0.1 Equation

$$-\nabla D\nabla u + \sigma_a u = f \tag{1}$$

$$2D\frac{\partial u}{\partial n} + u = 0 \tag{2}$$

where $D = 1/(3\sigma_t)$, $\sigma_t = \sigma_a + \sigma_s$, the weak form

$$\int (D\nabla u\nabla v + \sigma_a uv)\mathrm{d}x + \frac{1}{2}\int_\partial uv\mathrm{d}s = \int fv \tag{3}$$

0.2 Numerical Experiment

0.2.1 single internal data $H = u$

according to (3), the finite element space solution U satisfies

$$AU = F \tag{4}$$

and $A_{ij} = \int D\nabla\psi_i\nabla\psi_j + \int \sigma_a\psi_i\psi_j + \frac{1}{2}\int_\partial \psi_i\psi_j$, $F_j = \int f\psi_j$. Differentiate (4),

$$\frac{\partial A}{\partial\sigma_a}U + A\frac{\partial U}{\partial\sigma_a} = 0 \tag{5}$$

$$\frac{\partial U}{\partial\sigma_a} = -A^{-1}\frac{\partial A}{\partial\sigma_a}U \tag{6}$$

where $\frac{\partial A}{\partial\sigma_a}$ is linear operator $\mathbb{R}^{m\times n\times n} \to \mathbb{R}^{m\times n}$.

objective function without regularization

$$f(U) = \frac{1}{2}\int (U - H)^2 \tag{7}$$

suppose M is mass matrix, then

$$\frac{\partial f}{\partial\sigma_a} = (U - H)^T M\frac{\partial U}{\partial\sigma_a} = -(U - H)^T MA^{-1}\frac{\partial A}{\partial\sigma_a}U \tag{8}$$

taking the adjoint,

$$\frac{\partial f}{\partial\sigma_a} = -U^T\frac{\partial A}{\partial\sigma_a}^T A^{-T}M^T(U - H) \tag{9}$$

due to symmetric property of M and A, it is straightforward to write as

$$\frac{\partial f}{\partial\sigma_a} = -U^T\frac{\partial A}{\partial\sigma_a}\underline{A^{-1}M(U - H)} \tag{10}$$

Result

- domain $[0,1] \times [0,1]$, element size $1/40$.
- source $f(x) = \chi_{r<|x-x_0|<R}$
- $\sigma_s = 20$ as constant, $\sigma_a = 0.1$ as true value
- σ_s and σ_a are piecewise constant function.
- relative error calculated from $\|\sigma_a - \tilde{\sigma}_a\|/\|\sigma_a\|$
- initial value for BFGS $\sigma_{a0} = 0.06$ throughout domain.
- No regularization required.

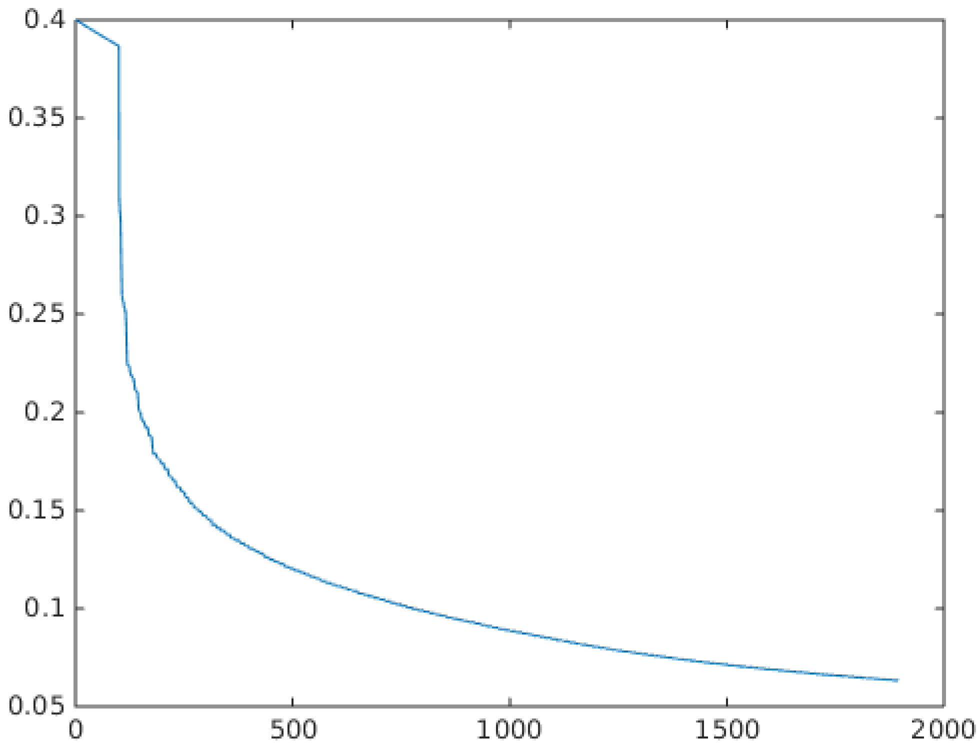

Figure 1: Relative error of σ_a with respect to iteration number

- The iteration continues to around 15000, and the final objective function can reach less than machine error(1e-19).
- The relative error on σ_a observed from the graph above is 6%.
- The converging rate is almost linear after 2000 iterations, but slow.

