Midterm Exam 2

Name: ____________ Student ID: ____________ Major: ______

Question 1 (ca. 11 marks)

Let $D = \{(x,y) \in \mathbb{R}^2; y^2 + x > 0\}$, and consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^2 + y}{y^2 + x}.$$

a) Sketch D.

b) Which symmetry-like property does f have? What can you conclude from this about the contours of f?

c) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{(x,y)\to(-1,1)} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibilities $\pm\infty$), or show that the limit does not exist.

d) Determine the critical point(s) of f.

e) Determine the k-contours of f for $k = -1, 0, 1$ and insert these into the sketch you drew in a).

f) Determine the slope of the graph G_f at $(1,-1)$ in north-eastern direction (NE), and the maximal slope/direction of G_f at $(1,-1)$.

Question 2 (ca. 4 marks)

A musketeer fires a bullet with angle of elevation $\alpha \approx 15°$ ($|\text{error}| \leq 0.5°$) and muzzle speed $v \approx 100\,\mathrm{m/s}$ ($|\text{error}| \leq 5\,\mathrm{m/s}$). The range of the bullet is given by the formula

$$r = \frac{v^2 \sin(2\alpha)}{g},$$

where $g \approx 10\,\mathrm{m/s^2}$ ($|\text{error}| \leq 0.2\,\mathrm{m/s^2}$) is the gravitational acceleration on the surface of the earth.

a) Determine the differential of r as a function of v, α, g.

b) Determine upper and lower bounds for the true range of the bullet.

Hint: It suffices to give non-rigorous estimates. Improved estimates (e.g., by using the Mean Value Theorem) will be honored by bonus marks.

Question 3 (ca. 5 marks)

a) Suppose all 2nd-order partial derivatives of $u\colon \mathbb{R}^2 \to \mathbb{R}$ (exist and) are zero everywhere. Show that there exist $a, b, c \in \mathbb{R}$ such $u(x,y) = a + bx + cy$ for $(x,y) \in \mathbb{R}^2$.

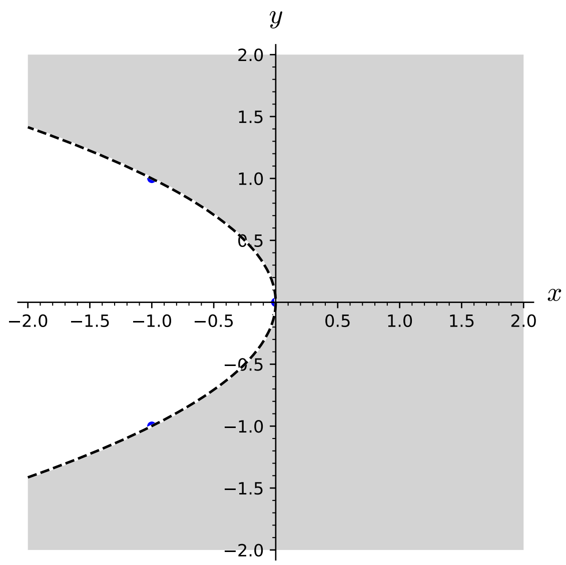
Hint: First consider u_x, u_y. Which property do these functions have?

b) Suppose $u\colon \mathbb{R}^2 \to \mathbb{R}$ is harmonic (i.e., $u_{xx} + u_{yy} = 0$) and has a strict local extremum in $(x_0, y_0) \in \mathbb{R}^2$. Show that all 2nd-order partial derivatives of u are zero at (x_0, y_0).

Can we conclude from a), b) that harmonic functions in $\mathbb{R}^2$ cannot have a strict local extremum? Justify your answer!

Solutions

1 a) D is shaded in the following plot:



$\boxed{\frac{1}{2}}$

b) $f(y,x) = \frac{y^2+x}{x^2+y} = 1/f(x,y)$, provided both points are in D $\boxed{\frac{1}{2}}$
$\Longrightarrow$ The $1/k$-contour of f is the mirror image of the k-contour with respect to the line $y = x$ (except that $k = 0$ is excluded, and some portion of the mirror image may fall outside D). $\boxed{1}$

c) On the line $y = mx$ we have

$$\begin{aligned} f(x,y) = f(x,mx) &= \frac{x^2+mx}{m^2x^2+x} = \frac{x+m}{m^2x+1}, \\ \lim_{x\downarrow 0} f(x,mx) &= m, \\ \lim_{x\to\infty} f(x,mx) &= \frac{1}{m^2}. \end{aligned}$$

If the limit $\lim_{(x,y)\to(0,0)} f(x,y)$ exists, the first limit should be independent of m, which is not the case. $\Longrightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ doesn't exist. $\boxed{1}$

