Midterm Exam 2

Name: ____________________ Student No.: ____________________

Question 1 (ca. 7 marks)

Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{4x}{1+x^2+2y^2}.$$

a) Show that $\lim_{|(x,y)|\to\infty} f(x,y) = 0$. What can you conclude from this about the k-contours of f with levels $k \neq 0$?

b) Derive equations for the k-contours of f with $k \neq 0$, and use this to find the range of f, i.e., the set $\{f(x,y); (x,y) \in \mathbb{R}^2\}$.

c) Which type of plane curve is formed by the points in which the corresponding contour of f has a horizontal tangent?

d) Sketch the contours of f for the levels $k \in \{0, 1, \sqrt{2}, 2\}$ together with the curve you found in c).

Question 2 (ca. 6 marks)

Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$ and consider the (differentiable) map $G\colon D \to \mathbb{R}^2$ defined by

$$G(x,y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right).$$

a) For $(x,y) \in D$ compute the Jacobi matrix $\mathbf{J}_G(x,y)$, and show that G is conformal.

b) Determine the fixed points of G, i.e., the solutions of $G(x,y) = (x,y)$.

c) Show that the image under G of a line not containing the origin $(0,0)$ is a circle through the origin (with the origin removed).

   Hint: Such a line has an equation of the form $ax+by = 1$. For $(u,v) = G(x,y)$ compute $au + bv$ and u^2+v^2.

d) Draw a figure (unit length at least 2 cm !) containing the unit circle, the three lines $x = \frac{1}{2}$, $y = 1$, $x - y = 1$ and their images under G, and the angles at intersection points $\neq (0,0)$ between the circles.

Question 3 (ca. 3 marks)

The height and mass ("weight") of a basketball player are known to be $h_0 = 2.00\,\mathrm{m}$ and $m_0 = 100\,\mathrm{kg}$, respectively, with a possible error of $0.5\,\mathrm{cm}$, respectively, $0.5\,\mathrm{kg}$.

a) Find the linear approximation of the *body mass index* $B(m,h) = m/h^2$ near (m_0, h_0).

b) Using the Mean Value Theorem, state a (tight) rigorous upper bound for the (absolute) error when the body mass index of the basketball player is calculated from the known data.

   Note: In b) it is not necessary to determine an explicit figure.

Question 4 (ca. 6 marks)

Consider the space curve C parametrized by

$$\mathbf{r}(t) = (t + 1/t,\, 2\ln t,\, t - 1/t)\,, \quad t \in (0, \infty).$$

a) Show that C is symmetric with respect to the x-axis.

b) Determine the (unique) point P on C closest to the origin $\mathbf{0} = (0, 0, 0)$.
Hint: This question has a short answer.

c) Find the z-coordinate of the (unique) point Q on C such that the arc from P to Q has length 1.

d) Determine the accompanying TNB frame at the point P.
Hint: After obtaining the unit tangent vector, one can use geometric arguments.

Solutions

1 a) Using polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, we have

$$\begin{aligned} |f(x,y)| &\leq \frac{4\,|x|}{1+x^2+y^2} && \boxed{\tfrac{1}{2}} \\ &= \frac{4r\,|\cos\phi|}{1+r^2} \leq \frac{4r}{1+r^2} \to 0 \quad \text{for } r \to \infty. && \boxed{1} \end{aligned}$$

This proves $\lim_{|(x,y)|\to\infty} f(x,y) = 0$ and implies that every k-contour, $k \neq 0$, is bounded. $\boxed{\tfrac{1}{2}}$

b) For $k \neq 0$ we have

$$\frac{4x}{1+x^2+2y^2} = k \iff x^2+2y^2-\frac{4x}{k} = -1 \iff \left(x-\frac{2}{k}\right)^2 + 2y^2 = \frac{4}{k^2} - 1. \quad \boxed{\tfrac{1}{2}}$$

$\Longrightarrow$ The k-contour of f is empty for $|k| > 2$, a point for $k = \pm 2$, and a certain ellipse for $-2 < k < 2$.
$\Longrightarrow$ The range of f is $[-2,2]$. $\boxed{1}$

c)

