Question 1 (ca. 12 marks)

Decide whether the following statements are true or false, and justify your answers.

a) The function $f(x,y) = \dfrac{\sin(x)\cos(y)}{1+x^2+y^2}$, $(x,y) \in \mathbb{R}^2$ attains a global maximum and a global minimum.

b) The function $g(x,y) = xy(x^2+2y^2-3)$, $(x,y) \in \mathbb{R}^2$ has at least 9 critical points.

c) Suppose you start at the point $(1,0)$ and move 0.5 units in the (x,y)-plane following (and continuously adjusting) the direction of steepest ascent of $h(x,y) = \dfrac{xy}{x^2+y^2}$. Afterwards you are closer to the y-axis than before.

d) If C^2-functions $f(u,v)$ and $g(x,y)$ are related by $g(x,y) = f(ax+by, cx+dy)$ ($a,b,c,d \in \mathbb{R}$, $ad-bc \neq 0$) then $g_{xx}g_{yy} - g_{xy}^2 = (ad-bc)(f_{uu}f_{vv} - f_{uv}^2)$.

e) The line integral of $(\sin y + y\sin x)\,\mathrm{d}x + (x\cos y - \cos x)\,\mathrm{d}y$ along the quarter circle $x^2+y^2=1$, $x \geq 0$, $y \geq 0$ (in the mathematically positive direction) is zero.

f) If P, Q are C^2-functions on $\mathbb{R}^2$ such that both $P\,\mathrm{d}x + Q\,\mathrm{d}y$ and $Q\,\mathrm{d}x - P\,\mathrm{d}y$ are exact, then P and Q solve Laplace's equation $\Delta u = u_{xx} + u_{yy} = 0$.

Question 2 (ca. 9 marks)

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = x^4 - 3x^2y + x^2 + 2y^2 - y.$$

a) Which symmetry property does f have? What can you conclude from this about the graph of f and the contours of f?

b) Determine the gradient $\nabla f(x,y)$ and the Hesse matrix $\mathbf{H}_f(x,y)$.

c) Determine all critical points of f and their types.
Hint: There are exactly 3 critical points.

d) Does f have a global extremum?

Question 3 (ca. 9 marks)

Consider the surface S in $\mathbb{R}^3$ with equation

$$xz - y^2 + 2y + 2 = 0.$$

a) Show that S is smooth.

b) Using the method of Lagrange multipliers, determine the point(s) on S that minimize(s) the distance from the origin $(0,0,0)$, and the corresponding distance d. (It need not be proved that at least one such point exists.)

c) Determine an equation for the tangent plane to S in the point $(2,-2,3)$.

d) The surface S is a (central) quadric. Determine its center and type.

Question 4 (ca. 6 marks)

Evaluate the integral

$$\int_0^1 \frac{t^{1010}+t-2}{\ln t}\,\mathrm{d}t\,.$$

Hint: Consider $F(x)=\displaystyle\int_0^1 \frac{t^x+t-2}{\ln t}\,\mathrm{d}t$, $x\in(-1,\infty)$. Show first that F is well-defined and can be differentiated under the integral sign.

Question 5 (ca. 6 marks)

a) Let K be the solid in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$x\geq 0,\quad y\geq 0,\quad x+y\leq 2,\quad 0\leq z\leq 1+x^2+2y.$$

Find the volume of K.

b) Let P be the surface in $\mathbb{R}^3$ consisting of all points (x,y,z) satisfying

$$z=x^2+y^2\leq 1.$$

Find the area of P.

Solutions

1 a) True. The function f is continuous, satisfies $\lim_{|(x,y)|\to\infty} f(x,y) = 0$ (since $|f(r\cos\phi, r\sin\phi)| \leq \frac{1}{1+r^2}$ in polar coordinates), and takes both positive and negative values. This implies the existence of a global maximum and a global minimum, as the following argument, e.g. for the minimum, shows.

