Name: ____________ Student ID: ____________ Major: ______

Question 1 (ca. 10 marks)

Let $D = \{(x,y) \in \mathbb{R}^2; x + 4y > 0\}$, and consider the function $f\colon D \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^2 + 4y^2}{x + 4y}.$$

a) Determine the limits

$$\lim_{(x,y)\to(0,0)} f(x,y), \qquad \lim_{\substack{(x,y)\to(0,0)\\ x>0,\, y>0}} f(x,y), \qquad \lim_{|(x,y)|\to\infty} f(x,y)$$

(including the possibility $+\infty$), or show that the limit does not exist.

b) Derive equations for the contours of f and describe their shapes as accurately as possible.

c) Determine the slope of the graph G_f at $(1,1)$ in the direction of the origin, and the maximal slope/direction of G_f at $(1,1)$.

d) The 1-contour of f passes through the points $(1,0)$, $(0,1)$, and $(1,1)$. Determine equations for the tangent lines of the 1-contour in these points, and sketch the contour together with the tangent lines and the boundary of D (unit length at least 3 cm!).

Question 2 (ca. 10 marks)

Consider the space curve C parametrized by

$$\mathbf{r}(t) = \left(3t+1, 3t^2, 2t^3\right), \quad t \in \mathbb{R}.$$

a) Is C contained in a plane?

b) Let E be the plane in $\mathbb{R}^3$ with equation $2x + 4y = 1$. Determine the (unique) point P on C that minimizes the distance to E, and the distance $d = d(P,E)$.

Hint: What can you say about the tangent line of C in P?

c) Let $Q = (x,y,z)$ be the (unique) point on C with $z > 0$ and such that the arc from $(1,0,0)$ to Q has length 5. Determine the osculating plane (in parametric form), center and radius of the osculating circle, and the TNB frame of C in Q.

Solutions

1 a) (i) This limit is the most difficult, and it is best to conclude its non-existence from the shapes of the contours near the origin $(0,0)$. Alternatively, we can argue as follows: D contains the positive coordinate axes except for $(0,0)$. Since $f(x,0)=x$, $f(0,y)=y$ tend to zero for $(x,y)\to(0,0)$, the limit, provided it exists, can only be zero. On the other hand, setting $x+4y=\delta$ we have

$$f(x,y)=\frac{(\delta-4y)^2+4y^2}{\delta}=\frac{\delta^2-8\delta y+20y^2}{\delta}=\delta-8y+\frac{20y^2}{\delta}.$$

Choosing $\delta=y^2$ gives

$$f(y^2-4y,y)=y^2-8y+20.$$

Thus, when moving along the curve $x=y^2-4y$ towards $(0,0)$, the limit is 20. Hence $\lim_{(x,y)\to(0,0)}f(x,y)$ doesn't exist. $\boxed{1\frac{1}{2}}$

(ii) From

$$0<\frac{x^2+4y^2}{x+4y}=\frac{x}{x+4y}\,x+\frac{4y}{x+4y}\,y<x+y\to 0\qquad\text{for } x,y\downarrow 0$$

we have $\lim_{\substack{(x,y)\to(0,0)\\ x>0,\,y>0}}f(x,y)=0$. $\boxed{1}$

(iii) The Cauchy-Schwarz Inequality gives

$$x+4y\leq|(1,4)|\,|(x,y)|=\sqrt{17}\sqrt{x^2+y^2}.$$

$$\implies\quad\frac{x^2+4y^2}{x+4y}\geq\frac{x^2+y^2}{x+4y}\geq\frac{\sqrt{x^2+y^2}}{\sqrt{17}}=\frac{|(x,y)|}{\sqrt{17}}$$

$\implies\lim_{|(x,y)|\to\infty}f(x,y)=+\infty$ $\boxed{1\frac{1}{2}}$

Alternatively, in polar coordinates we have

$$f(r\cos\phi,r\sin\phi)=\frac{r^2\cos^2\phi+4r^2\sin^2\phi}{r\cos\phi+4r\sin\phi}=\frac{r(1+3\sin^2\phi)}{\cos\phi+4\sin\phi}\geq\frac{r}{5},$$

which tends to $+\infty$ for $r\to+\infty$.

b) For $k\leq 0$ the k-contours of f are empty. For $k>0$ we have

$$f(x,y)=k\iff x^2+4y^2=k(x+4y)\iff\left(x-\frac{k}{2}\right)^2+4\left(y-\frac{k}{2}\right)^2=\frac{5}{4}k^2$$
$$\iff\frac{(x-k/2)^2}{\left(\frac{\sqrt{5}}{2}k\right)^2}+\frac{(y-k/2)^2}{\left(\frac{\sqrt{5}}{4}k\right)^2}=1.$$

This shows that for $k>0$ the k-contour of f is an ellipse with center $\left(\frac{k}{2},\frac{k}{2}\right)$ and semi-axes $a=\frac{\sqrt{5}}{2}k$ (in x-direction), $b=\frac{\sqrt{5}}{4}k$ (in y-direction), of which the origin $(0,0)$ has been removed. $\boxed{2}$