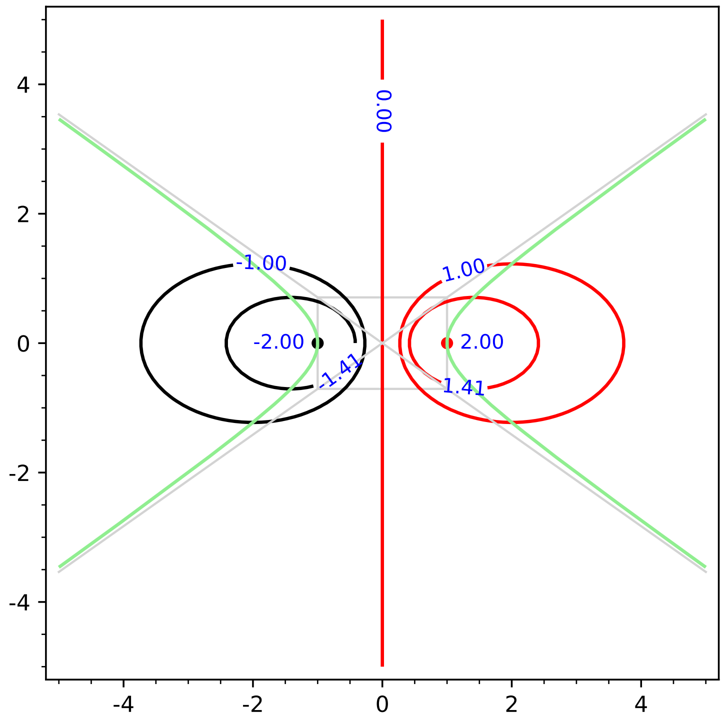
$$\begin{aligned} f_x &= \frac{4(1+x^2+2y^2)-(2x)(4x)}{(1+x^2+2y^2)^2} = \frac{4(1-x^2+2y^2)}{(1+x^2+2y^2)^2} \\ &= 0 \iff x^2 - 2y^2 = 1 \qquad \boxed{1} \end{aligned}$$

Hence the points in which the corresponding contour of f has a horizontal tangent form an hyperbola. $\boxed{\tfrac{1}{2}}$

The hyperbola is centered at the origin, has semi-axes $a = 1$, $b = \frac{1}{2}\sqrt{2} \approx 0.7$, and asymptotes $y = \pm\frac{1}{2}\sqrt{2}\,x$). Strictly speaking, the critical points $(\pm 1, 0)$ should be removed, because the ± 2-contour has no tangent.

d) The 0-contour is the x-axis.
For $k = 2$ the equation derived in b) is $(x-1)^2 + 2y^2 = 0$, and hence the 2-contour is the single point $(1,0)$.
For $k = 1$ the equation in b) is $(x-2)^2 + 2y^2 = 3$, which yields an ellipse with center $(2,0)$ and semi-axes $a = \sqrt{3} \approx 1.7$, $b = \frac{1}{2}\sqrt{6} \approx 1.2$.
For $k = \sqrt{2}$ the equation in b) is $(x-\sqrt{2})^2 + 2y^2 = 1$, which yields an ellipse with center $(\sqrt{2}, 0) \approx (1.4, 0)$ and semi-axes $a = 1$, $b = \frac{1}{2}\sqrt{2} \approx 0.7$.

In the subsequent plot the relevant contours are colored red and hyperbola obtained in c) green.



$\boxed{2}$

Remarks: Many students failed to answer a) correctly. It is not sufficient to prove $\lim_{x\to\pm\infty} f(x, mx) = 0$ for all $m \in \mathbb{R}$ and $\lim_{y\to\pm\infty} f(0, y) = 0$, as the example

$$f(r\cos\phi, r\sin\phi) = \begin{cases} 1/(r\phi) & \text{for } r > 0,\ 0 < \phi < 2\pi, \\ 0 & \text{for } r = 0 \text{ or } \phi = 0 \end{cases}$$

shows.

Similarly, it is not sufficient to show $\lim_{x\to\pm\infty} f(x, y) = 0$ for every $y \in \mathbb{R}$ and $\lim_{y\to\pm\infty} f(x, y) = 0$ for every $x \in \mathbb{R}$. Our old friend $g(x, y) = \frac{xy}{x^2+y^2} = \frac{1}{2}\sin(2\phi)$ in polar coordinates provides a counterexample. It satisfies $|g(x, y)| \leq \min\left\{\frac{|x|}{|y|}, \frac{|y|}{|x|}\right\}$, for $x \neq 0$, $y \neq 0$, from which $\lim_{x\to\pm\infty} g(x, y) = 0$ for every $y \in \mathbb{R}$ and $\lim_{y\to\pm\infty} g(x, y) = 0$ for every $x \in \mathbb{R}$. But the polar coordinate form shows that $\lim_{|(x,y)|\to\infty} g(x, y)$ doesn't exist.

