Recovery of transport equation coefficients

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2 Mathematical Formulation

We first consider transport equation with scattering known.

$$
\begin{aligned}
v \cdot \nabla u + (\sigma_a + \sigma_s)u &= \sigma_s \langle u \rangle \quad \text{in } \Omega && (1)\\
u &= g \quad \text{on } \partial\Omega \times V^- && (2)
\end{aligned}
$$

Given data as $H(x) = \sigma_a \langle u \rangle$ for recovering σ_a only or $H(x) = \gamma\sigma_a\langle u \rangle$ for recovering γ and σ_a both.

Definition 1. *Function E as*

$$E(x, v, \sigma) = \int_0^{\tau(x,v)} \exp\Big(-\sigma(x - \tau(x, v)v + sv)\Big)\mathrm{d}s$$

Corresponding operator as $I : C^\infty(+; \partial\Omega \times V^-) \to C^\infty(\Omega \times V)$ such that

$$Ig = g(x - \tau(x, v)v, v) \int_0^{\tau(x,v)} \exp\Big(-\sigma(x - \tau(x, v)v + sv)\Big)\mathrm{d}s$$

and define norm on non-negative function space $C^\infty(+; \partial\Omega \times V^-)$ as

$$\|g\| = \max_{x,v} \|g(x - \tau(x, v)v, v)\| \tag{3}$$

and norm on $C^\infty(\Omega \times V)$ is

$$\|f\| = \max_{x,v} \|f(x, v)\| \tag{4}$$

Definition 2. *Operator $R : C^\infty(\Omega) \to C^\infty(\Omega \times V)$ as*

$$R(f(x))(x, v) = \int_0^{\tau(x,v)} \frac{E(x, v, \sigma)}{E(x - \tau(x, v)v + sv, v, \sigma)} f(x - \tau(x, v)v + sv)\mathrm{d}s$$

For short, we simply write

$$E(x - \tau(x, v)v + sv, v, \sigma) = E(s) \tag{5}$$

Definition 3. *Operator $S : C^\infty(\Omega \times V) \to C^\infty(\Omega)$ as averaging operator on angular space.*

$$S(g(x, v))(x) = \frac{1}{|V|} \int_V g(x, v)\mathrm{d}v$$

Definition 4. *If a function $g \in C^\infty(+; \partial\Omega \times V^-)$ belongs to function class $G(h, \sigma)$, then*

$$SIh = SIg \tag{6}$$

We say two boundary conditions are the same if they belongs to the same class.

Remark 1. *A function $g \in C^\infty(+; \partial\Omega \times V^-)$ can belong to multiple classes with different σ and represent element.*

Corollary 1. *By using Hölder inequality, we can obtain a coarse estimate,*

$$\|R\|_{L^\infty} \leq \operatorname{diam}(\Omega)$$

and

$$\|SI\|_{L^\infty} \leq \operatorname{diam}(\Omega)$$

Also we have a trivial estimate

$$\|SIg\| \geq \min_{x,v} \|g(x - \tau(x,v)v, v)\| \exp(-\tau \max_x \sigma)$$

Corollary 2. *The solution to the (1) satisfies*

$$u = Ig + R(\sigma_s Su)$$

and $\psi = Su$ satisfies

$$\psi = S(Ig) + SR\sigma_s\psi$$

Theorem 1. *Averaged solution operator for (1) is*

$$Su = (I - SR\sigma_s)^{-1} S(Ig) = Q^{-1} SIg \tag{7}$$

when $Q = (I - SR\sigma_s)$ is invertible, and this is always true when

$$\operatorname{diam}(\Omega)\|\sigma_s\| \leq 1$$

Proof. From corollary 2, this is trivial. $\square$

Theorem 2. *Consider equation (1), if $\sigma_s = 0$ and data is $f = \langle u \rangle$, not $\sigma_a\langle u \rangle$, then σ_a can be recovered when source is beam source $g = g(x)\delta(v - v')$.*