In particular, all k-contours with $k>0$ contain points arbitrarily close to $(0,0)$, showing again that $\lim_{(x,y)\to(0,0)}f(x,y)$ doesn't exist.

c) We have

$$\begin{aligned}
f_x &= \frac{2x(x+4y)-1(x^2+4y^2)}{(x+4y)^2} = \frac{x^2+8xy-4y^2}{(x+4y)^2},\\
f_y &= \frac{8y(x+4y)-4(x^2+4y^2)}{(x+4y)^2} = \frac{16y^2+8xy-4x^2}{(x+4y)^2},\\
\nabla f &= \left(\frac{x^2+8xy-4y^2}{(x+4y)^2}, \frac{16y^2+8xy-4x^2}{(x+4y)^2}\right), && \boxed{1}\\
\nabla f(1,1) &= \left(\tfrac{1}{5}, \tfrac{4}{5}\right),\\
\nabla f(1,1)\cdot \tfrac{1}{\sqrt{2}}(-1,-1) &= -\tfrac{1}{2}\sqrt{2}.
\end{aligned}$$

$\Longrightarrow$ The slope of G_f at $(1,1)$ in the direction of the origin (SW) is $-\frac{1}{2}\sqrt{2}$, $\boxed{1}$
and the maximal slope of G_f at $(1,1)$ (in the direction of the gradient, or $(1,4)$) is $|\nabla f(1,1)| = \frac{1}{5}\sqrt{17}$. $\boxed{1}$

d) The 1-contour C of f has equation $x^2-x+4y^2-4y=0$. Since it is the 0-contour of $g(x,y)=x^2-x+4y^2-4y$, the tangent to C in (x_0,y_0) has equation

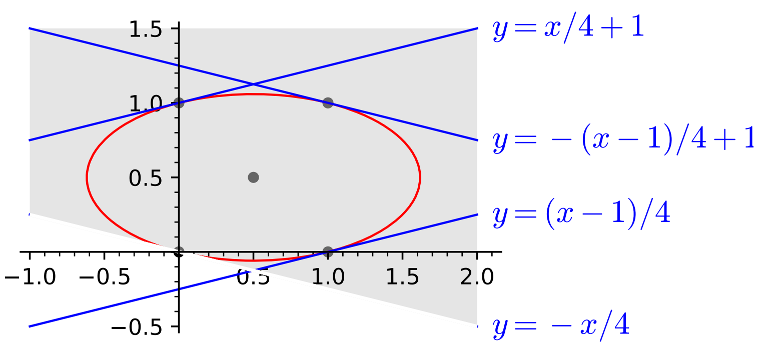
$$\nabla g_x(x_0,y_0)(x-x_0)+\nabla g_y(x_0,y_0)(y-y_0) = (2x_0-1)(x-x_0)+(8y_0-4)(y-y_0)=0.$$

For the indicated points the tangent equations are:

$$\begin{aligned}
(x_0,y_0)&=(1,0): & x-1-4y&=0,\\
(x_0,y_0)&=(0,1): & -x+4(y-1)&=0,\\
(x_0,y_0)&=(1,1): & x-1+4(y-1)&=0. && \boxed{1\tfrac{1}{2}}
\end{aligned}$$

Since $(0,0)$ is the only point of C outside D, the boundary of D, viz. $x+4y=0$, is tangent to C in $(0,0)$. $\boxed{\tfrac{1}{2}}$

For the sketch we use that the semi-axes of C are $\sqrt{5}/2 \approx 1.12$, $b=\sqrt{5}/2\approx 0.56$ ($a\approx 1.1$, $b\approx 0.55$ are also accepted).



$\boxed{1}$

$$\sum_1 = 10+2$$

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2 a) No. A plane E in $\mathbb{R}^3$ has equation $ax+by+cz=d$ for some $a,b,c,d\in\mathbb{R}$ with at least one of a,b,c nonzero. If $\mathbf{r}(t)\in E$ for all $t\in\mathbb{R}$, $a(3t+1)+3bt^2+2ct^3=d$ for all $t\in\mathbb{R}$. This is a cubic/quadratic/linear equation in t (cubic if $c\neq 0$, quadratic if $c=0\wedge b\neq 0$, linear if $b=c=0\wedge a\neq 0$, which can have at most 3/2/1 zeros. $\boxed{1}$

There are other ways to answer this question. For example, since $\mathbf{r}(0)=(1,0,0)$, $\mathbf{r}'(0)=(3,0,0)$, $\mathbf{r}''(0)=(0,6,0)$, cf. b), the osculating plane in $\mathbf{r}(0)$ has equation $z=0$. But there exist points on C with $z\neq 0$, and hence C cannot be planar.

b) The tangent to C in P must be parallel to E and hence its direction orthogonal to the normal vector $(2,4,0)$. Since $\mathbf{r}'(t)=(3,6t,6t^2)$, this is the case iff $(2,4,0)\cdot(3,6t,6t^2)=6+24t=0$, i.e., $t=-1/4$. The corresponding point on C is $P=\mathbf{r}(-1/4)=\left(\frac{1}{4},\frac{3}{16},\frac{1}{32}\right)$. $\boxed{1}$

Since there must exist a point on C minimizing the distance to E (since $t\mapsto d\big(\mathbf{r}(t),E\big)$ is continuous and approaches ∞ for $t\mapsto\pm\infty$) and P is the only candidate, P has the required property.