Here is an alternative proof without polar coordinates:

$$|f(x, y)| \leq \frac{4\,|x|}{x^2 + y^2} = \frac{|x|}{\sqrt{x^2 + y^2}} \cdot \frac{4}{\sqrt{x^2 + y^2}} \leq \frac{4}{\sqrt{x^2 + y^2}} \to 0 \quad \text{for } |(x, y)| \to \infty,$$

because $|(x, y)| = \sqrt{x^2 + y^2}$.

But it is incorrect to conclude the required result from $|f(x, y)| \leq \frac{4}{|x|} \to 0$ for $|x| \to \infty$, $|f(x, y)| \leq \frac{4|x|}{2y^2} \to 0$ for $|y| \to \infty$ and "one of $|x|$ or $|y|$ must tend to infinity if $|(x, y)| \to \infty$"; cf. the counterexample g above.

In b) some students excluded 0 from the range of f, but it is obvious that $f(0, y) = 0$ and hence $0 \in \text{range}(f)$.

$$\sum_1 = 7$$

2 a) Writing $G=(G_1,G_2)$, we have

$$(G_1)_x=\frac{1(x^2+y^2)-(2x)x}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2},\quad (G_1)_y=\frac{-2xy}{(x^2+y^2)^2},$$

and hence, using symmetry,

$$\begin{aligned}\mathbf{J}_G(x,y)&=\begin{pmatrix}(G_1)_x & (G_1)_y\\ (G_2)_x & (G_2)_y\end{pmatrix}=\begin{pmatrix}\frac{y^2-x^2}{(x^2+y^2)^2} & -\frac{2xy}{(x^2+y^2)^2}\\ -\frac{2xy}{(x^2+y^2)^2} & \frac{x^2-y^2}{(x^2+y^2)^2}\end{pmatrix}\\ &=\frac{1}{(x^2+y^2)^2}\begin{pmatrix}y^2-x^2 & -2xy\\ -2xy & x^2-y^2\end{pmatrix}. \qquad \boxed{1}\\ \implies \mathbf{J}_G(x,y)^{\mathsf{T}}\mathbf{J}_G(x,y)&=\mathbf{J}_G(x,y)^2\\ &=\frac{1}{(x^2+y^2)^4}\begin{pmatrix}(x^2+y^2)^2 & 0\\ 0 & (x^2+y^2)^2\end{pmatrix}\\ &=\begin{pmatrix}(x^2+y^2)^{-2} & 0\\ 0 & (x^2+y^2)^{-2}\end{pmatrix}. \qquad \boxed{\tfrac{1}{2}}\end{aligned}$$

Since for all $(x,y)\neq(0,0)$ this has the form $\lambda\,\mathrm{I}_2$ with $\lambda=(x^2+y^2)^{-2}\neq 0$, G is conformal. $\boxed{\tfrac{1}{2}}$

Alternatively, it suffices to observe that the columns of $\mathbf{J}_G(x,y)$ are orthogonal and have the same length, viz. $\frac{1}{x^2+y^2}$, or make the analogous observation about the matrix $\begin{pmatrix}y^2-x^2 & -2xy\\ -2xy & x^2-y^2\end{pmatrix}$.

b) $G(x,y)=(x,y)\iff \frac{x}{x^2+y^2}=x\wedge\frac{y}{x^2+y^2}=y\iff x^2+y^2=1$,
i.e., the fixed points of G are those on the unit circle in $\mathbb{R}^2$. $\boxed{1}$

c) Denoting the line $ax+by=1$ by L, we have for $(x,y)\in L$ and $(u,v)=G(x,y)$ the equations

$$\begin{aligned}au+bv&=a\,\frac{x}{x^2+y^2}+b\,\frac{y}{x^2+y^2}=\frac{ax+by}{x^2+y^2}=\frac{1}{x^2+y^2},\\ u^2+v^2&=\frac{x^2}{(x^2+y^2)^2}+\frac{y^2}{(x^2+y^2)^2}=\frac{1}{x^2+y^2}.\\ &\implies u^2+v^2-au-bv=0,\end{aligned}$$

which is the equation of a circle with center $(a/2,b/2)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$. Clearly this circle contains $(0,0)$. $\boxed{1}$

