1.59$, and for determining the shape near $(0,0)$ it is advisable to use polar coordinates.

e) The function f is homogeneous of degree 1. What can you conclude from this about the remaining contours of f?

Question 2 (ca. 6 marks)

Consider the (differentiable) map $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x,y) = \bigl(2x - x^2 + y^2, 2y - 2xy\bigr).$$

a) For $(x,y) \in \mathbb{R}^2$ compute the Jacobi matrix $\mathbf{J}_G(x,y)$.

b) In which points $(x,y) \in \mathbb{R}^2$ is G conformal?

c) Determine the fixed points of G, i.e., the solutions of $G(x,y) = (x,y)$.

d) Determine the images under G of the three lines with equations $x = -1$, $x = 0$, $x = 1$.

e) Draw a figure containing the three curves determined in d) and the image of the unit circle under G. Indicate the angles of intersection between intersecting curves.

Question 3 (ca. 6 marks)

Consider the space curve C parametrized by

$$\mathbf{r}(t) = \bigl(6t, 3t^2, t^3\bigr), \quad t \in \mathbb{R}.$$

a) Which symmetry property does C have?

b) Suppose we walked from the origin along the curve in the positive ($t > 0$) direction exactly 20 units. At which point $\mathbf{p}$ are we now?

c) Find a parametrization of the osculating circle of C in the point $\mathbf{p}$ determined in b).

Solutions

1 a) Since $f(x,y)=f(y,x)$, the function f is invariant under a reflection at the line $y=x$ (and G_f is symmetric with respect to the plane spanned by this line and the z-axis).
$\Longrightarrow$ The contours of f are symmetric with respect to the line $y=x$. $\boxed{1}$

b) On the line $y=x$ we have $f(x,x)=2x\to 0$ for $x\to 0$. On the curve $y=x^2$ we have $f(x,x^2)=1+x^3\to 1$ for $x\to 0$. Hence $\lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist. $\boxed{1}$

For $(x,y)\to(1,0)$ we have $x^3+y^3\to 1$, $xy\to +0$, and hence $f(x,y)\to \frac{1}{+0}=+\infty$. Thus $\lim_{(x,y)\to(1,0)} f(x,y)=+\infty$. $\boxed{1}$

Since

$$f(x,y)>\frac{x^2}{y}\geq x \quad \text{if } x\geq y$$

and, similarly, $f(x,y)>y$ for $x\leq y$, the values of f outside any square $(0,R]\times(0,R]$, $R>0$, are $>R$. Hence $\delta=R$ can serve as a response to $R>0$ in a proof of $\lim_{|(x,y)|\to\infty} f(x,y)=+\infty$. Thus $\lim_{|(x,y)|\to\infty} f(x,y)=+\infty$. $\boxed{1}$

c) $\nabla f(x,y)=\left(\frac{2x}{y}-\frac{y^2}{x^2},\frac{2y}{x}-\frac{x^2}{y^2}\right)=(0,0) \iff 2x^3=y^3\wedge 2y^3=x^3$. Both equations imply $4x^3=2y^3=x^3$ and hence $x=0$. But in the domain of f we have $x>0$ and hence there is no solution.

Since $\nabla f(x,y)$ is nonzero everywhere, all contours of f are smooth. $\boxed{1}$

d) Since $f(x,x)=2x=2 \iff x=1$, the 2-contour intersects the line $y=x$ in $(1,1)$.

The condition for a horizontal tangent in (x,y) is $f_x=0\wedge f_y\neq 0$, which reduces to $2x^3=y^3$, i.e., $y=\sqrt[3]{2}\,x$. Inserting this in $f(x,y)=2$ gives

$$\frac{3x^3}{\sqrt[3]{2}\,x^2}=2 \Longrightarrow x=\tfrac{2}{3}\sqrt[3]{2}\approx 0.84,\ y=\tfrac{2}{3}\sqrt[3]{4}\approx 1.06$$

Then, by symmetry, there also is exactly one point on the 2-contour with a vertical tangent, viz. $\left(\frac{2}{3}\sqrt[3]{4},\frac{2}{3}\sqrt[3]{2}\right)\approx(1.06,0.84)$.

