Tutorial-4

Q1) for each of the following recurrences, give an expression for the runtime T(n) if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply:

(1) $T(n) = 3T(n/2) + n^2$

→ According to Master's Theorem,

if $T(n) = aT(n/b) + f(n)$; $a \geq 1, b > 1$

and $c = \log_b a$, then if:

- $f(n) < n^c \Rightarrow T(n) = \Theta(n^c)$
- $f(n) = n^c \Rightarrow T(n) = \Theta(n^c \log n)$
- $f(n) > n \Rightarrow T(n) = \Theta(f(n))$

Here, $a = 3, b = 2$

$c = \log_2 3 \approx 1.58$

$f(n) = n^2, \; n^c = n^{1.58}$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \Theta(n^2)}$

(2) $T(n) = 4T(n/2) + n^2$

→ Here, $a = 4, b = 2$

$c = \log_2 4 = 2$

$f(n) = n^2, n^c = n^2$

$\because f(n) = n^c \Rightarrow \boxed{T(n) = \Theta(n^2 \log n)}$

(3) $T(n) = T(n/2) + 2^n$

→ Here, $f(n) = 2^n$ which is not a polynomial.

$\therefore$ Master's Theorem does not apply.

4. $T(n) = 2^n T(n/2) + n^n$

→ Here, $a = 2^n$ which is not a constant.

$\therefore$ Master's Theorem does not apply.

5. $T(n) = 16T(n/4) + n$

→ Here, $a = 16,\ b = 4$

$c = \log_4 16 = 2$

$f(n) = n,\quad n^c = n^2$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \Theta(n^2)}$

6. $T(n) = 2T(n/2) + n\log n$

→ According to extended Masters Theorem,

If $T(n) = aT(n/b) + \Theta(n^k \log^p n)$; $a \geq 1$; $b > 1$, $k \geq 0$; $p \in \mathbb{R}$

then if :

- $a > b \Rightarrow$ $T(n) = \Theta(n^{\log_b a})$
- $a = b^k \Rightarrow$ Then if :
  - ✓ $p > -1 \Rightarrow T(n) = \Theta(n^{\log_b a} \log^{p+1} n)$
  - ✓ $p = -1 \Rightarrow T(n) = \Theta(n^{\log_b a} \log\log n)$
  - ✓ $p < -1 \Rightarrow T(n) = \Theta(n^{\log_b a})$
- $a < b^k$, then if :
  - ✓ $p \geq 0 \Rightarrow T(n) = \Theta(n^k \log^p n)$
  - ✓ $p < 0 \Rightarrow T(n) = \Theta(n^k)$

Here, $a = 2,\ b = 2,\ k = 1,\ p = 1$

Now, $b^k = 2^1 = 2 = a$

and $p > -1$

$\Rightarrow T(n) = \Theta(n^{\log_2 2} \log^{1+1} n)$

$\Rightarrow \boxed{T(n) = \Theta(n \log^2 n)}$

7 $T(n) = 2T(n/2) + n/\log n$

$\rightarrow$ using extended Master's Theorem,

$T(n) = aT(n/b) + \Theta(n^k \log^k n)$

Here, $a = 2, b = 2, k = 1, p = -1$

Now, $b^k = 2^1 = 2 = a$

and $p = -1$

$\Rightarrow T(n) = \Theta(n^{\log_2 2} \log \log n)$

$\Rightarrow T(n) = \Theta(n \log \log n)$

$\Rightarrow \boxed{T(n) = \Theta(n \log \log n)}$

8 $T(n) = 2T(n/4) + n^{0.51}$

$\rightarrow$ Here, $a = 2, b = 4$

$c = \log_4 2 = 0.5$

$f(n) = n^{0.51}, n^c = n^{0.5}$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \Theta(n^{0.51})}$

9 $T(n) = 0.5T(n/2) + 1/n$

$\rightarrow$ Here, $a < 1$

$\therefore$ Master's Theorem does not apply.