Conversely, suppose $(u,v)\neq(0,0)$ satisfies $u^2+v^2-au-bv=0$. Using the second equation, we can solve $G(x,y)=(u,v)$ for x,y:

$$u=\frac{x}{x^2+y^2}=x(u^2+v^2)\implies x=\frac{u}{u^2+v^2},$$

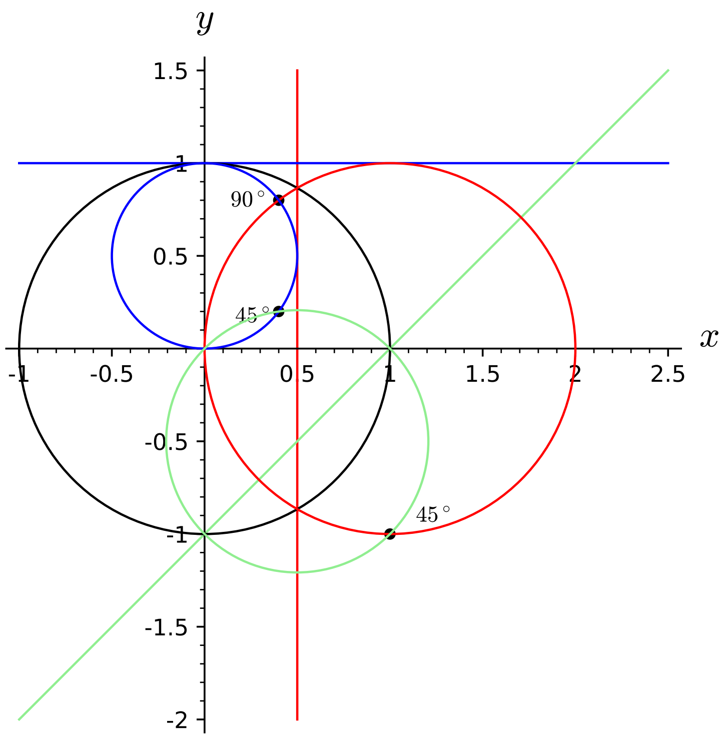
and similarly $y = \frac{v}{u^2+v^2}$. (This says that the map G is its own inverse.) With this definition of x, y we have

$$ax + by = \frac{au + bv}{u^2 + v^2} = 1.$$

$\Longrightarrow$ The preimage (or image) under G of the circle $u^2 + v^2 - au - bv = 0$ is the line $ax + by = 1$. $\boxed{1}$

d) In the subsequent plot a line L and its image $G(L)$ have the same color. The circle $G(L)$ is determined by $(0,0)$ and the images of two points on L, which in the case of the red and green line can be taken as the intersection points with the unit circle; by b), these points are their own images. The blue circle must contain $(0,0)$ and $(0,1)$ and touch the unit circle in $(0,1)$ (since the blue line does this and tangents are "preserved" by a differentiable map. This determines the circle uniquely. Another way to obtain it is to use a third point, e.g., the image of $(1,1)$, which is $\left(\frac{1}{2}, \frac{1}{2}\right)$.

The three lines intersect at angles of 45°, 45°, 90°. Since G is conformal, the angles at the intersection points $\neq (0,0)$ of the circles must also be 45°, 45°, 90°; cf. the picture.



$\boxed{2}$

Remarks: In c) virtually all students forgot to check that every point $\neq (0,0)$ on the circle is indeed the image of some point on the line. This was in a way anticipated and the corresponding mark treated as a bonus mark.

$$\sum_2 = 7$$

3 a) We use m and kg as units of measurement.

$$\begin{aligned} \mathrm{d}B &= h^{-2}\mathrm{d}m - 2mh^{-3}\mathrm{d}h & \boxed{1}\\ \implies B(m,h) &\approx \frac{100}{2^2} + \frac{1}{2^2}\,\Delta m - \frac{2\cdot 100}{2^3}\,\Delta h \\ &= 25 + \frac{1}{4}\,\Delta m - 25\,\Delta h \quad [\mathrm{kg\,m^{-2}}], & \boxed{1} \end{aligned}$$

where $\Delta m = m - m_0$, $\Delta h = h - h_0$.

b) The Mean Value Theorem gives that the error $\Delta B = B(m,h) - B(100,2)$ is equal to $h'^{-2}\Delta m - 2m'h'^{-3}\Delta h$ for some $m' \in [99.5, 100.5]$ and $h' \in [1.995, 2.005]$.