Similarly, if $\lim_{|(x,y)|\to\infty} f(x,y)$ exists, the second limit should be independent of m, which isn't the case either. $\Longrightarrow \lim_{|(x,y)|\to\infty} f(x,y)$ doesn't exist. $\boxed{1}$

For $(x,y) \to (-1,1)$ we have $x^2 + y \to 2$, $y^2 + x \to 0+$ (because $y^2 + x > 0$ in D). $\Longrightarrow \lim_{(x,y)\to(-1,1)} f(x,y) = +\infty$. $\boxed{1}$

d) We have

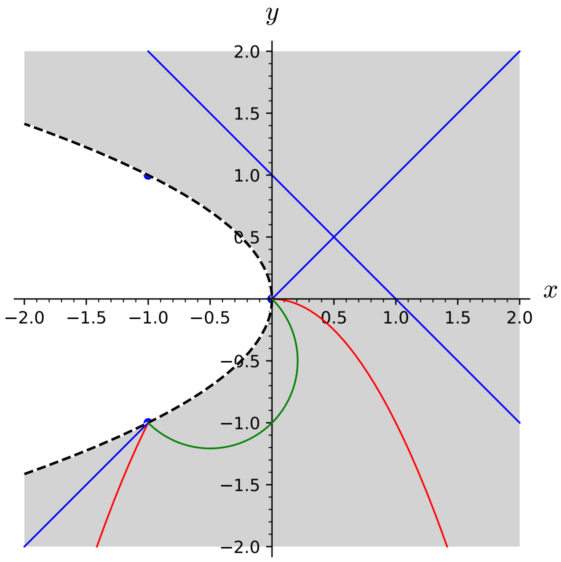
$$\begin{aligned} f_x &= \frac{2x(y^2+x) - (x^2+y)}{(y^2+x)^2} = \frac{2xy^2 + x^2 - y}{,} \\ f_y &= \frac{y^2+x-2y(x^2+y)}{(y^2+x)^2} = \frac{-2x^2y - y^2 + x}{,} \end{aligned}$$ $\boxed{1}$

and $\nabla f(x,y)=0$ iff $2xy^2+x^2-y=0$ and $-2x^2y-y^2+x$. Multiplying the 1st equation by x, the 2nd equation by y and adding gives $x^3-y^3=0$, i.e. $x=y$, and further $(2x^3+x^2-x)=2(x-1/2)(x+1)=0$. Since the points $(0,0)$ and $(-1,-1)$ are not in D, the only critical point of f is $\left(\frac{1}{2},\frac{1}{2}\right)$. $\boxed{1}$

e) $\underline{k=-1:}$ $x^2+y=-(y^2+x) \iff x^2+y^2+y+x=0 \iff \left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2=\frac{1}{2}$ $\implies$ The -1-contour is a circle with center $\left(-\frac{1}{2},-\frac{1}{2}\right)$ and radius $\frac{1}{2}\sqrt{2}$, from which the arc outside D is removed.

$\underline{k=0:}$ The 0-contour is $y=-x^2$ (a parabola), again with the arc outside D removed.

$\underline{k=1:}$ $x^2+y=y^2+x \iff (x^2-y^2)=(x-y)(x+y)=x-y$
$\implies$ The 1-contour is the union of the lines $x=y$ and $x+y=1$, with the line segment outside D removed.



$\boxed{3}$

f) We have

$$\nabla f(1,-1)=\left(1,\tfrac{1}{2}\right),$$
$$\nabla f(1,-1)\cdot\tfrac{1}{\sqrt{2}}(1,1)=\left(1,\tfrac{1}{2}\right)\cdot\tfrac{1}{\sqrt{2}}(1,1)=\tfrac{3}{2\sqrt{2}}.$$

$\implies$ The slope of G_f at $(1,-1)$ in the direction NE is is $\frac{3}{2\sqrt{2}}$. $\boxed{1}$

The maximal slope of G_f at $(1,-1)$, in the direction $\mathbb{R}^+\left(1,\frac{1}{2}\right)$ of the gradient, is $|\nabla f(1,-1)|=\frac{1}{2}\sqrt{5}$. $\boxed{1}$

Remarks:

$$\sum_1 = 12$$

2 a) We have

$$r(v,\alpha,g)=\frac{v^2\sin(2\alpha)}{g},$$
$$\mathrm{d}r=\frac{2v\sin(2\alpha)}{g}\,\mathrm{d}v+\frac{2v^2\cos(2\alpha)}{g}\,\mathrm{d}\alpha-\frac{v^2\sin(2\alpha)}{g^2}\,\mathrm{d}g. \qquad \boxed{2}$$

b) We use m, s, rad as unit of measurement. Since $1^\circ = \pi/180\,\text{rad}$, the approximation for the angle is $\pi/12$, and the error bound $\pi/360$.

$$\begin{aligned}\implies \Delta r &= r(v,\alpha,g) - r\,(100,\pi/12,10)\\ &\approx \frac{\partial r}{\partial v}\,(100,\pi/12,10)\,\Delta v + \frac{\partial r}{\partial \alpha}\,(100,\pi/12,10)\,\delta\alpha + \frac{\partial r}{\partial g}\,(100,\pi/12,10)\,\Delta g\end{aligned}$$

with $\Delta v = v - 100$, $\Delta\alpha = \alpha - \pi/12$, $\Delta g = g - 10$. (Compared with the lecture, the sign of adsolute errors Δr, Δv, $\Delta\alpha$, Δg is the opposite (true value – approx. value), which is more convenient to use here.)