- The absolute error is large on corners, the relative L^∞ error is over 30%.
- The absolute error in center is much more better.

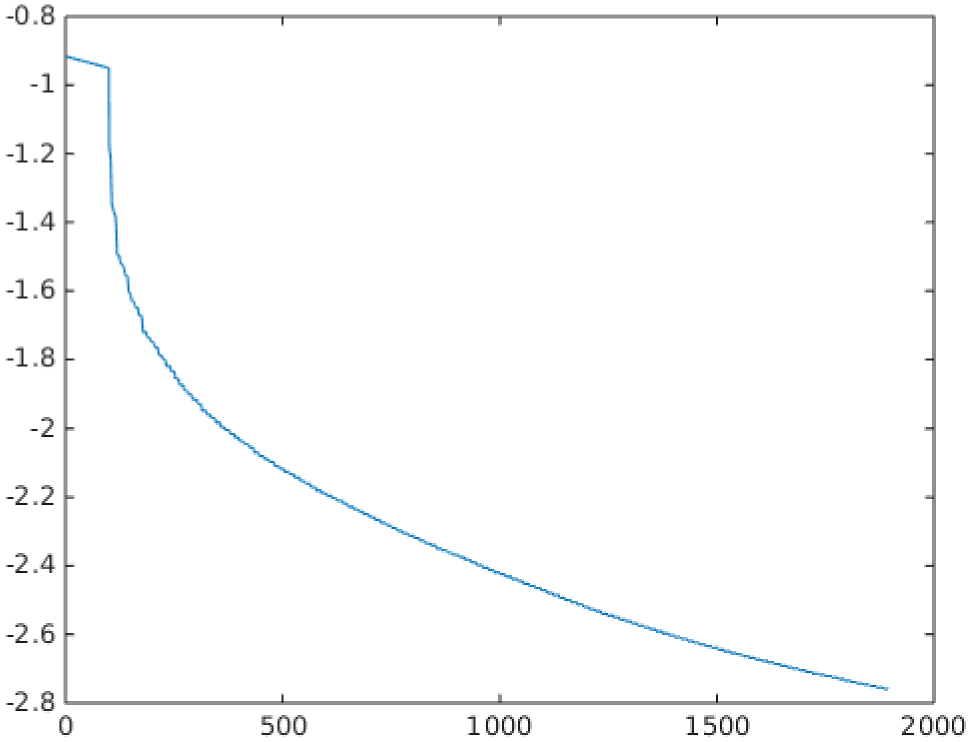

Figure 2: Log of relative error of σ_a with respect to iteration number

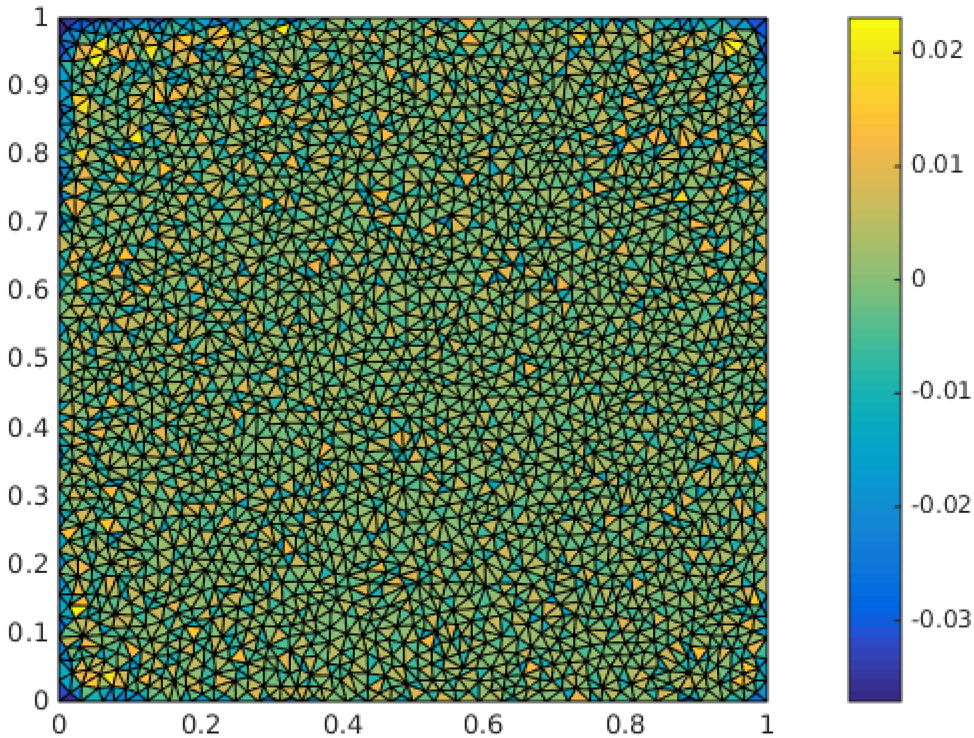

Figure 3: Absolute error of σ_a

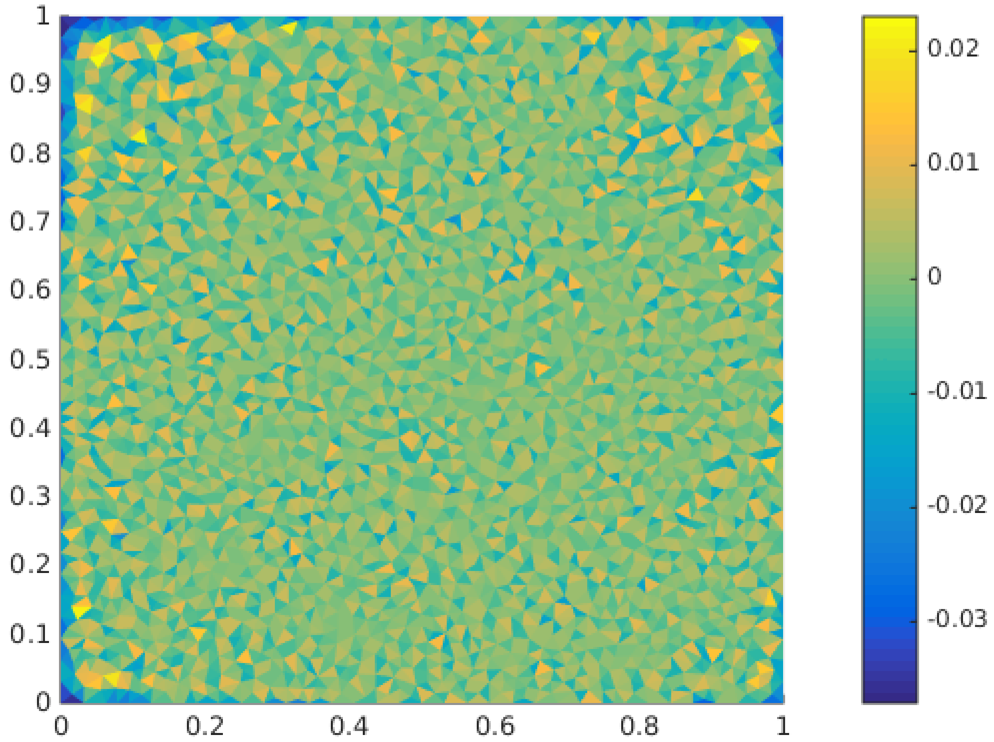