We have $f(\pi/2,\pi) = \frac{-1}{1+5\pi^2/4} < 0$. Since $\lim_{|(x,y)|\to\infty} f(x,y) = 0$, there exists $R > 0$ such that $f(x,y) > \frac{-1}{1+\pi^2/4}$ for all points (x,y) with $|(x,y)| > R$. On the disk $\overline{\mathrm{B}_R(0,0)} = \{(x,y); x^2+y^2 \leq R^2\}$, which is compact (i.e., closed and bounded) and contains $(\pi/2,0)$, the continuous function f attains a minimum value, which must be $\leq \frac{-1}{1+\pi^2/4}$, and hence also $\leq f(x,y)$ for every point $(x,y) \in \mathbb{R}^2 \setminus \overline{\mathrm{B}_R(0,0)}$. $\boxed{2}$

b) True. The 0-contour of g is the union of the lines $x = 0$, $y = 0$, and the ellipse $x^2 + 2y^2 = 3$. The 5 points where two of these curves intersect, viz. $(0,0)$, $(\pm a, 0)$, $(0, \pm b)$ with $a = \sqrt{3}$, $b = \sqrt{3/2}$ (the semi-axes of the ellipse) must be critical, since the 0-contour isn't smooth there. Further, the coordinate axes divide the solid ellipse into 4 compact regions K_1, K_2, K_3, K_4, on which g (which is continuous) must attain both a maximum and a minimum. One of these is zero and attained on the boundary of K_i, but the other is nonzero and attained in the inerior of K_i. Since such points are critical, there are at least 4 further critical points. $\boxed{2}$

Remark: In fact there are exactly 9 critical points.

c) True. The direction of steepest ascent in (x,y) is that of the gradient $\nabla h(x,y)$. We have

$$\begin{aligned} h_x &= \frac{y(x^2+y^2) - 2x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}, \\ h_y &= \frac{x(x^2-y^2)}{(x^2+y^2)^2} \qquad \text{(by symmetry)} \end{aligned}$$

Thus $\nabla h(1,0) = (0,1)$, implying that you move from $(1,0)$ north and enter the 1st quadrant. In the sector of the 1st quadrant specified by $0 < \theta < 45°$ in polar coordinates we have $h_x < 0$, $h_y > 0$, i.e., gradients point in the direction NW there. Thus you will approach the x-axis as long as you don't cross the line $y = x$. But this impossible, since the distance between $(1,0)$ and the line is $\sqrt{2}/2 > 0.5$. $\boxed{2}$

d) False. The correct relation is $g_{xx}g_{yy} - g_{xy}^2 = (ad-bc)^2(f_{uu}f_{vv} - f_{uv}^2)$. This can be proved as follows:

$$\begin{aligned} g_x &= f_u a + f_v c, \\ g_y &= f_u b + f_v c, \\ g_{xx} &= f_{ux}a + f_{vx}c = (f_{uu}a + f_{uv}c)a + (f_{vu}a + f_{vv}c)c = a^2 f_{uu} + 2ac\, f_{uv} + c^2 f_{vv}, \\ g_{xy} &= f_{uy}a + f_{vy}c = (f_{uu}b + f_{uv}d)a + (f_{vu}b + f_{vv}d)c = ab\, f_{uu} + (ad+bc)f_{uv} + cd\, f_{vv}, \\ g_{yx} &= g_{xy}, \\ g_{yy} &= f_{uy}b + f_{vy}d = (f_{uu}b + f_{uv}d)b + (f_{vu}b + f_{vv}d)d = b^2 f_{uu} + 2bd\, f_{uv} + d^2 f_{vv}, \end{aligned}$$

which just says

$$\begin{pmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{pmatrix} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} f_{uu} & f_{uv} \\ f_{vu} & f_{vv} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Taking the determinant on both sides proves the claim. $\boxed{2}$

That the relation with the un-squared determinant $ad - bc$ can't be true, can also be seen as follows: Suppose f has a strict local minimum in $(0,0)$ and $a, b, c, d \in \mathbb{R}$ satisfy $ad - bc = -1$. Then we would have $g_{xx}g_{yy} - g_{xy}^2 = -(f_{uu}f_{vv} - f_{uv}^2) < 0$ at $(x,y) = (0,0)$, which corresponds to $(u,v) = (0,0)$, and hence g would have a saddle point in $(0,0)$. But (bijective) linear coordinate changes clearly preserve the type of critical points; contradiction. (If you want a concrete counterexample, take $f(u,v) = u^2 + v^2$, $g(x,y) = f(x,-y) = x^2 + y^2$, which satisfy $f_{uu}f_{vv} - f_{uv}^2 = g_{xx}g_{yy} - g_{xy}^2 = 4$ everywhere but $a = 1$, $d = -1$, $b = c = 0$, and hence $ad - bc = -1$.)