For any point $P_0\in E$, the distance $d(P,E)$ is equal to the length of the orthogonal projection of $P-P_0$ to $\mathbb{R}(2,4,0)$ Choosing $P_0=\left(\frac{1}{2},0,0\right)$,

$$\left|\frac{\begin{pmatrix}-1/4\\3/16\\1/32\end{pmatrix}\cdot\begin{pmatrix}2\\4\\0\end{pmatrix}}{\begin{pmatrix}2\\4\\0\end{pmatrix}\cdot\begin{pmatrix}2\\4\\0\end{pmatrix}}\begin{pmatrix}2\\4\\0\end{pmatrix}\right|=\frac{1}{4\sqrt{20}}=\frac{1}{8\sqrt{5}}=\frac{\sqrt{5}}{40}. \qquad \boxed{2}$$

c) The arc length function of C, measured from $(1,0,0)=\mathbf{r}(0)$, is

$$s(t)=\int_0^t|\mathbf{r}'(\tau)|\,\mathrm{d}\tau=\int_0^t\sqrt{9+36\tau^2+36\tau^4}\mathrm{d}\tau=\int_0^t 3+6\tau^2\mathrm{d}\tau=3t+2t^3.$$

Since $s(\pm1)=\pm5$, $\mathbf{r}(1)=(4,3,2)$, $\mathbf{r}(-1)=(-2,3,-2)$, we must have $Q=(4,3,2)$. $\boxed{1}$

Since $\mathbf{r}'(t)=(3,6t,6t^2)$, $\mathbf{r}''(t)=(0,6,12t)$ the osculating plane of C in Q is

$$Q+\mathbb{R}\,\mathbf{r}'(1)+\mathbb{R}\mathbf{r}''(1)=\begin{pmatrix}4\\3\\2\end{pmatrix}+\mathbb{R}\begin{pmatrix}3\\6\\6\end{pmatrix}+\mathbb{R}\begin{pmatrix}0\\6\\12\end{pmatrix}=\begin{pmatrix}4\\3\\2\end{pmatrix}+\mathbb{R}\begin{pmatrix}1\\2\\2\end{pmatrix}+\mathbb{R}\begin{pmatrix}0\\1\\2\end{pmatrix}. \quad \boxed{1}$$

Further, we obtain

$$\mathbf{T}(1) = \frac{1}{3}\begin{pmatrix}1\\2\\2\end{pmatrix}, \qquad \boxed{1}$$

$$\mathbf{N}(1) = \lambda\left((0,1,2) - \frac{(0,1,2)\cdot(1,2,2)}{|(1,2,2)|^2}(1,2,2)\right) \qquad (\lambda > 0)$$

$$= \lambda\left(\begin{pmatrix}0\\1\\2\end{pmatrix} - \frac{6}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}\right) = \lambda\begin{pmatrix}-2/3\\-1/3\\2/3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-2\\-1\\2\end{pmatrix}, \qquad \boxed{1}$$

$$\kappa(1) = \frac{|\mathbf{r}'(1)\times\mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \frac{|(3,6,6)\times(0,6,12)|}{|(3,6,6)|^3} = \frac{18\,|(1,2,2)\times(0,1,2)|}{27\,|(1,2,2)|^3}$$

$$= \frac{2\,|(2,-2,1)|}{3\cdot 27} = \frac{2}{27}. \qquad \boxed{1}$$

$\Longrightarrow$ The radius of the osculating circle of C in Q is $\frac{27}{2}$, $\boxed{\tfrac{1}{2}}$
and its center is

$$Q + \tfrac{27}{2}\,\mathbf{N}(1) = \begin{pmatrix}4\\3\\2\end{pmatrix} + \frac{9}{2}\begin{pmatrix}-2\\-1\\2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}-10\\-3\\22\end{pmatrix}. \qquad \boxed{\tfrac{1}{2}}$$

Finally, we have

$$\mathbf{B}(1) = \mathbf{T}(1)\times\mathbf{N}(1) = \frac{1}{9}\begin{pmatrix}1\\2\\2\end{pmatrix}\times\begin{pmatrix}-2\\-1\\2\end{pmatrix} = \frac{1}{9}\begin{pmatrix}6\\-6\\3\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2\\-2\\1\end{pmatrix}. \qquad \boxed{1}$$

$$\sum_{2} = 10 + 1$$

General Remarks:

$$\sum_{\text{Midterm 2}} = 20 + 3$$