Proof. This is trivial. □

Theorem 3. *Consider equation (1), if $\sigma_s = 0$ and data is $f = \langle u \rangle$, not $\sigma_a\langle u\rangle$, then σ_a can be uniquely recovered when source is beam combination $g = \sum_{i=1}^n g_i(x)\delta(v - v_i)$, where for any two directions v_k and v_l, they have common incoming source point. i.e. there exists a open set P on $\partial\Omega$, such that $\{v_1, \cdots, v_n\} \subset V^-(P)$.*

Proof. It is easy to show

$$f = \sum_i^n g_i(x - \tau_i v_i) \exp\Big(- \int_0^{\tau_i} \sigma(x - \tau_i v_i + s v_i) \mathrm{d}s \Big) \tag{8}$$

take Frechét derivative w.r.t σ,

$$\frac{\partial f}{\partial \sigma}(\psi) = \sum_i^n -g_i(x - \tau_i v_i) E(\tau_i, v_i) J(\tau_i, v_i) \tag{9}$$

where

$$J(\tau_i, v_i) = \int_0^{\tau_i} \psi(x - \tau_i v_i + s v_i) \mathrm{d}s \tag{10}$$

$$E(\tau_i, v_i) = \exp\Big(- \int_0^{\tau_i} \sigma(x - \tau_i v_i + s v_i) \mathrm{d}s \Big) \tag{11}$$

If putting $\dfrac{\partial f}{\partial \sigma}(\psi) = 0$, we want to show only $\psi = 0$ can make it.
Denote $K(\tau) = \dfrac{\partial f}{\partial \sigma}(\psi)$, and consider directional derivative along v_k, then

$$\begin{aligned} \frac{\partial K}{\partial \tau_k} &= \sum_{i=1}^n -\frac{\partial g_i(x - \tau_i v_i)}{\partial \tau_k} E_i J_i - g_i \frac{\partial E_i}{\partial \tau_k} J_i - g_i E_i \frac{\partial J_i}{\partial \tau_k} \qquad (12) \\ &= \sum_{i=1}^n -\Big(\frac{\partial g_i(x - \tau_i v_i)}{\partial \tau_k} E_i + g_i \frac{\partial E_i}{\partial \tau_k} \Big) J_i - g_i E_i \left(\int_0^{\tau_i} \frac{\partial \psi}{\partial \tau_k} + \psi(x) \frac{\partial \tau_i}{\partial \tau_k} \right) \end{aligned}$$

evaluate above expression at the common incoming source point set P, then $\tau_j = 0$ for all $j = 1, \cdots, n$, which indicates $\psi|_P = 0$, if we assume $\left(\dfrac{\partial \tau_i}{\partial \tau_k} \right)_{ik}$ is nonsingluar. Then by taking multiple directional derivatives, we can extract more information of higher order Taylor expansion coefficients on P as zero. Thus if ψ is analytic, we can show $\psi = 0$. □

3 Zero Scattering $\sigma_s = 0$

3.1 Recover σ_a only

This is trivial.

3.2 Recover σ_a and γ both

If $H = \gamma\sigma_a\langle u\rangle$, then two datasets H_1 and H_2 for different boundary conditions would be enough for recovering σ_a and γ. Considering

$$\theta(x) = \frac{H_1}{H_2} = \frac{Su_1}{Su_2} = \frac{Ig_1}{Ig_2} \tag{13}$$

3.2.1 Two Beam sources

Suppose we have $g_1 = g_1(x)\delta(v - v_1)$ and $g_2 = g_2(x)\delta(v - v_2)$ as two different beam sources. Then from (13),

$$\int_0^{\tau_1} \sigma(x - \tau_1 v_1 + s v_1)\mathrm{d}s = \tilde{\theta} + \int_0^{\tau_2} \sigma(x - \tau_2 v_1 + s v_2)\mathrm{d}s \tag{14}$$

where $\tilde{\theta}$ is modified from θ with known information.

