# Computation Note for 2D Transport Equation

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## 1 Transport Equation

$$\theta \cdot \nabla u + \mu u = \mu_s \int k(\theta) u(x, y, \theta) \mathrm{d}\theta + f \tag{1}$$

with vanishing boundary condition. To simplify the computing, $k(\theta)$ is taken as a constant and

$$\int k(\theta) \mathrm{d}\theta = 1 \tag{2}$$

## 2 Source Iteration

Solving transport equation with source iteration:

$$\theta \cdot \nabla u^{(l+1)} + \mu u^{(l+1)} = \mu_s \int k(\theta) u^{(l)}(x, y, \theta) \mathrm{d}\theta + f \tag{3}$$

with initial guess $u^{(0)} = 0$.

### 2.1 FEM Approach

Using FEM and source iteration, then we need to discretize the angular space. On each node, there is a mesh on angular space, see the integration of scattering part. However, for iteration, updating all the value of all nodes and all angles will be a huge problem, since the ordering of nodes now make difference. This is caused by angular mesh, and will cause ray effect.

## 2.2 Fourier Approach

Consider $f$ as finite banded source, represented as

$$f = \sum_{-N}^{N} f_k \exp(\mathrm{i}k\theta) \tag{4}$$

Consider the solution with form

$$u = \sum_{-N}^{N} u_k \exp(\mathrm{i}k\theta) = \sum u_k \psi_k \tag{5}$$

Thus

$$\frac{1}{2}(\psi_1 + \psi_{-1}, -j(\psi_1 - \psi_{-1})) \cdot (\sum \psi_k \nabla u_k) + \mu \sum \psi_k u_k = \mu_s u_0 + \sum \psi_k f_k \tag{6}$$

simplify as

$$\sum \frac{1}{2}(\psi_{k+1} + \psi_{k-1}, -j(\psi_{k+1} - \psi_{k-1})) \cdot \nabla u_k + \mu \sum \psi_k u_k = \mu_s u_0 + \sum \psi_k f_k \tag{7}$$

Consider coefficient of $\psi_k$, when $k \neq 0$

$$\frac{1}{2}\frac{\partial(u_{k+1} + u_{k-1})}{\partial x} + \frac{1}{2}j\frac{\partial(u_{k+1} - u_{k-1})}{\partial y} + \mu u_k = f_k \tag{8}$$

when $k = 0$

$$\frac{1}{2}\frac{\partial(u_{k+1} + u_{k-1})}{\partial x} + \frac{1}{2}j\frac{\partial(u_{k+1} - u_{k-1})}{\partial y} + \mu u_k = \mu_s u_0 + f_k \tag{9}$$

Thus take $U = \{u_k\}$ as a vector of functions.

$$AU_x + BU_y + \mu U = \mu_s u_0 E_0 + F \tag{10}$$

Port above equation into iteration

$$AU_x^{(l+1)} + BU_y^{(l+1)} + \mu U^{(l+1)} = \mu u_0^{(l)} E_0 + F \tag{11}$$

Here one can adjust the dimension of matrices $A$ and $B$ by choosing the expected band width.

### 2.2.1 FEM continue

$$\int_\Omega AU_x V + BU_y V + \mu UV = E_0 \int \mu_s u_0 v + FV \tag{12}$$

On the targeted mesh, one can solve the above FEM problem to get all modes of $U$ evaluated on all mesh points. The size of this problem is size of band times size of mesh.

#### 2.2.2 Improvement

On a fine mesh will make the problem extremely unsolvable. If we know $F$ is sparse or only consists on some smooth modes, or only a few high frequency modes, then we simply solve our problem near the corresponding $k$.

# 3 Recovering $F$ for point sources

Since $F$ will contain all modes, then it is not realistic to use Fourier approach. However, if $\mu$ and $\mu_s$ are already known, then using source iteration with full boundary data will tell the location of point sources effectively.

#### 3.0.3 Integrated data

Consider integrated data on boundary

$$H(x) = \int_\theta u(x,\theta)\mathrm{d}\theta$$

And we are going to recover the point sources from given $H(x)$ on boundary, with knowledge that input data is zero. So all output is from inside source. We first show the result on uniqueness.

## 3.1 Some Uniqueness

Consider the equation on space $\Omega V = \Omega \times V$, where $V = \{|\mathbf{v}| = 1\}$, and $\Omega \subset \mathbb{R}^n$ is compact and convex with smooth boundary.

$$\mathbf{v}\cdot\nabla u(\mathbf{x},\mathbf{v}) + \mu u = \mu_s \int_{\Omega_x V} k(x,\mathbf{v}',\mathbf{v})u(\mathbf{x},\mathbf{v}')\mathrm{d}\omega + f \tag{13}$$

and boundary condition given as

$$u|_{\Omega\times V^-} = 0 \tag{14}$$

data collected as for all $\mathbf{x} \in \partial\Omega$

$$I(\mathbf{x}) = \int_V |u(\mathbf{x},\mathbf{v})|^2\mathrm{d}\omega \tag{15}$$

Here $\Omega_x V$ is the tangent bundle space at $x$, can be taken as $T_x\Omega V$.

### 3.1.1 Estimation

So from the above setting $\Omega$ is $n$-dimensional and $V$ is one dimension less than $\Omega$. First we consider the transport operator.

$$Tu = \mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v})$$

Let's make some analogue with Poincare inequality here and then generalize the inequality.

$$\int_{\Omega V} |u|^2 \mathrm{d}\mathbf{x}\mathrm{d}\omega \leq C(\Omega) \int_{\Omega V} |Tu|^2 \mathrm{d}\mathbf{x}\mathrm{d}\omega \tag{16}$$

**Proof 3.1.1** *First we introduce transverse time $\tau^+(\mathbf{x}, \mathbf{v})$ and $\tau^-(\mathbf{x}, \mathbf{v})$ with usual definition. Also we notice that the geodesic trace $\gamma_{\mathbf{x},\mathbf{v}}(t)$ is a straight line with initial constraints on $\partial\Omega \times V^-$:*

$$\gamma_{\mathbf{x},\mathbf{v}}(0) = \mathbf{x} \tag{17}$$
$$\dot{\gamma}_{\mathbf{x},\mathbf{v}}(0) = \mathbf{v} \tag{18}$$