In polar coordinates the 2-contour is given by

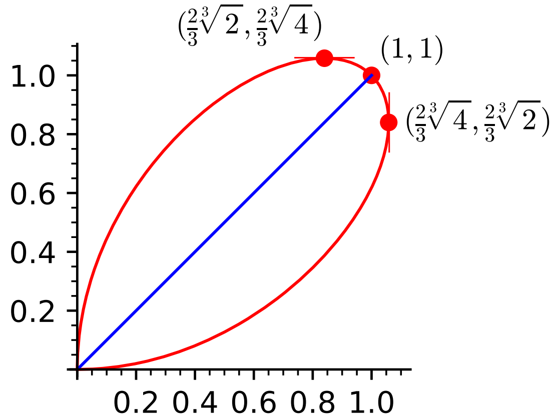
$$r^3\cos^3\phi+r^3\sin^3\phi=2(r\cos\phi)(r\sin\phi) \iff r=r(\phi)=\frac{\sin(2\phi)}{\cos^3\phi+\sin^3\phi}.$$

Since $x>0$ and $y>0$, the angle ϕ is restricted to $(0,\pi/2)$. Since

$$\lim_{\phi\downarrow 0}\frac{\sin(2\phi)}{\cos^3\phi+\sin^3\phi}=\lim_{\phi\uparrow\pi/2}\frac{\sin(2\phi)}{\cos^3\phi+\sin^3\phi}=0,$$

the 2-contour in the limit has both a horizontal tangent and a vertical tangent at $(0,0)$. $\boxed{1}$

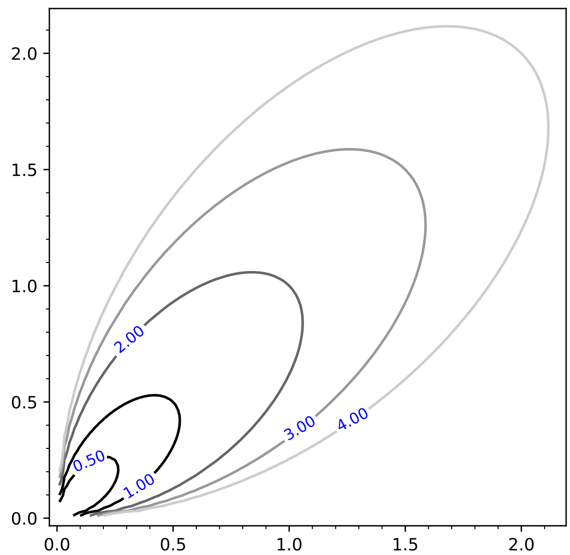
In particular $r(\phi)$ can be continuously extended to $[0,\pi/2]$ by setting $r(0)=r(\pi/2)=0$.



$\boxed{2}$

e) Since $f(kx, ky) = k\, f(x, y)$ for all (x, y) in the domain of f, the map $(x, y) \mapsto (kx, ky)$ maps the 1-contour to the k-contour (and the 2-contour to the $2k$-contour). Hence all contours are similar (i.e., arise from the 2-contour by scaling it). $\boxed{+1}$

This is visible in the following contour plot of f, which is not required in the exam.



Remarks:

$$\sum_1 = 8 + 1$$

2 a) Writing $G = (u, v)$, we have

$$\mathbf{J}_G(x, y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2 - 2x & 2y \\ -2y & 2 - 2x \end{pmatrix}.$$ $\boxed{1}$

b) Using a), we have

$$\mathbf{J}_G(x, y)^{\mathsf{T}} \mathbf{J}_G(x, y) = \begin{pmatrix} (2 - 2x)^2 + 4y^2 & 0 \\ 0 & (2 - 2x)^2 + 4y^2 \end{pmatrix}.$$