Figure 4: Absolute error of σ_a

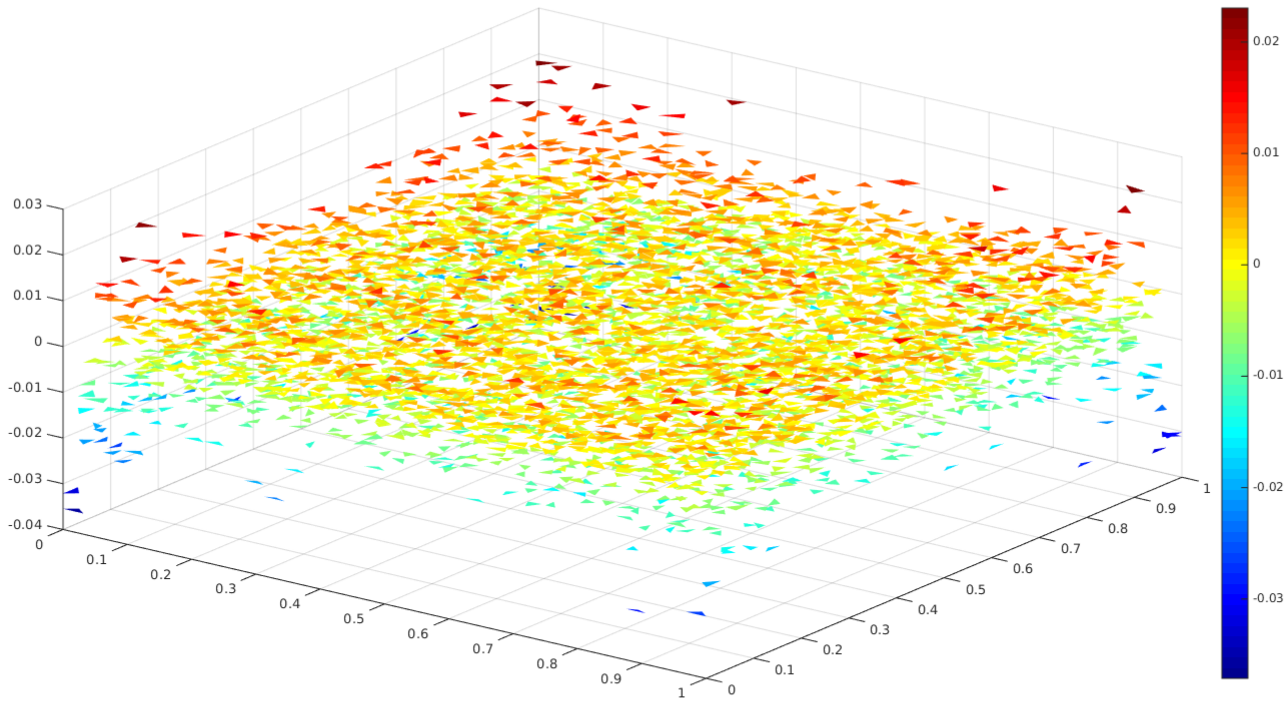

Figure 5: 3D view of absolute error of σ_a

objective function with regularization

$$f(U) = \int (U - H)^2 + \alpha R(\sigma_a) \tag{11}$$

where $R(\sigma_a)$ is calculated through following algorithm

for $1 \le i \le nelem$ **do**
  for j is i's neighbor **do**
    $R \leftarrow R + (\sigma_a(i) - \sigma_a(j))^2$