Since $\sin(\pi/6) = 1/2$, we obtain

$$\begin{aligned} r(v,\alpha,g) - 500 &\approx 10\,\Delta v + 1000\sqrt{3}\,\Delta\alpha - 50\,\Delta g,\\ |r(v,\alpha,g) - 500| &\lessapprox 50 + \frac{1000\sqrt{3}\,\pi}{360} = 60 + \frac{100\sqrt{3}\,\pi}{36} + 10, \end{aligned} \qquad \boxed{2}$$

which numerically is ≈ 75. Thus the range of the rocket is between 425 m and 575 m.

For rigorous bounds one needs to bound the partial derivatives over the whole uncertainty region $95 \leq v \leq 105$, $\pi/12 - \pi/360 \leq \alpha \leq \pi/12 + \pi/360$, $-9.8 \leq g \leq 10.2$. This gives

$$|r(v,\alpha,g) - 500| \leq \frac{105^2\sin(31\pi/180)}{9.8} + \frac{2\,105^2\cos(29\pi/180)}{9.8} + \frac{105^2\sin(31\pi/180)}{9.8^2}. \qquad \boxed{1}$$

Remarks:

$$\sum_2 = 5$$

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3 a) We have $\nabla u_x = (u_{xx}, u_{xy}) = (0,0)$ and $\nabla u_y = (u_{yx}, u_{yy}) = (0,0)$. Since $\mathbb{R}^2$ is connected, there exist constants $b, c \in \mathbb{R}$ such that

$$u_x(x,y) = b \quad \text{and} \quad u_y(x,y) = c \quad \text{for all } (x,y) \in \mathbb{R}^2. \qquad \boxed{1}$$

Then consider the function $f(x,y) = u(x,y) - bx - cy$. We have

$$\begin{aligned} f_x &= u_x - b = 0,\\ f_y &= u_y - c = 0 \end{aligned}$$

throughout $\mathbb{R}^2$, i.e., $\nabla f = 0$. By the same token as above, there exists a constant $a \in \mathbb{R}$ such that $u(x,y) - bx - cy = f(x,y) = a$ for $(x,y) \in \mathbb{R}^2$, as desired. $\boxed{2}$

b) By Clairaut's Theorem and since u is harmonic, its Hesse matrix has the form

$$\mathbf{H}_u(x,y) = \begin{pmatrix} u_{xx}(x,y) & u_{xy}(x,y) \\ u_{yx}(x,y) & u_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} u_{xx}(x,y) & u_{xy}(x,y) \\ u_{xy}(x,y) & -u_{xx}(x,y) \end{pmatrix}.$$

In particular we have

$$\det \mathbf{H}_u(x_0,y_0) = -u_{xx}(x_0,y_0)^2 - u_{xy}(x_0,y_0)^2. \qquad \boxed{1}$$

Hence, if one of $u_{xx}(x_0,y_0)$, $u_{xy}(x_0,y_0)$ is nonzero, we obtain $\det \mathbf{H}_u(x_0,y_0) < 0$, and so (x_0,y_0) would be a saddle point instead of a local extremum; contradiction. This shows $\mathbf{H}_u(x_0,y_0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. $\boxed{1}$

(This conclusion also holds for a non-strict local extremum.)

Although it is in fact true, we can't conclude from a), b) that u has no strict local extremum. The functions $(x,y) \mapsto a + bx + cy$, $a,b,c \in \mathbb{R}$, don't have a strict local extremum, but in order to conclude that u is of this form, $\mathbf{H}_u$ should vanish at all points, whereas the assumption of a strict local extremum and b) guarantee this only for one point.

The $\mathbf{H}_u(x_0,y_0) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ can't be used either to obtain a contradiction, since there exist functions $f(x,y)$ with, e.g., a strict local minimum in $(0,0)$ and $\mathbf{H}_f(0,0) = \mathbf{0}$. One such example is $f(x,y) = x^4 + y^4$. $\boxed{1}$

Remarks:

$$\sum_{3} = 6$$

General Remarks:

$$\sum_{\text{Midterm 2}} = 23 = 20 + 3$$