Math 241 Calculus III.
Prof. Honold

# Midterm Exam 2

April 21, 2022
18:00–19:30

Name: ____________ Student ID: ____________ Major: ____

## Question 1 (ca. 11 marks)

Consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) = \frac{3x}{x^2+xy+y^2}.$$

Here $D \subseteq \mathbb{R}^2$ is the maximum possible domain for $f$.

a) Determine $D$.

b) Which obvious symmetry property does $f$ have? What can you conclude from this about the graph and the contours of $f$?

c) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\,y>0}} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibilities $\pm\infty$), or show that the limit does not exist.

d) Show that $f$ has no critical point (i.e., no point at which $\nabla f$ vanishes).

e) Sketch the 1-contour of $f$ as accurately as possible. Your drawing should include the points with a horizontal or vertical tangent.

f) Determine the slope of the graph $G_f$ at $(1,-1)$ in the direction of the origin (NW), and the maximal slope/direction of $G_f$ at $(1,-1)$.

## Question 2 (ca. 7 marks)

Consider the differentiable map $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$G(x,y) = \bigl(2xy-2x, x^2-y^2+2y\bigr).$$

a) For $(x,y) \in \mathbb{R}^2$ compute the Jacobi matrix $\mathbf{J}_G(x,y)$.

b) In which points $(x,y) \in \mathbb{R}^2$ is $G$ conformal?

c) Determine the $G$-image $S = G(Q)$ of the unit square $Q = [0,1]^2$, and graph $Q$ and the region $S$ on paper (unit length at least 2 cm).

*Hint:* It suffices to determine the $G$-images of the vertices and edges of $Q$. The edges are segments of the coordinate lines with equations $x=0$, $x=1$, $y=0$, $y=1$. You should indicate the correspondence between edges and their $G$-images by using the same color (or line style such as "dashed", "dotted", etc.).

d) Is the figure obtained in c) compatible with the result in b)? Justify your answer!

## Question 3 (ca. 4 marks)

The total resistance $R$ produced by two conductors with resistances $R_1$ and $R_2$ connected in a parallel electrical circuit is given by the formula

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$

a) Find the linear approximation of $R = R(R_1, R_2)$ near $(R_1, R_2)$.

b) Suppose $R_1 = 10\,\Omega$, $R_2 = 5\,\Omega$ with a possible tolerance of $\pm 0.5\,\Omega$. Using the Mean Value Theorem, state (tight) rigorous upper and lower bounds for the true value of $R$.

*Note:* In b) it is not necessary to determine explicit figures.

# Solutions

**1** a) $x^2+xy+y^2=\left(x+\frac{1}{2}y\right)^2+\frac{3}{4}y^2=0$ has only the trivial solution $x=y=0$.
$\implies D=\mathbb{R}^2\setminus\{(0,0)\}$ $\boxed{\frac{1}{2}}$

b) $f(-x,-y)=-f(x,y)$ for $(x,y)\in D$.
$\implies$ $\mathrm{G}_f$ is point-symmetric with respect to the origin in $\mathbb{R}^3$, and the $-k$-contour of $f$ is obtained from the $k$-contour by reflection at the origin of $\mathbb{R}^2$ (i.e., the map $(x,y)\mapsto(-x,-y)$). $\boxed{1\frac{1}{2}}$

c) Since $f(x,0)=3/x\to+\infty$ for $x\downarrow 0$ and $f(x,0)\to-\infty$ for $x\uparrow 0$, the first limit doesn't exist. $\boxed{1}$

Since $f(x,0)=3/x\to+\infty$ for $x\downarrow 0$ and $f(y^2,y)=\frac{3y^2}{y^4+y^3+y^2}=\frac{3}{1+y+y^2}\to 3$ for $y\downarrow 0$, the second limit doesn't exist either. $\boxed{1}$

The third limit is zero, since $x^2+xy+y^2\geq\frac{1}{2}(x^2+y^2)$ and hence

$$|f(x,y)|\leq\frac{6\,|x|}{x^2+y^2}\leq\frac{6}{\sqrt{x^2+y^2}}=\frac{6}{|(x,y)|}. \qquad \boxed{1}$$

Alternatively, in polar coordinates we have

$$f(r\cos\phi,r\sin\phi)=\frac{3r\cos\phi}{r^2(\cos^2\phi+\cos\phi\sin\phi+\sin^2\phi)}=\frac{3\cos\phi}{r\left(1+\frac{1}{2}\sin(2\phi)\right)},$$