3.2.2 Condition of recovery

When v_1 and v_2 are not opposite directions, i.e. $v_1 + v_2 \neq \mathbf{0}$, then we either need more information of σ to achieve a unique recovery or have requirement on geometry of domain,

$$\Theta(\mathbf{x}) = \int_0^{\tau_1} \sigma(x - \tau_1 v_1 + s v_1)\mathrm{d}s - \int_0^{\tau_2} \sigma(x - \tau_2 v_2 + s v_2)\mathrm{d}s \tag{15}$$

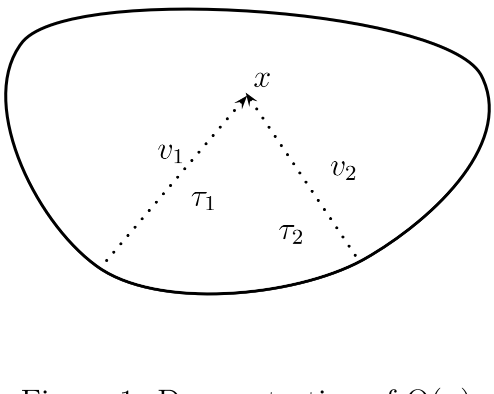

Figure 1: Demonstration of $\Theta(x)$

- when g_1 and g_2 are opposite beams sources, then

$$\Theta(\mathbf{x}) = \int_0^{\tau_1} \sigma(x - \tau_1 v_1 + s v_1)\mathrm{d}s - \int_0^{\tau-\tau_1} \sigma(x + (\tau - \tau_1)v_1 - s v_1)\mathrm{d}s$$

then taking derivative w.r.t τ_1 will be enough. And σ can be recovered with loss of one derivative only.

- When g_1 and g_2 does not satisfy opposite beam condition, then it is hard to recover all information of σ without any further data. We first show a trivial example.

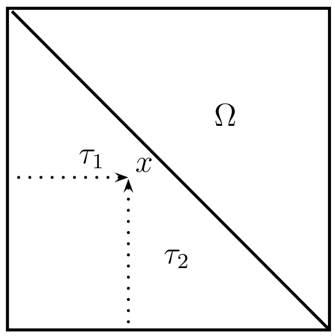

Figure 2: Demonstration of $\Theta(x)$

Suppose v_1 and v_2 are along axis directions, then the data can be written as

$$\Theta(x, y) = \int_0^{x} \sigma(s, y)\mathrm{d}s - \int_0^{y} \sigma(x, t)\mathrm{d}t \tag{16}$$

While when $\sigma = x + y$ throughout the domain $\Omega = [0, 1] \times [0, 1]$,

$$\Theta(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$$

when $\sigma = x + y$ on lower triangle part and $\sigma = 0$ on upper part, then

$$\Theta(x, y) = \frac{x^2}{2} - \frac{y^2}{2}$$

as well. It is easy to show we can have a *unique* and *stable* recovery on lower triangle part. And on other part, there is no uniqueness if we do not impose analytic property or smoothness onto σ.

3.2.3 More information

From above argument, we need some more assumption on the geometry of Ω or smoothness of σ or some further information of σ.

- If we assume non-smoothness of σ and no information of *over-cut-cone*(see explanation below the figure) boundary condition, then a unique recovery depends on specific domain shape. It is easy to prove *cut-cone* property is necessary by appropriate extension and affine transform(since it is angular-independent). Numerical experiments are trivial here and easy to imagine.

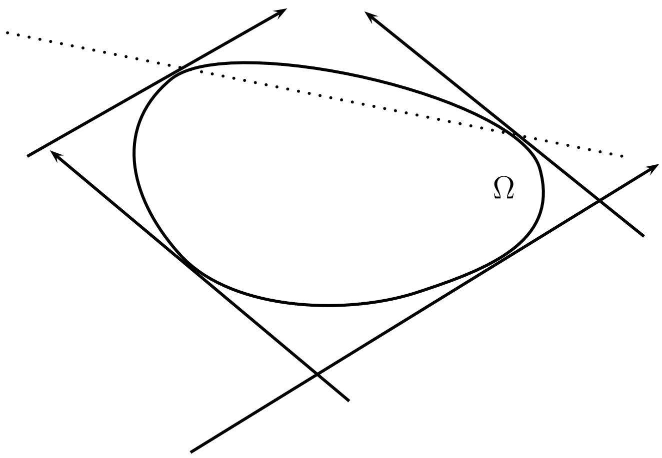

Figure 3: *cut-cone* property, If Ω is *all* under the dashed line, then satisfies *cut-cone* property, *over-cut-cone* boundary is the boundary above the *cut-cone* part.