$\Longrightarrow$ G is conformal in all points satisfying $(2 - 2x)^2 + 4y^2 \neq 0$, i.e., in all points of $\mathbb{R}^2$ except $(1, 0)$. $\boxed{1}$

c) $G(x,y)=(x,y) \iff 2x-x^2+y^2=x \wedge 2y-2xy=y \iff x-x^2+y^2=y(1-2x)=0$
The solutions are $y=0$, $x=0,1$; $x=\frac{1}{2}$ leads to $\frac{1}{4}+y^2=0$, which is not solvable.
$\Longrightarrow$ The fixed points of G are $(0,0)$ and $(1,0)$. $\boxed{1}$

d) $\underline{x=-1:}$ Here $(u,v)=(-3+y^2,4y)$, which is the parabola $u=-3+\frac{1}{16}v^2$. $\boxed{\frac{1}{2}}$

$\underline{x=0:}$ Here $(u,v)=(y^2,2y)$, which is the parabola $u=\frac{1}{4}v^2$. $\boxed{\frac{1}{2}}$

$\underline{x=1:}$ Here $(u,v)=(1+y^2,0)$, which is the part of the u-axis to the left of $(1,0)$. $\boxed{\frac{1}{2}}$

e) Since G is conformal, the image curve C of the unit circle touches the parabola $u=-3+\frac{1}{16}v^2$ in $G(-1,0)=(-3,0)$ and is orthogonal to the parabola $u=\frac{1}{14}v^2$ in $G(0,\pm 1)=(1,\pm 2)$. In the fixed point $(1,0)$, in which G is not conformal, the shape of C is not clear, but we can argue informally as follows: G "folds" the line $x=1$ at $(1,0)$ to obtain the half-line $L^+=\{(u,0);u\geq 1\}$. Since the two half circles of the unit circle meeting at $(1,0)$ both touch the line $x=1$, the image curve C must touch L^+ both from above and below. (Parametrizing C in polar coordinates also reveals this.) $\boxed{+\frac{1}{2}}$

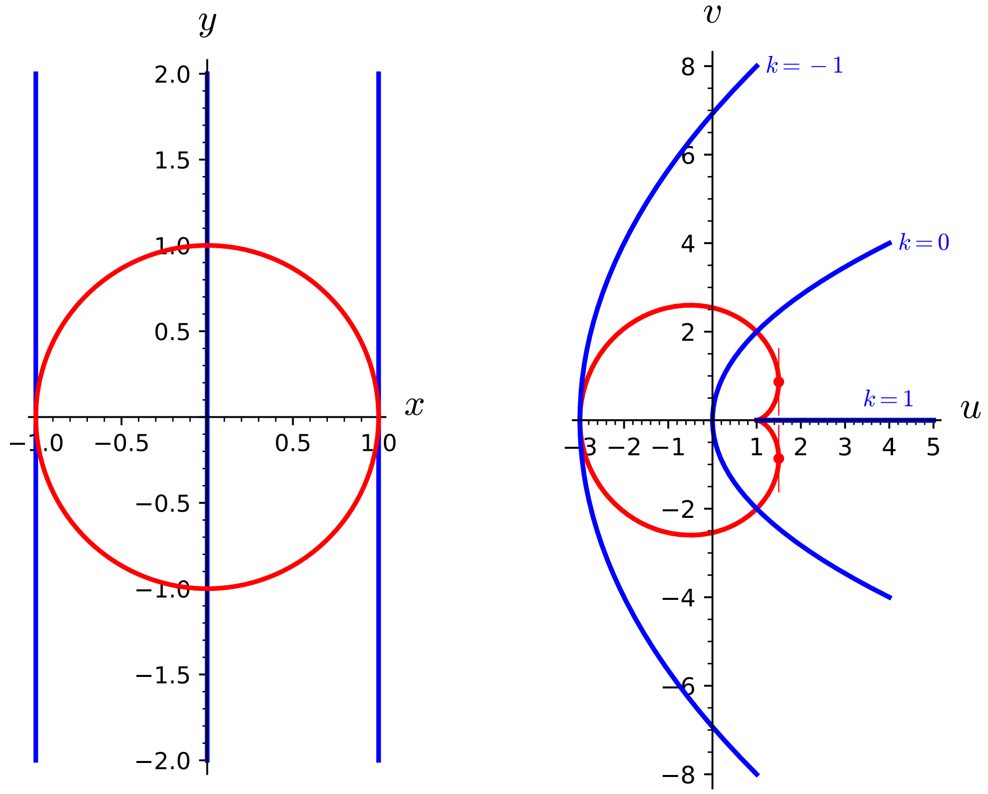
Since $x^2+y^2=1$ is equivalent to $y=\pm\sqrt{1-x^2}$, we have

$$C=\left\{\left(1+2x-2x^2,\pm 2(1-x)\sqrt{1-x^2}\right);-1\leq x\leq 1\right\}.$$

From this it is immediate that C is symmetric with respect to the u-axis. $\boxed{+\frac{1}{2}}$

This is enough information for a sketch of C; cf. the plot below.