e) False. Since $\frac{\partial}{\partial y}(\sin y + y \sin x) = \cos y + \sin x = \frac{\partial}{\partial x}(x \cos y - \cos x)$, the given form ω is exact in $\mathbb{R}^2$ (which is simply connected). An anti-derivative f of ω is obtained by the usual method (or can just be guessed): $f_x = \sin y + y \sin x \implies f(x,y) = x \sin y - y \cos x + h(y) \implies f_y = x \cos y - \cos x + h'(y) = x \cos y - \cos x \iff h'(y) = 0$, i.e., we can take $f(x,y) = x \sin y - y \cos x$. Then the Fundamental Theorem for Line Integrals gives

$$\int_C \omega = f(0,1) - f(1,0) = 0 \cdot \sin 1 - 1 \cdot \cos 0 - (1 \cdot \sin 0 - 0 \cdot \cos 1) = -1 \neq 0. \qquad \boxed{2}$$

f) True. Exactness implies $P_y = Q_x$ and $Q_y = -P_x$. Hence $P_{yy} = Q_{xy} = Q_{yx} = (-P_x)_x = -P_{xx}$, using Clairaut's Theorem and the linearity of partial derivatives. Similarly, $Q_{xx} = P_{yx} = P_{xy} = (-Q_y)_y = -Q_{yy}$. $\boxed{2}$

$$\sum_1 = 12$$

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2 a) $f(-x,y) = f(x,y)$ for $(x,y) \in \mathbb{R}^2$ $\boxed{1}$

This implies $(x,y,z) \in \mathrm{G}_f \iff (-x,y,z) \in \mathrm{G}_f$, i.e., the graph of f is symmetric with respect to the (y,z)-plane in $\mathbb{R}^3$, and the contours of f are symmetric with respect to the y-axis in $\mathbb{R}^2$. $\boxed{1}$

b) We compute

$$\begin{aligned} f(x,y) &= x^4 - 3x^2y + x^2 + 2y^2 - y, \\ \nabla f(x,y) &= (4x^3 - 6xy + 2x, -3x^2 + 4y - 1) \qquad \boxed{1} \\ &= \bigl(x(4x^2 - 6y + 2), -3x^2 + 4y - 1\bigr), \\ \mathbf{H}_f(x,y) &= \begin{pmatrix} 12x^2 - 6y + 2 & -6x \\ -6x & 4 \end{pmatrix}. \qquad \boxed{1} \end{aligned}$$

c) The system $f_x = f_y = 0$ is equivalent to

$$\bigl(x = 0 \wedge -3x^2 + 4y - 1 = 0\bigr) \vee \bigl(4x^2 - 6y + 2 = 0 \wedge -3x^2 + 4y - 1 = 0\bigr).$$

The first alternative has the solution $(x,y)=\left(0,\frac{1}{4}\right)$.
The second alternative is a linear system of equations for x^2, y, which can be solved, e.g., by Gaussian eliminiation:

$$\left[\begin{array}{rr|r} 4 & -6 & -2 \\ -3 & 4 & 1 \end{array}\right] \to \left[\begin{array}{rr|r} 4 & -6 & -2 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

The solution is $y=1$, $x^2=(-2+6x_2)/4=1$, so that $(x,y)=(\pm 1,1)$.
In all there are three critical points, viz.,

$$\mathbf{p}_1=\left(0,\tfrac{1}{4}\right),\quad \mathbf{p}_2=(1,1),\quad \mathbf{p}_3=(-1,1).$$ $\boxed{1\frac{1}{2}}$

Further we have

$$\mathbf{H}_f(\mathbf{p}_1)=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 4 \end{pmatrix},\quad \mathbf{H}_f(\mathbf{p}_2)=\begin{pmatrix} 8 & -6 \\ -6 & 4 \end{pmatrix},\quad \mathbf{H}_f(\mathbf{p}_3)=\begin{pmatrix} 8 & 6 \\ 6 & 4 \end{pmatrix}.$$

Since $\mathbf{H}_f(\mathbf{p}_1)$ is positive definite, $\mathbf{p}_1$ is a local minimum of f. $\boxed{\frac{1}{2}}$