- If we do not have geometric restriction or *over-cut-cone* boundary information of σ, then we need $\sigma \in C^\infty(\Omega)$. And we can show this is necessary and sufficient condition for recovering σ theoretically. However, this requirement seems to be *difficult* for numerical computing.

- If we have *over-cut-cone* boundary information, then the recovery will be unique and stable, this is trivial and easy to verify by numerical experiment.

3.3 Generalized beam sources

Consider beam source $g = \sum_{i=1}^n g_i(x)\delta(v - v_i)$.

4 Numerical Approach when $\sigma_s = 0$

4.1 Optimization

4.2 Fix point iteration

4.2.1 Beam sources

Since fix point iteration is weaker than optimization, we have to make prior assumption on geometry or σ to sustain uniqueness. From the Theorem 2, we can see a natural fix iteration scheme as

- initialize σ^0.
- update u^k by solving

$$\begin{aligned} v \cdot \nabla u_1^k + \sigma^k u_1^k &= 0 & (17) \\ u_1^k &= g_1 & (18) \end{aligned}$$

- update $\langle u_2^k \rangle$ by

$$\langle u_2^k \rangle = \frac{\langle u_1^k \rangle}{\theta(x)} \tag{19}$$

- update σ^{k+1} by recovering σ^{k+1} from

$$\begin{aligned} v \cdot \nabla u_2^k + (\alpha\sigma^{k+1} + (1-\alpha)\sigma^k)u_2^k &= 0 & (20) \\ u_2^k &= g_2 & (21) \end{aligned}$$

  where α is a weight factor.

4.2.2 Possible Convergence

- For beam sources $g_1 = g_1(x)\delta(v - v_1)$ and $g_2 = g_2(x)\delta(v - v_2)$. We can get iteration scheme as

$$\int_0^{\tau_1} \Delta\sigma^k = (1-\alpha)\int_0^{\tau_2} \Delta\sigma^{k+1} + \alpha\int_0^{\tau_2} \Delta\sigma^k \tag{22}$$

  when the directions are <u>opposite</u>, a.k.a $v_1 + v_2 = \mathbf{0}$, then

$$\int_0^{\tau_1} \Delta\sigma^k = (1-\alpha)\int_0^{\tau-\tau_1} \Delta\sigma^{k+1} + \alpha\int_0^{\tau-\tau_1} \Delta\sigma^k \tag{23}$$

Differentiate w.r.t τ_1, then

$$\Delta\sigma^{k+1} = \frac{\alpha+1}{\alpha-1}\Delta\sigma^k \tag{24}$$

When $\alpha < 0$, we get a linear convergence. Especially, we can obtain a one-step convergence by setting $\alpha = -1$.

- For non-opposite beam sources, we consider the same scheme (22).

$$\int_0^{\tau_1} \Delta\sigma^k = (1-\alpha)\int_0^{\tau_2} \Delta\sigma^{k+1} + \alpha\int_0^{\tau_2} \Delta\sigma^k \tag{25}$$

Since this does not have *angular-averaged* operator included, we can use affine transform to make the directions orthogonal. Later we will see an good initial guess would be necessary.

Again, we take the unit square as an example.

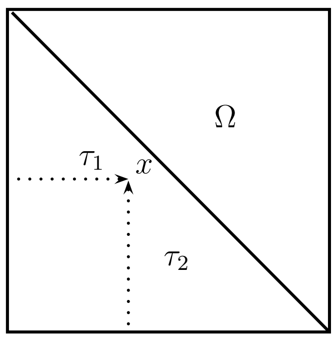

Figure 4: Iteration scheme on unit square

Take $\psi^k = \Delta\sigma^k$ for convenience.

$$\int_0^x \psi^k(s,y)\mathrm{d}s = (1-\alpha)\int_0^y \psi^{k+1}(x,t)\mathrm{d}t + \alpha\int_0^y \psi^k(x,t)\mathrm{d}t \tag{26}$$

Differentiate w.r.t y, then

$$\psi^{k+1}(x,y) = \frac{-\alpha}{1-\alpha}\psi^k(x,y) + \frac{1}{1-\alpha}\int_0^x \frac{\partial\psi^k}{\partial y}(s,y)\mathrm{d}s \tag{27}$$

We consider the following *strict* conditions.