Since the vertex of the parabola $y=1+2x-2x^2$ is at $x=1/2$, we can also easily determine the remaining two points on C with a vertical tangent, viz. $\left(\frac{3}{2},\pm\frac{1}{2}\sqrt{3}\right)$.)



$\boxed{1\frac{1}{2}}$

Remarks:

$$\sum_2 = 6 + 1$$

---

3 a) $\mathbf{r}(t) = (x, y, z) \in C$ implies $\mathbf{r}(-t) = (-x, y, -z) \in C$. Hence C is symmetric with respect to the y-axis. $\boxed{1}$

b)

$$\begin{aligned}
\mathbf{r}'(t) &= (6, 6t, 3t^2),\\
\mathbf{r}''(t) &= (0, 6, 6t), && \text{(for later use)}\\
|\mathbf{r}'(t)| &= \sqrt{36 + 36t^2 + 9t^4} = 3\sqrt{4 + 4t^2 + t^4} = 3(t^2 + 2),\\
s(t) &= \int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau = \int_0^t 3\tau^2 + 6\,\mathrm{d}\tau = \left[\tau^3 + 6\tau\right]_0^t = t^3 + 6t = 20 \iff t = 2.
\end{aligned}$$

The point is thus $\mathbf{p} = \mathbf{r}(2) = (12, 12, 8)$. $\boxed{2}$

c) Using a mix of row and column vector notation (to save space), we obtain

$$\mathbf{T}(2) = \frac{(6,12,12)}{|(6,12,12)|} = \frac{(1,2,2)}{|(1,2,2)|} = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix},$$

$$\mathbf{N}(2) = \lambda\left(\mathbf{r}''(2) - \frac{\mathbf{r}''(2)\cdot(1,2,2)}{|(1,2,2)|^2}(1,2,2)\right) \qquad (\lambda > 0)$$

$$= \lambda\left(\begin{pmatrix}0\\6\\12\end{pmatrix} - \frac{36}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}\right) = \lambda\begin{pmatrix}-4\\-2\\4\end{pmatrix} = \lambda\begin{pmatrix}-2\\-1\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\-1\\2\end{pmatrix}, \qquad \boxed{1}$$

$$\kappa(2) = \frac{|\mathbf{r}'(2)\times\mathbf{r}''(2)|}{|\mathbf{r}'(2)|^3} = \frac{|(6,12,12)\times(0,6,12)|}{|(6,12,12)|^3} = \frac{6^2\,|(1,2,2)\times(0,1,2)|}{6^3\,|(1,2,2)|^3}$$

$$= \frac{|(2,-2,1)|}{6\cdot 27} = \frac{1}{54}. \qquad \boxed{1}$$

$\Longrightarrow$ The center of the osculating circle is $\mathbf{q} = \mathbf{p} + 54\,\mathbf{N}(2) = (-24,-6,44)$, and a parametrization of the osculating circle is

$$c(t) = \mathbf{q} - 54\cos t\,\mathbf{N}(2) + 54\sin t\,\mathbf{T}(2)$$

$$= \begin{pmatrix}-24\\-6\\44\end{pmatrix} - 18\cos t\begin{pmatrix}-2\\-1\\2\end{pmatrix} + 18\sin t\begin{pmatrix}1\\2\\2\end{pmatrix}$$

$$= \begin{pmatrix}-24+36\cos t+18\sin t\\-6+18\cos t+36\sin t\\44-36\cos t+36\sin t\end{pmatrix}. \qquad \boxed{1}$$

Remarks:

$$\sum_3 = 6$$

General Remarks:

$$\sum_{\text{Midterm 2}} = 20 + 2$$