10 $T(n) = 16T(n/4) + n!$

$\rightarrow$ Here, $a = 16, b = 4$

$c = \log_4 16 = 2$

$f(n) = n!, n^c = n^2$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \Theta(n!)}$

11. $T(n) = 4T(n/2) + \log n$

→ Using extended Masters Theorem,

$$T(n) = aT(n/b) + \Theta(n^k \log^p n)$$

Here, $a=4, b=2, k=0, p=1$

Now, $b^k = 2^0 = 1 < a$

$\Rightarrow T(n) = \Theta(n^{\log_2 4})$

$\Rightarrow \boxed{T(n) = \Theta(n^2)}$

12. $T(n) = \sqrt{n}\, T(n/2) + \log n$

→ Here, $a = \sqrt{n}$ which is not a constant.

$\therefore$ Masters Theorem does not apply.

13. $T(n) = 3T(n/2) + n$

→ Here, $a = 3, b = 2$

$c = \log_2 3 = 1.58$

$f(n) = n,\ n^c = n^{1.58}$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \Theta(n^{1.58})}$

14. $T(n) = 3T(n/3) + \sqrt{n}$

→ Here, $a = 3, b = 3$

$c = \log_3 3 = 1$

$f(n) = \sqrt{n},\ n^c = n$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \Theta(n)}$

15. $T(n) = 4T(n/2) + kn$

→ Here, $a = 4, b = 2$

$c = \log_2 4 = 2$

$f(n) = kn,\ n^c = n^2$

$\because f(n) < n^c \Rightarrow \boxed{T(n) = \Theta(n^2)}$

(16) $T(n) = 3T(n/4) + n\log n$

→ Using extended Masters Theorem,

$T(n) = aT(n/b) + \theta(n^k \log^p n)$

Here, $a = 3, b = 4, k = 1, p = 1$

Now, $b^k = 4^1 = 4 > a$

and $p \geq 0$

$\Rightarrow T(n) = \theta(n^1 \log^1 n)$

$\Rightarrow \boxed{T(n) = \theta(n\log n)}$

(17) $T(n) = 3T(n/3) + n/2$

→ Here, $a = 3, b = 3$

$c = \log_3 3 = 1$

$f(n) = n/2 \approx n, n^c = n$

$\because f(n) = n^c \Rightarrow \boxed{T(n) = \theta(n\log n)}$

(18) $T(n) = 6T(n/3) + n^2\log n$

→ using extended Masters Theorem,

$T(n) = aT(n/b) + \theta(n^k \log^p n)$

Here, $a = 6, b = 3, k = 2, p = 1$

Now, $b^k = 3^2 = 9 > a$

and $p \geq 0$

$\Rightarrow T(n) = \theta(n^2 \log^1 n)$

$\Rightarrow \boxed{T(n) = \theta(n^2\log n)}$

(19) $T(n) = 4T(n/2) + n/\log n$.

→ Using extended Masters Theorem,

$T(n) = aT(n/b) + \theta(n^k \log^p n)$

Here, $a = 4, b = 2, k = 1, p = -1$

Now, $b^k = 2^1 = 2 < a \Rightarrow T(n) = \theta(n^{\log_2 4}) \Rightarrow \boxed{T(n) = \theta(n^2)}$

20 $T(n) = 64T(n/8) - n^2 \log n$

→ Here, $f(n) = -n^2 \log n$ which is negative.

∴ Masters Theorem does not apply.

21 $T(n) = 7T(n/3) + n^2$

→ Here, $a = 7, b = 3$

$c = \log_3 7 = 1.77$

$f(n) = n^2, \quad n^c = n^{1.77}$

$\because f(n) > n^c \Rightarrow \boxed{T(n) = \Theta(n^2)}$

22 $T(n) = T(n/2) + n(2 - \cos n)$

→ Here, the regularity condition is violated.

$\{a f(n/b) \leq c f(n);\ c < 1\}$

∴ Masters Theorem does not apply.