Conditon 1.

For each $k \geq 0$ and $n \geq 1$, $\lambda \frac{\partial^n \psi^k}{\partial x \partial y^{n-1}} \geq \frac{\partial^n \psi^k}{\partial y^n} \geq 0$, $\lambda \in (0,1)$.

Suppose the above condition holds at $k = 0$. then by deduction, assuming this holds at $k = m$, consider $k = m + 1$,

$$\psi^{m+2}(x,y) = \frac{-\alpha}{1-\alpha}\psi^{m+1}(x,y) + \frac{1}{1-\alpha}\int_0^x \frac{\partial \psi^{m+1}}{\partial y} \mathrm{d}s \tag{28}$$

and taking derivative w.r.t x,

$$\frac{\partial \psi^{m+2}}{\partial x} = \frac{-\alpha}{1-\alpha}\frac{\partial \psi^{m+1}}{\partial x} + \frac{1}{1-\alpha}\frac{\partial \psi^{m+1}}{\partial y} \tag{29}$$

$$\frac{\partial \psi^{m+2}}{\partial y} = \frac{-\alpha}{1-\alpha}\frac{\partial \psi^{m+1}}{\partial y} + \frac{1}{1-\alpha}\int_0^x \frac{\partial \psi^{m+1}}{\partial y^2} \tag{30}$$

Since $\alpha < 0$, and by assumption,

$$\lambda \frac{-\alpha}{1-\alpha}\frac{\partial \psi^{m+1}}{\partial x} \geq \frac{-\alpha}{1-\alpha}\frac{\partial \psi^{m+1}}{\partial y} \tag{31}$$

also

$$\lambda \int_0^x \frac{\partial^2 \psi^{m+1}}{\partial x \partial y} \geq \int_0^x \frac{\partial \psi^{m+1}}{\partial y^2} \tag{32}$$

Thus

$$\lambda \frac{\partial \psi^{m+2}}{\partial x} \geq \frac{\partial \psi^{m+2}}{\partial y} \tag{33}$$

for higher ordered derivative, this method also works.

So if initial guess can satisfy this super strict condition, then each output will also satisfy this condition. However, we can prove

$$|\psi^{m+2}| \leq \frac{-\alpha}{1-\alpha}|\psi^{m+1}| + \lambda \frac{1}{1-\alpha}|\psi^{m+1}| \leq \frac{\lambda - \alpha}{1-\alpha}|\psi^{m+1}| \tag{34}$$

So we conclude

Theorem 4. *For the case of recovering sigma and γ both. When using non-opposite single beam sources and the problem setting forces unique solution(see 3.2.3), also initial guess satisfies Condition 1, iteration scheme will converge.*

Corollary 3. *Using opposite beam sources can be recovered using one-step.*

5 Nonzero Scattering

5.1 Recover σ_a only

If $H = \sigma_a \langle u \rangle = \sigma_a S u$, then one measurement is enough for recovering σ_a. Consider iteration scheme,

$$\sigma_a^{k+1} = \frac{H}{(I - SR^k\sigma_s)^{-1}S(I^k g)} = \frac{H}{(Q^k)^{-1}S(I^k g)} \tag{35}$$

Then the error estimate as following

$$\frac{\Delta\sigma_a^{k+1}}{\sigma_a^{k+1}}(Q)^{-1}SIg = (Q^k)^{-1}\Big(S(I^k - I) + S(R^k\sigma_s - R\sigma_s)(Q)^{-1}SI\Big)g \tag{36}$$

$$\begin{aligned} &\|(Q^k)^{-1}\Big(S(I^k - I) + S(R^k - R)\sigma_s(Q)^{-1}SI\Big)g\| \\ &\leq \|(Q^k)^{-1}\|(\|S(I^k - I)g\| + \|S(R^k - R)\sigma_s(Q)^{-1}SIg\|) \end{aligned} \tag{37}$$