Since $\det \mathbf{H}_f(\mathbf{p}_2)=\det \mathbf{H}_f(\mathbf{p}_3)=8\cdot 4-(\pm 6)^2=-4<0$, the points $\mathbf{p}_2$, $\mathbf{p}_3$ are saddle points of f. $\boxed{1}$

d) No. A global extremum must be a critical point. Since saddle points are not even local extrema, the only remaining possibility is that $\mathbf{p}_1$ is a global minimum. But

$$\begin{aligned} f(\mathbf{p}_1)&=2\left(\tfrac{1}{4}\right)^2-\tfrac{1}{4}=-\tfrac{1}{8}, \\ f(2,3)&=2^4-3\cdot 2^2\cdot 3+2^2+2\cdot 3^2-3=-1<-\tfrac{1}{8}, \end{aligned}$$

and hence $\mathbf{p}_1$ is not a global minimum. $\boxed{2}$

$$\sum_2 = 9$$

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3 a) S is a level set of $g(x,y,z)=xz-y^2+2y$, which has $\nabla g(x,y,z)=(z,-2y+2,x)$. Evidently, the only point at which ∇g vanishes is $(0,1,0)$, but $(0,1,0)$ is not on S. Hence S is smooth. $\boxed{1}$

b) Let $f(x,y,z)=x^2+y^2+z^2$. The task is equivalent to solving the optimization problem "minimize $x^2+y^2+z^2$ subject to $g(x,y,z)=-2$".

Since S has no singular points, the theorem on Langrange multipliers is applicable everywhere and yields that every minimum must satisfy $\nabla f(x,y,z)=\lambda\nabla g(x,y,z)$ for some $\lambda\in\mathbb{R}$. Since $\nabla f(x,y,z)=(2x,2y,2z)$ is a multiple of (x,y,z) (the factor 2 can be discarded), this gives the system of equations

$$\begin{aligned} x&=\lambda z, \\ y&=\lambda(-2y+2), \\ z&=\lambda x, \\ xz-y^2+2y&=-2. \end{aligned}$$ $\boxed{2}$

The 1st and 3rd equation give $x = \lambda^2 x$, i.e., $x = 0 \vee \lambda = \pm 1$.

$\underline{x = 0}$: From the 3rd equation $z = 0$, and then from the 4th equation $y^2 - 2y - 2 = 0$, i.e., $y = 1 \pm \sqrt{3}$. This gives the two points $\bigl(0, 1 \pm \sqrt{3}, 0\bigr)$. $\boxed{1}$

$\underline{\lambda = 1}$: Here $x = z$, the 2nd equation gives $y = 2/3$ and the 4th equation $x^2 = xz = 4/9 - 4/3 - 2 < 0$. Thus there is no solution in this case.

$\underline{\lambda = -1}$: Here $x = -z$, the 2nd equation gives $y = 2$, and the 4th equation $x^2 = -xz = -2^2 + 2 \cdot 2 + 2 = 2$. This gives the two points $\bigl(\pm\sqrt{2}, 2, \mp\sqrt{2}\bigr)$. $\boxed{1}$

The distance of these points from the origin is $2\sqrt{2}$. Since obviously $\sqrt{3} - 1 < 2\sqrt{2}$, the unique point on S minimizing the distance from the origin is $\bigl(0, 1 - \sqrt{3}, 0\bigr)$, and $d = \sqrt{3} - 1$. $\boxed{1}$

c) The tangent plane to S in $(2, -2, 3)$ has equation $\nabla g(2, -2, 3) \cdot (x - 2, y + 2, z - 3) = 0$. Since $\nabla g(2, -2, 3) = (3, 6, 2)$, this gives $3(x - 2) + 6(y + 2) + 2(z - 3) = 0$, i.e., $3x + 6y + 2z = 0$. (This is also clear from the requirements that $\nabla g(2, -2, 3)$ must be a normal vector of the plane and the point $(2, -2, 3)$ must be on the plane.) $\boxed{1}$

d) Rewriting the equation for S as $xz - (y - 1)^2 + 3 = 0$ shows that the center is $(0, 1, 0)$. $\boxed{\tfrac{1}{2}}$

S is thus affinely equivalent to the quadric $xz - y^2 + 3 = 0$. The further coordinate change $x = x' + z'$, $y = y'$, $z = x' - z'$ turns the latter into $x'^2 - z'^2 - y'^2 + 3 = 0$ or, after dropping primes and normalizing such that the right-hand side is positive, $-x^2 + y^2 + z^2 = 3$, which reveals that S is a hyperboloid of one sheet. $\boxed{1\tfrac{1}{2}}$

$$\sum_3 = 9$$

4 The integrand

$$f(x, t) = \frac{t^x + t - 2}{\ln t}$$

is not defined for $t \in \{0, 1\}$. The singularity at $t = 1$ can be removed, since the numerator is zero for $t = 1$ and, e.g., l'Hospital's Rule can be applied: $\lim_{t \to 1} f(x, t) = \lim_{t \to 1} \frac{x t^{x-1} + 1}{1/t} = x + 1$. The singularity at $t = 0$ cannot be removed. However, since $\int_0^1 t^x \, \mathrm{d}x = \frac{1}{x+1}$ exists for $x > -1$, it is clear that $\int_0^1 f(x, t) \, \mathrm{d}t$ exists for $x > -1$ as well. $\boxed{1}$

