Numerical Approach to Cauchy Elliptic Problem

Yimin Zhong

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1 Brief Intro

The aimed Cauchy problem stated as

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi\frac{\mathrm{d}u}{\mathrm{d}x}\right)+\frac{\mathrm{d}}{\mathrm{d}y}\left(\phi\frac{\mathrm{d}u}{\mathrm{d}y}\right)=0 \tag{1}$$

with Cauchy condition as

$$u = h \quad \text{on} \quad \Gamma_1 \tag{2}$$
$$\nabla u\cdot\mathbf{n} = g \quad \text{on} \quad \Gamma_1 \tag{3}$$

where Γ_1 is part of the boundary of the domain Ω.

1.1 Uniqueness & Stability

It is well known the problem is ill-posed, the solution is extreme sensitive to small perturbation on data. The famous example is

EXAMPLE 1.1.

$$\triangle u = 0 \tag{4}$$

with Cauchy data as

$$u(x,0) = 0 \tag{5}$$
$$u_y(x,0) = A_n \sin nx \tag{6}$$

for all $(x,y)\in\mathbb{R}\times\mathbb{R}^+$. *Since the solution is given as*

$$u_n(x,y)=\frac{A_n}{n}\sin nx\sinh ny\to\infty \tag{7}$$

as $n\to\infty$.

2 Numerical Illustration

There are multiple methods for solving Cauchy Problem.

2.1 Spectral

2.2 Inverse Cauchy Problem

We restate our problem in an optimization sense. Regard the boundary as our data and we impose some other boundary-condition as variables to recover the given data. So we impose

$$u = \widetilde{h} \quad \text{on} \quad \Gamma_2 \tag{8}$$

$$\nabla u \cdot \mathbf{n} = g \quad \text{on} \quad \Gamma_1 \tag{9}$$

where $\Gamma_2 = \partial\Omega - \Gamma_1$. We set the objective functional as

$$F(\widetilde{h}) = \min_{\widetilde{h} \in V} \|u(x) - h\|_{\Gamma_1}^2 + \text{Regularization} \tag{10}$$

Since ψ, ϕ are positive functions, we can define a norm on Γ_1 as

$$\| \cdot \|_{\Gamma_1} = \langle \phi * \cdot, \cdot \rangle$$

2.2.1 BFGS/LBFGS,CG,GMRES

For **BFGS**, we need to find the general gradient of functional F, by the help of Green's formula, we can do it in this way. Assuming there is no regularization term here.

$$\frac{\partial F}{\partial \widetilde{h}} = \int_{\Gamma_1} \phi \frac{\partial u}{\partial \widetilde{h}} (u(x) - h)) \mathrm{d}s \tag{11}$$

Well, consider another PDE **w.r.t** v.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi \frac{\mathrm{d}v}{\mathrm{d}x}\right) + \frac{\mathrm{d}}{\mathrm{d}y}\left(\phi \frac{\mathrm{d}v}{\mathrm{d}y}\right) = 0 \tag{12}$$

with boundary condition as

$$v = 0 \quad \text{on} \quad \Gamma_2 \tag{13}$$

$$\frac{\partial u}{\partial \mathbf{n}} = u - h \quad \text{on} \quad \Gamma_1 \tag{14}$$

Then by Green's formula

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}\left(\psi \frac{\mathrm{d}v}{\mathrm{d}x}\right) u + \frac{\mathrm{d}}{\mathrm{d}y}\left(\phi \frac{\mathrm{d}v}{\mathrm{d}y}\right) u = 0 \tag{15}$$

$$\int_\Omega \frac{\mathrm{d}}{\mathrm{d}x}\left(\psi \frac{\mathrm{d}u}{\mathrm{d}x}\right) v + \frac{\mathrm{d}}{\mathrm{d}y}\left(\phi \frac{\mathrm{d}u}{\mathrm{d}y}\right) v = 0 \tag{16}$$