Computation Note for 2D Transport Equation

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1 Transport Equation

$$\theta \cdot \nabla u + \mu u = \mu_s \int k(\theta) u(x, y, \theta) \mathrm{d}\theta + f \tag{1}$$

with vanishing boundary condition. To simplify the computing, $k(\theta)$ is taken as a constant and

$$\int k(\theta) \mathrm{d}\theta = 1 \tag{2}$$

2 Source Iteration

Solving transport equation with source iteration:

$$\theta \cdot \nabla u^{(l+1)} + \mu u^{(l+1)} = \mu_s \int k(\theta) u^{(l)}(x, y, \theta) \mathrm{d}\theta + f \tag{3}$$

with initial guess $u^{(0)} = 0$.

2.1 FEM Approach

Using FEM and source iteration, then we need to discretize the angular space. On each node, there is a mesh on angular space, see the integration of scattering part. However, for iteration, updating all the value of all nodes and all angles will be a huge problem, since the ordering of nodes now make difference. This is caused by angular mesh, and will cause ray effect.

2.2 Fourier Approach

Consider f as finite banded source, represented as

$$f = \sum_{-N}^{N} f_k \exp(\mathrm{i}k\theta) \tag{4}$$

Consider the solution with form

$$u = \sum_{-N}^{N} u_k \exp(\mathrm{i}k\theta) = \sum u_k \psi_k \tag{5}$$

Thus

$$\frac{1}{2}(\psi_1 + \psi_{-1}, -j(\psi_1 - \psi_{-1})) \cdot (\sum \psi_k \nabla u_k) + \mu \sum \psi_k u_k = \mu_s u_0 + \sum \psi_k f_k \tag{6}$$

simplify as

$$\sum \frac{1}{2}(\psi_{k+1} + \psi_{k-1}, -j(\psi_{k+1} - \psi_{k-1})) \cdot \nabla u_k + \mu \sum \psi_k u_k = \mu_s u_0 + \sum \psi_k f_k \tag{7}$$

Consider coefficient of ψ_k, when $k \neq 0$

$$\frac{1}{2}\frac{\partial(u_{k+1} + u_{k-1})}{\partial x} + \frac{1}{2}j\frac{\partial(u_{k+1} - u_{k-1})}{\partial y} + \mu u_k = f_k \tag{8}$$

when $k = 0$

$$\frac{1}{2}\frac{\partial(u_{k+1} + u_{k-1})}{\partial x} + \frac{1}{2}j\frac{\partial(u_{k+1} - u_{k-1})}{\partial y} + \mu u_k = \mu_s u_0 + f_k \tag{9}$$

Thus take $U = \{u_k\}$ as a vector of functions.

$$AU_x + BU_y + \mu U = \mu_s u_0 E_0 + F \tag{10}$$

Port above equation into iteration

$$AU_x^{(l+1)} + BU_y^{(l+1)} + \mu U^{(l+1)} = \mu u_0^{(l)} E_0 + F \tag{11}$$

Here one can adjust the dimension of matrices A and B by choosing the expected band width.

2.2.1 FEM continue

$$\int_\Omega AU_x V + BU_y V + \mu UV = E_0 \int \mu_s u_0 v + FV \tag{12}$$

On the targeted mesh, one can solve the above FEM problem to get all modes of U evaluated on all mesh points. The size of this problem is size of band times size of mesh.

2.2.2 Improvement

On a fine mesh will make the problem extremely unsolvable. If we know F is sparse or only consists on some smooth modes, or only a few high frequency modes, then we simply solve our problem near the corresponding k.

3 Recovering F for point sources

Since F will contain all modes, then it is not realistic to use Fourier approach. However, if μ and μ_s are already known, then using source iteration with full boundary data will tell the location of point sources effectively.

3.0.3 Integrated data

Consider integrated data on boundary

$$H(x) = \int_\theta u(x,\theta)\mathrm{d}\theta$$

And we are going to recover the point sources from given $H(x)$ on boundary, with knowledge that input data is zero. So all output is from inside source. We first show the result on uniqueness.

3.1 Some Uniqueness

Consider the equation on space of unit tangent bundle $S\Omega = \Omega \times V$, where $V = \{|\mathbf{v}| = 1\}$, and $\Omega \subset \mathbb{R}^n$ is compact and convex with smooth boundary.

$$\mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v}) + \mu u = \mu_s \int_{\Omega_x V} k(x, \mathbf{v}', \mathbf{v}) u(\mathbf{x}, \mathbf{v}')\mathrm{d}\omega + f \tag{13}$$

and boundary condition given as

$$u|_{\partial\Omega \times V^-} = 0 \tag{14}$$

data collected as for all $\mathbf{x} \in \partial\Omega$

$$I(\mathbf{x}) = \int_V |u(\mathbf{x}, \mathbf{v})|^2 \mathrm{d}\omega \tag{15}$$

3.1.1 Estimation

So from the above setting Ω is n-dimensional and V is one dimension less than Ω. First we consider the transport operator.

$$Tu = \mathbf{v} \cdot \nabla u(\mathbf{x}, \mathbf{v})$$

Let's make some analogue with Poincare inequality here and then generalize the inequality.