And

$$(I^k - I) = I^k\Big(1 - \exp(-\int_0^{\tau(x,v)} \Delta\sigma_a^k(s)\mathrm{d}s)\Big) \leq \mathrm{diam}(\Omega)\|\Delta\sigma_a^k\| \tag{38}$$

From Definition 2,

$$\|(R^k - R)\sigma_s(f)\| = \|\int_0^{\tau(x,v)}\Big(\frac{E^k(\tau)}{E(s)} - \frac{E(\tau)}{E(s)}\Big)\sigma_s f(s)\| \tag{39}$$

$$\leq \|\sigma_s\|\Big(\|\int_0^{\tau}\frac{E^k(\tau) - E(\tau)}{E^k(s)}f(s)\| + \|E(\tau)\int_0^{\tau(x,v)}\frac{E(s) - E^k(s)}{E(s)E^k(s)}f(s)\|\Big)$$

$$\leq \|\sigma_s\|\Big(\|\int_0^{\tau}\frac{E^k(\tau) - E(\tau)}{E^k(s)}f(s)\| + \|\int_0^{\tau(x,v)}\frac{E(s) - E^k(s)}{E^k(s)}f(s)\|\Big) \tag{40}$$

$$\leq C(\sigma_a^k)\|\sigma_s\|\mathrm{diam}(\Omega)\|\Delta\sigma_a^k\|\|f\| \tag{41}$$

Thus

$$\frac{\Delta\sigma_a^{k+1}}{\sigma_a^{k+1}}\frac{\min\|g\|\exp(-\mathrm{diam}(\Omega)\max\sigma_a)}{\|Q\|} \leq C(\sigma_a^k)\mathrm{diam}(\Omega)\|\Delta\sigma_a^k\|\|\sigma_s\|\|g\| \tag{42}$$

$$\frac{\|\Delta\sigma_a^{k+1}\|}{\|\Delta\sigma_a^k\|} \leq C(\sigma_a^k, \mathrm{diam}(\Omega))\|Q\|\|\sigma_s\|\mathrm{diam}(\Omega)\frac{\max\|g\|}{\min\|g\|} \tag{43}$$

which gives a convergent series under <u>some</u> condition.

5.2 Recover σ_a and γ both

Theorem 5. *In general, σ_a can be recovered from data $f = \langle u \rangle$ if source is under* <u>*strict*</u> *condition.*

Proof. We simply consider $\dfrac{\partial f}{\partial \sigma_a}(\psi)$ for some nonzero ψ. Since

$$(I - SR\sigma_s)f = SIg \tag{44}$$

thus

$$(I - SR\sigma_s)\frac{\partial f}{\partial \sigma_a}(\psi) - S\frac{\partial R\sigma_s f}{\partial \sigma_a}(\psi) = -SJ_\tau E_\tau g(x - \tau(x,v)v, v) \tag{45}$$

where

$$J_s = \int_0^s \psi(x - \tau(x,v)v)\mathrm{d}s \tag{46}$$

And

$$\frac{\partial R\sigma_s f}{\partial \sigma_a} = -E_\tau \int_0^\tau \frac{(J_\tau - J_s)\sigma_s f(s)}{E_s}\mathrm{d}s \tag{47}$$

which only needs to show

$$L(x) = S\Big(E_\tau \int_0^\tau \frac{(J_\tau - J_s)\sigma_s f}{E_s} + E_\tau J_\tau g\Big) \neq 0 \tag{48}$$

Moreover, take $K_\tau(v)$ as

$$K_\tau(v) = \int_0^\tau \frac{(J_\tau - J_s)\sigma_s f}{E_s} + J_\tau g(x - \tau(x,v)v, v) \tag{49}$$

then

$$\frac{\partial K_\tau}{\partial \tau} = \psi\Big(\int_0^\tau \frac{\sigma_s f}{E_s}\mathrm{d}s + g\Big) = \psi\frac{u(x,v)}{E_\tau} \tag{50}$$