Differentiating under the integral sign gives

$$F'(x) = \int_0^1 f_x(x, t) \, \mathrm{d}t = \int_0^1 \frac{(\ln t) t^x}{\ln t} \, \mathrm{d}t = \left[\frac{t^{x+1}}{x + 1} \right]_0^1 = \frac{1}{x + 1}. \qquad \boxed{1}$$

It can be justified as follows: For small $\delta > 0$ and $0 < t < 1$ we have

$$t^x = \mathrm{e}^{x \ln t} \leq \begin{cases} 1 & \text{if } x \geq 0, \\ t^{-1+\delta} & \text{if } -1 + \delta < x < 0. \end{cases}$$

Thus $t^x = |t^x| \leq t^{-1+\delta} =: \Phi(t)$ for all $x \in (-1 + \delta, \infty)$ and $t \in (0, 1)$, which provides an integrable bound independent of x on account of $\int_0^1 t^{-1+\delta} = \bigl[t^\delta / \delta\bigr]_0^1 = 1/\delta$. $\boxed{2}$

It follows that $F(x) = \ln(x+1) + C$ for some $C \in \mathbb{R}$. The constant C can be determined from

$$F(0) = \int_0^1 \frac{t-1}{\ln t}\,\mathrm{d}t,$$
$$F(1) = \int_0^1 \frac{2t-2}{\ln t}\,\mathrm{d}t = 2\,F(0),$$

which gives $\ln(2) + C = 2C$, i.e., $C = \ln 2$, and $F(x) = \ln(x+1) + \ln 2 = \ln(2x+2)$.

$$\implies \int_0^1 \frac{t^{1010} + t - 2}{\ln t}\,\mathrm{d}t = F(1010) = \ln(2022). \qquad \boxed{2}$$

$$\sum_4 = 6$$

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5 a) The volume of K is

$$V = \int_K 1\,\mathrm{d}^3(x,y,z) = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{1+x^2+2y} \mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \qquad \boxed{1}$$
$$= \int_0^2\int_0^{2-x} 1 + x^2 + 2y\,\mathrm{d}y\,\mathrm{d}x$$
$$= \int_0^2 \left[(1+x^2)y + y^2\right]_{y=0}^{2-x}\,\mathrm{d}x$$
$$= \int_0^2 (1+x^2)(2-x) + (2-x)^2\,\mathrm{d}x$$
$$= \int_0^2 6 - 5x + 3x^2 - x^3\,\mathrm{d}x$$
$$= \left[6x - \tfrac{5}{2}\,x^2 + x^3 - \tfrac{1}{4}\,x^4\right]_0^2 = 12 - 10 + 8 - 4 = 6. \qquad \boxed{2}$$

b) Denoting the unit disk in $\mathbb{R}^2$ by D, the surface P is the graph of $f(x,y) = x^2 + y^2$, $(x,y) \in D$. Using the formula for such surfaces, or going the long way using the parametrization $\gamma(x,y) = \big(x, y, f(x,y)\big)$, we obtain the surface area as

$$A = \int_D \sqrt{1 + |\nabla f(x,y)|^2}\,\mathrm{d}^2(x,y) = \int_D \sqrt{1 + |(2x,2y)|^2}\,\mathrm{d}^2(x,y)$$
$$= \int_D \sqrt{1 + 4x^2 + 4y^2}\,\mathrm{d}^2(x,y) \qquad \boxed{1}$$
$$= \int_{\substack{0\leq r\leq 1\\ 0\leq\phi\leq 2\pi}} r\sqrt{1+4r^2}\,\mathrm{d}^2(r,\phi) = 2\pi\int_0^1 r\sqrt{1+4r^2}\,\mathrm{d}r \qquad \boxed{1}$$
$$= \frac{\pi}{6}\left[(1+4r^2)^{3/2}\right]_0^1 = \frac{\pi}{6}\left(5\sqrt{5} - 1\right). \qquad \boxed{1}$$

$$\sum_5 = 6$$

$$\sum_{\text{Final Exam}} = 42$$