Theorem 3.1.1 (Poincare Inequality) *If given $u|_{\partial\Omega\times V^-} = 0$, then we have*

$$\int_{S\Omega} \lambda |u|^2 \mathrm{d}\Sigma \leq \lambda_0 \int_{S\Omega} |Tu|^2 \mathrm{d}\Sigma \tag{16}$$

To simplify the proof, we claim the following lemma.

Lemma 3.1.1 *Consider a function $\phi \in C(S\Omega)$ with support as $\Omega_\phi V$. If*

$$\sup_{(\mathbf{x},\mathbf{v})\Omega_\phi V} |T\phi(\mathbf{x}, \mathbf{v})| < \infty \tag{17}$$

And $\phi|_{\partial\Omega\times V^-} = 0$, then

$$\int_{S\Omega} \lambda |\phi|^2 \mathrm{d}\Sigma \leq \lambda_0 \int_{\Omega_\phi V} |T\phi|^2 \mathrm{d}\Sigma \tag{18}$$

Proof 3.1.1 *First we introduce transverse time $\tau^+(\mathbf{x}, \mathbf{v})$ and $\tau^-(\mathbf{x}, \mathbf{v})$ with usual definition. Also we notice that the geodesic defined as $\gamma_{\mathbf{x},\mathbf{v}}(t)$ with initial constraints on $\partial\Omega \times V^-$:*

$$\gamma_{\mathbf{x},\mathbf{v}}(0) = \mathbf{x} \tag{19}$$
$$\dot{\gamma}_{\mathbf{x},\mathbf{v}}(0) = \mathbf{v} \tag{20}$$

Consider cylinder domain $\Theta = \{(\mathbf{x}, \mathbf{v}, t)|0 \leq t \leq \tau^+(\mathbf{x}, \mathbf{v})\}$ on $\partial\Omega \times V^-$. And consider mapping β as geodesic flow $\beta(\mathbf{x}, \mathbf{v}, t) = (\gamma_{\mathbf{x},\mathbf{v}}(t), \dot{\gamma}_{\mathbf{x},\mathbf{v}}(t))$, this maps cylinder domain Θ diffeomorphically onto $\Omega \times V$. Therefore,

$$\int_{S\Omega} \lambda |\phi|^2 \mathrm{d}\Sigma = \int_{\Theta} (\lambda \circ \beta) |\phi \circ \beta|^2 \beta^*(\mathrm{d}\Sigma) \tag{21}$$

Here β^* is the pull back.

Remark 3.1.1 *$\beta(\cdot,\cdot,t)$ as a geodesic flow on unit tangent bundle maps $S\Omega$ to $S\Omega$. Moreover, geodesic flow preserves symplectic form, hence the volume form $\mathrm{d}\Sigma$.*

The vector field $\dfrac{\partial}{\partial t}$ preserves the volume form $\beta^*(\mathrm{d}\Sigma)$. Thus $\beta^*(\mathrm{d}\Sigma) = d\sigma dt$ for some volume form $\mathrm{d}\sigma$ on $\partial\Omega \times V^-$. Then

$$\int_{S\Omega} \lambda|\phi|^2 \mathrm{d}\Sigma = \int_{\partial\Omega\times V^-} \mathrm{d}\sigma \int_0^{\tau^+} (\lambda \circ \beta)|\phi \circ \beta|^2 \mathrm{d}t \tag{22}$$

By the same way, and observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi \circ \beta) = \frac{\mathrm{d}}{\mathrm{d}t}(\phi(\gamma_{x,v}(t), \dot{\gamma}_{x,v}(t))) = \nabla\phi \cdot \dot{\gamma} = T(\phi) \circ \beta \tag{23}$$

$$\int_{\Omega_\phi V} |T\phi|^2 \mathrm{d}\Sigma = \int_{\Theta_\phi} |\frac{\partial}{\partial t}(\phi \circ \beta)|^2 \mathrm{d}t\mathrm{d}\sigma \tag{24}$$

Where Θ_ϕ is the corresponding support. Compare Eq:22 and Eq:24, we only have to prove:

$$\int_{\partial\Omega\times V^-} \mathrm{d}\sigma \int_0^{\tau^+} (\lambda \circ \beta)|\phi \circ \beta|^2 \mathrm{d}t \leq \lambda_0 \int_{\Theta_\phi} |\frac{\partial}{\partial t}(\phi \circ \beta)|^2 \mathrm{d}t\mathrm{d}\sigma \tag{25}$$

Then Cauchy Inequality will be enough.