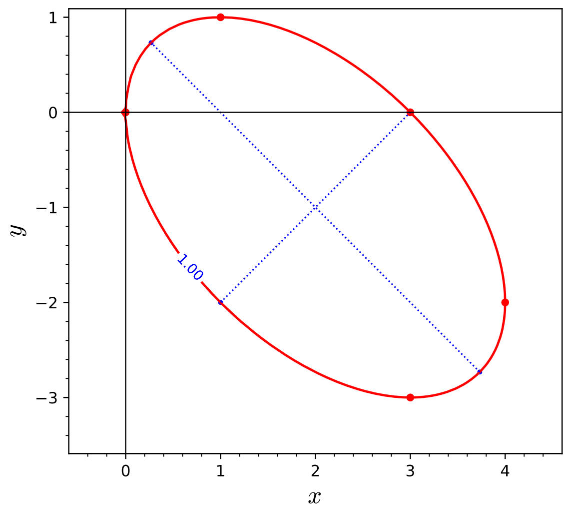
from which $|f(r\cos\phi,r\sin\phi)|\leq 6/r\to 0$ for $r\to\infty$ follows likewise.

d) We have

$$\begin{aligned}
f_x&=\frac{3(x^2+xy+y^2)-3x(2x+y)}{(x^2+xy+y^2)^2}=\frac{3(y^2-x^2)}{(x^2+xy+y^2)^2}, && \boxed{\tfrac{1}{2}}\\
f_y&=\frac{-3x(x+2y)}{(x^2+xy+y^2)^2}, && \boxed{\tfrac{1}{2}}\\
\nabla f&=(f_x,f_y).
\end{aligned}$$

Clearly the only solution of $y^2-x^2=3x(x+2y)=0$ is $(x,y)=(0,0)$, but $(0,0)\notin D$. Hence $f$ has no critical point. $\boxed{1}$

e) The 1-contour has the equation $g(x,y):=x^2+xy+y^2-3x=0$, provided we include the origin $(0,0)$ among its points. From $\nabla g=(2x+y-3,x+2y)$, $\nabla g(0,0)=(-3,0)$ the tangent at $(0,0)$ is vertical. Other points with vertical tangent must have $f_y=0$, which implies $x=-2y$ and $4y^2-2y^2+y^2+6y=0$, i.e., $3y^2+6y=0$, giving the point $(4,-2)$. Points with horizontal tangent must have $f_x=0$, i.e. $x=\pm y$ and $x^2\pm x^2+x^2-3x=0$, giving the points $(1,1)$ and $(3,-3)$. For the drawing it is also good to observe that the 1-contour intersects the $x$-axis in $(3,0)$.




$\boxed{2}$

f) We have

$$\nabla f(1,-1) = (0,3),$$
$$\nabla f(1,-1)\cdot \tfrac{1}{\sqrt{2}}(-1,1) = \tfrac{3}{\sqrt{2}}.$$

$\Longrightarrow$ The slope of $\mathrm{G}_f$ at $(1,-1)$ in the direction of the origin (NW) is $\frac{3}{\sqrt{2}}$, $\boxed{1}$
and the maximal slope of $\mathrm{G}_f$ at $(1,-1)$, in the direction of the gradient (N), is $|\nabla f(1,-1)| = 3$. $\boxed{1}$

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$$\sum_1 = 11$$

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**2** a) Writing $G = (u,v)^{\mathsf{T}}$ with $u(x,y) = 2xy - 2x$, $v(x,y) = x^2 - y^2 + 2y$, we have

$$\mathbf{J}_G(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 2y-2 & 2x \\ 2x & -2y+2 \end{pmatrix}. \qquad \boxed{1}$$

b) Using a), we have

$$\mathbf{J}_G(x,y)^{\mathsf{T}}\mathbf{J}_G(x,y) = \begin{pmatrix} (2y-2)^2+4x^2 & 0 \\ 0 & (2y-2)^2+4x^2 \end{pmatrix}.$$

$\Longrightarrow$ $G$ is conformal in all points satisfying $(2y-2)^2 + 4x^2 \neq 0$, i.e., in all points of $\mathbb{R}^2$ except $(0,1)$. $\boxed{1}$

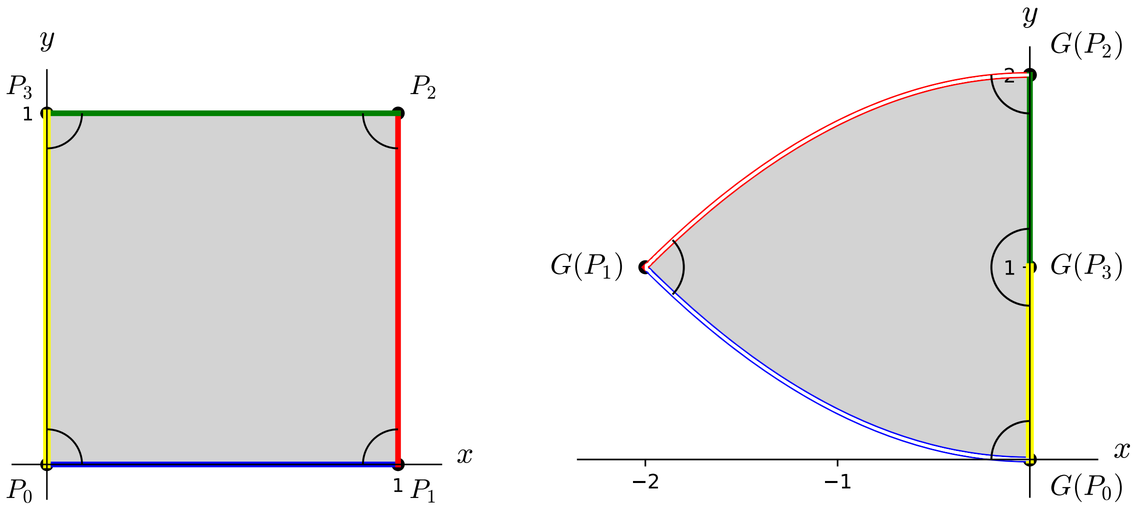
c) Writing $P_0 = (0,0)$, $P_1 = (1,0)$, $P_2 = (1,1)$, $P_3 = (0,1)$ for the vertices of $Q$ and $[P_0, P_1]$, etc., for the edges of $Q$, we have $G(0,0) = (0,0)$, $G(1,0) = (-2,1)$, $G(1,1) = (0,2)$, and further:

(i) $G(x,0) = (-2x, x^2) = \bigl(-2x, \frac{1}{4}(-2x)^2\bigr)$, $0 \leq x \leq 1$
$\Longrightarrow G\bigl([P_0,P_1]\bigr)$ is the arc of the parabola $y = \frac{1}{4}x^2$, $-2 \leq x \leq 0$. $\boxed{\tfrac{1}{2}}$

(ii) $G(1,y) = (2y-2, 1-y^2+2y) = \bigl(2(y-1), 2-(y-1)^2\bigr)$, $0 \leq y \leq 1$
$\implies G\bigl([P_1,P_2]\bigr)$ is the arc of the parabola $y = 2 - \frac{1}{4}x^2$, $-2 \leq x \leq 0$. $\boxed{\frac{1}{2}}$

(iii) $G(x,1) = (0, x^2+1)$, $0 \leq x \leq 1$
$\implies G\bigl([P_2,P_3]\bigr)$ is the line segment from $(0,1)$ to $(0,2)$. $\boxed{\frac{1}{2}}$

(iv) $G(0,y) = (0, -y^2+2y) = \bigl(0, 1-(y-1)^2\bigr)$, $0 \leq y \leq 1$
$\implies G\bigl([P_3,P_0]\bigr)$ is the line segment from $(0,0)$ to $(0,1)$. $\boxed{\frac{1}{2}}$




(a) $Q$ (b) $G(Q)$

$\boxed{2}$

d) Since $G$ is conformal in all points except $P_3 = (0,1)$, the $90°$-angles between the edges at $P_0, P_1, P_2$ are mapped to $90°$-angles between the arcs/edges at $G(P_0)$, $G(P_1)$, $G(P_2)$. The $90°$-angle between the edges at $P_3$ is mapped to a $180°$-angle between the edges at $G(P_3)$; since $G$ is not conformal in $P_3$, this doesn't contradict the result in b). $\boxed{1}$

$$\sum_2 = 7$$

**3** a) We use Ohm ($\Omega$) as unit of measurement.

$$\begin{aligned}
R &= \frac{R_1R_2}{R_1+R_2},\\
\frac{\partial R}{\partial R_1} &= \frac{R_2(R_1+R_2) - R_1R_2}{(R_1+R_2)^2} = \frac{R_2^2}{(R_1+R_2)^2},\\
\frac{\partial R}{\partial R_2} &= \frac{R_1^2}{(R_1+R_2)^2},\\
\Delta R &= R(R_1+\Delta R_1, R_2+\Delta R_2) - R(R_1,R_2)\\
&\approx \frac{R_2^2}{(R_1+R_2)^2}\,\Delta R_1 + \frac{R_1^2}{(R_1+R_2)^2}\,\Delta R_2
\end{aligned}$$ $\boxed{2}$

for small $\Delta R_1$, $\Delta R_2$.

b) The Mean Value Theorem gives

$$\Delta R = R - \frac{10 \cdot 5}{10+5} = \frac{r_2^2}{(r_1+r_2)^2}\,\Delta R_1 + \frac{r_1^2}{(r_1+r_2)^2}\,\Delta R_2$$

with $|\Delta R_1| \leq 0.5$, $|\Delta R_2| \leq 0.5$, $9.5 \leq r_1 \leq 10.5$, $4.5 \leq r_2 \leq 5.5$. $\boxed{1}$

Hence we have

$$\begin{aligned} R &\leq \frac{10}{3} + \frac{0.5(5.5^2+10.5^2)}{14^2} \approx 3.69\,[\Omega], \\ R &\geq \frac{10}{3} - \frac{0.5(5.5^2+10.5^2)}{14^2} \approx 2.975\,[\Omega]. \end{aligned} \qquad \boxed{1}$$

The explicit figures were not required. A 100 % rigorous estimate is then $2.97 \leq R \leq 3.70\,[\Omega]$.

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$$\sum_{3} = 4$$

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$$\sum_{\text{Midterm 2}} = 22 = 20 + 2$$

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