We can rewrite $L(x)$ as

$$L(x) = S\Big(E_{\tau,v}\int_0^{\tau(x,v)} \psi\frac{u(s,v)}{E_{s,v}}\mathrm{d}s\Big) \tag{51}$$

Here we assume the source is also beam-like as we proved in previous section, $g(x,v) = g(x)\delta(v - v')$. We know the solution can be decomposed into two parts.

$$u(x,v) = g(x - \tau' v')\delta(v - v')E(\tau', v') + \Upsilon \tag{52}$$

Where Υ is non-singular function in v, and bounded by $\|\sigma_s\|\mathrm{diam}(\Omega)$, linear in σ_s and g.

$$\Upsilon = R\sigma_s f \tag{53}$$

rewrite $L(x)$ as

$$\begin{aligned} L(x) &= E_{\tau',v'} g(x-\tau'v') \int_0^{\tau'} \psi(x-\tau'v'+sv')\mathrm{d}s + S\left(E_{\tau,v}\int_0^{\tau}\psi\frac{\Upsilon}{E_{s,v}}\mathrm{d}s\right) \\ &= L_1(x) + L_2(x) \end{aligned} \tag{54}$$

We can kind of prove the case when σ_s is tiny.

$$\frac{\partial L_1(x)}{\partial \tau'} = E_{\tau',v'} g(x-\tau'v')\Big(-\sigma(x)\int_0^{\tau'}\psi\mathrm{d}s + \psi(x)\Big) \tag{55}$$

$$\frac{\partial L_2(x)}{\partial \tau'} = S\Big(\frac{\partial E_{\tau,v}}{\partial \tau'}\int_0^{\tau}\psi\frac{\Upsilon}{E_{s,v}}\mathrm{d}s + E_{\tau,v}\int_0^{\tau} + \Big) \tag{56}$$

□

Proof. Another proof. Since we have already

$$\begin{aligned} Su' &= SI'g + SR'\sigma_s\langle u'\rangle \\ Su &= SIg + SR\sigma_s\langle u\rangle \end{aligned} \tag{57, 58}$$

Thus

$$S(u'-u) = S(I'-I)g + S(R'\sigma_s Su' - R\sigma_s Su) \tag{59}$$

$$S(u'-u) = S(I'-I)g + SR'\sigma_s S(u'-u) + S(R'-R)\sigma_s Su \tag{60}$$

$$\|(I - SR'\sigma_s)S(\Delta u)\| = \|S(I'-I)g + S(R'-R)\sigma_s Su\| \tag{61}$$

It is easy to show $S(\Delta u)$ is bounded from above by $\|\Delta\sigma\|$. On the other hand, it is not easy at all.

$$\|S(I'-I)g + S(R'-R)\sigma_s Su\| \geq \|S(I'-I)g\| - \|S(R'-R)\sigma_s Su\| \tag{62}$$
$$\|S(I'-I)g + S(R'-R)\sigma_s Su\| \geq \|S(R'-R)\sigma_s Su\| - \|S(I'-I)g\| \tag{63}$$

$$\|S(I'-I)g\| = \langle E_\tau(1-\exp(-\int_0^\tau \Delta\sigma))g\rangle \leq \|g\|\mathrm{diam}(\Omega)\|\Delta\sigma\| \tag{64}$$

$$\|S(I'-I)g\| \geq C\mathrm{diam}(\Omega)\min_x g\|S(1-\exp(-\int_0^\tau \Delta\sigma))\| \tag{65}$$

$$\|S(R'-R)\sigma_s Su\| \leq C\|\sigma_s\|\|\Delta\sigma\|\mathrm{diam}(\Omega)\|g\| \tag{66}$$

Now we need to show there is a relationship between

$$\|S(1-\exp(-\int_0^\tau \Delta\sigma))\| \geq \max|\Delta\sigma| \tag{67}$$

or first order approximation

$$\|S\int_0^\tau \Delta\sigma\| \geq \lambda \max|\Delta\sigma| \tag{68}$$

for some $\lambda > 0$. When this is satisfied, we can prove this forces a stability estimate. $\square$