

Computation Note for 2D Transport Equation

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1 Transport Equation

$$\theta \cdot \nabla u + \mu u = \mu_s \int k(\theta) u(x, y, \theta) d\theta + f \quad (1)$$

with vanishing boundary condition. To simplify the computing, $k(\theta)$ is taken as a constant and

$$\int k(\theta) d\theta = 1 \quad (2)$$

2 Source Iteration

Solving transport equation with source iteration:

$$\theta \cdot \nabla u^{(l+1)} + \mu u^{(l+1)} = \mu_s \int k(\theta) u^{(l)}(x, y, \theta) d\theta + f \quad (3)$$

with initial guess $u^{(0)} = 0$.

2.1 FEM Approach

2.2 Fourier Approach

Consider f as finite banded source, represented as

$$f = \sum_{-N}^N f_k \exp(ik\theta) \quad (4)$$

Consider the solution with form

$$u = \sum_{-N}^N u_k \exp(ik\theta) = \sum u_k \psi_k \quad (5)$$

Thus

$$\frac{1}{2}(\psi_1 + \psi_{-1}, -j(\psi_1 - \psi_{-1})) \cdot (\sum \psi_k \nabla u_k) + \mu \sum \psi_k u_k = \mu_s u_0 + \sum \psi_k f_k \quad (6)$$

simplify as

$$\sum \frac{1}{2}(\psi_{k+1} + \psi_{k-1}, -j(\psi_{k+1} - \psi_{k-1})) \cdot \nabla u_k + \mu \sum \psi_k u_k = \mu_s u_0 + \sum \psi_k f_k \quad (7)$$

Consider coefficient of ψ_k , when $k \neq 0$

$$\frac{1}{2} \frac{\partial(u_{k+1} + u_{k-1})}{\partial x} + \frac{1}{2} j \frac{\partial(u_{k+1} - u_{k-1})}{\partial y} + \mu u_k = f_k \quad (8)$$

when $k = 0$

$$\frac{1}{2} \frac{\partial(u_{k+1} + u_{k-1})}{\partial x} + \frac{1}{2} j \frac{\partial(u_{k+1} - u_{k-1})}{\partial y} + \mu u_k = \mu_s u_0 + f_k \quad (9)$$

Thus take $U = \{u_k\}$ as a vector of functions.

$$AU_x + BU_y + \mu U = \mu_s u_0 E_0 + F \quad (10)$$

Port above equation into iteration

$$AU_x^{(l+1)} + BU_y^{(l+1)} + \mu U^{(l+1)} = \mu u_0^{(l)} E_0 + F \quad (11)$$

Here one can adjust the dimension of matrices A and B by choosing the expected band width.

2.2.1 FEM continue

$$\int_{\Omega} AU_x V + BU_y V + \mu UV = E_0 \int \mu_s u_0 v + FV \quad (12)$$

On the targeted mesh, one can solve the above FEM problem to get all modes of U evaluated on all mesh points. The size of this problem is size of band times size of mesh.

2.2.2 Improvement

On a fine mesh will make the problem extremely unsolvable. If we know F is sparse or only consists on some smooth modes, or only a few high frequency modes, then we simply solve our problem near the corresponding k .

3 Recovering F for point sources

Since F will contain all modes, then it is not realistic to use Fourier approach. However, if μ and μ_s are already known, then using source iteration with full boundary data will tell the location of point sources effectively.

3.0.3 Integrated data

Consider integrated data on boundary $H(x) = \int_{\theta} u(x, \theta) d\theta$.