From this, using mm as the unit of measurement, we obtain

$$\begin{aligned} \implies |\Delta B| &\leq \frac{0.5}{1.995^2} + \frac{2\cdot 100.5\cdot 0.005}{1.995^3} & \boxed{1}\\ &\approx 0.252199269727261 \quad [\mathrm{kg\,m^{-2}}]. \end{aligned}$$

(This should be compared with the corresponding non-rigorous estimate, which is $|\Delta B| \leq \frac{1}{8} + \frac{1}{8} = 0.25$ [kg m^{-2}].)

Remarks: Many students had difficulties with this question. In a) the value $B(100,2) = 25$ must be present (this is part of the definition of "linear approximation"). In b) when estimating $|\Delta B|$ the sign of both summands must be +, because the signs of Δm, Δh are arbitrary. (For the largest possible negative error $\Delta h = -0.005$ the sign of the 2nd summand becomes +. The triangle inequality for differences is $|x - y| \leq |x| + |y|$ and not "$|x - y| \leq |x| - |y|$".)

$$\sum_3 = 3$$

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4 a) We have

$$r(1/t) = (t + 1/t,\ 2\ln(1/t),\ 1/t - t) = (t + 1/t,\ -2\ln t,\ -(t - 1/t)).$$

Thus C (considered as a non-parametric curve) is invariant under the map $(x,y,z) \mapsto (x,-y,-z)$, i.e., symmetric with respect to the x-axis. $\boxed{1}$

b) Since $t + 1/t$ takes its minimum at $t = 1$ (with value 2), all three coordinate functions are simultaneously minimized in absolute value for $t = 1$. $\implies$ The point on C closest to the origin is $\mathbf{r}(1) = (2,0,0)$. $\boxed{1}$

c)

$$\begin{aligned} \mathbf{r}'(t) &= \left(1 - \frac{1}{t^2}, \frac{2}{t}, 1 + \frac{1}{t^2}\right), \\ |\mathbf{r}'(t)|^2 &= \left(1 - \frac{1}{t^2}\right)^2 + \frac{4}{t^2} + \left(1 + \frac{1}{t^2}\right)^2 = 2 + \frac{2}{t^4} + \frac{4}{t^2} \\ &= 2\left(1 + \frac{1}{t^2}\right)^2, \\ |\mathbf{r}'(t)| &= \sqrt{2}\left(1 + \frac{1}{t^2}\right) & \boxed{1} \end{aligned}$$

$\Longrightarrow$ The arc length function, measured from $P = \mathbf{r}(1)$ is

$$s(t) = \int_1^t \sqrt{2}\left(1 + \frac{1}{\tau^2}\right) \mathrm{d}\tau = \sqrt{2}\left[\tau - \frac{1}{\tau}\right]_1^t = \sqrt{2}\left(t - \frac{1}{t}\right).$$

The point $Q = \mathbf{r}(t_1)$ is characterized by $s(t_1) = \sqrt{2}\left(t_1 - \frac{1}{t_1}\right) = 1$ and hence has z-coordinate $t_1 - \frac{1}{t_1} = 1/\sqrt{2} = \frac{1}{2}\sqrt{2}$. $\boxed{1}$

d) We have

$$\begin{aligned} \mathbf{T}(t) &= \frac{\left(1 - \frac{1}{t^2}, \frac{2}{t}, 1 + \frac{1}{t^2}\right)}{= \sqrt{2}\left(1 + \frac{1}{t^2}\right)} = \frac{1}{\sqrt{2}}\left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}, 1\right), \\ \mathbf{T}(1) &= \frac{(0,1,1)}{\sqrt{2}}. \end{aligned} \qquad \boxed{1}$$

Since $\mathbf{T}'(t) = (*, *, 0)$, the unit normal vector $\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|}$ must be contained in the x, y-plane and orthogonal to $\mathbf{T}(1)$. This leaves only the possibilities $\mathbf{N}(1) = (\pm 1, 0, 0)$. Since $t \mapsto \mathbf{r}(t)$ has in P its “perihelion”, $\mathbf{N}(1)$ points away from the origin. $\boxed{1}$

$\Longrightarrow \mathbf{N}(1) = (1, 0, 0)$. $\boxed{\frac{1}{2}}$

$$\Longrightarrow \mathbf{B}(1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \qquad \boxed{\tfrac{1}{2}}$$

which can also be seen geometrically.

Remarks: In contrast with the other questions, many students obtained a full score for this question.

$$\sum_4 = 7$$

General Remarks:

$$\sum_{\text{Midterm 2}} = 24$$