  end for
end for

result

- Regularization parameter $\alpha = 1e-8$
- other settings are the same as previous settings
- After regularization, relative error gets to 1.19%, after 150 iterations.

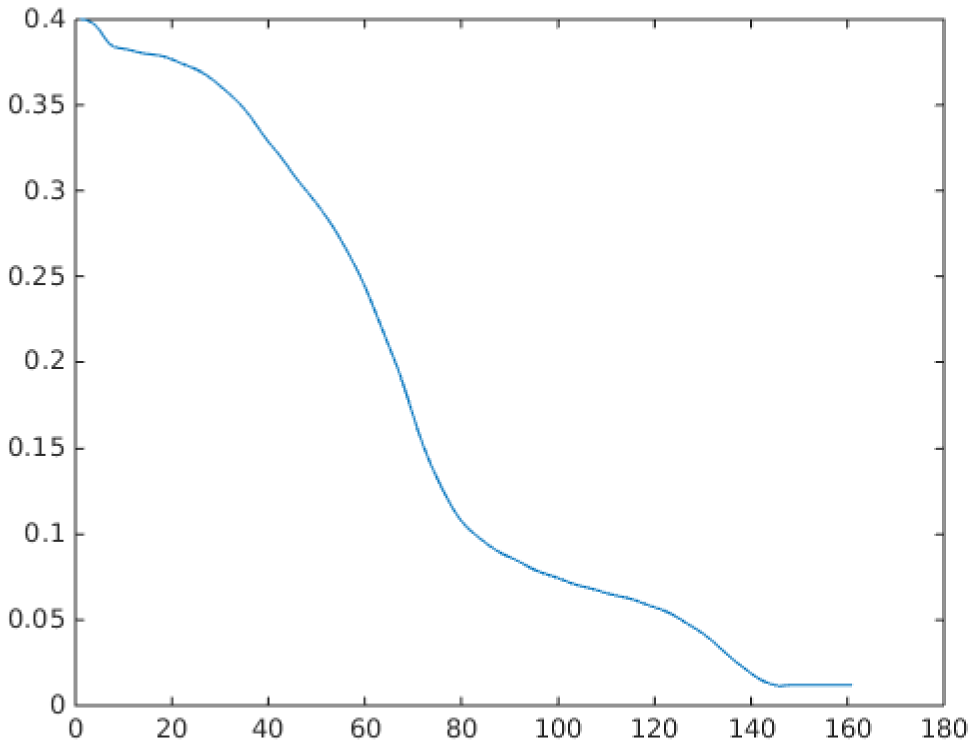

Figure 6: relative error of σ_a with respect to iteration number with regularization

0.2.2 More cases

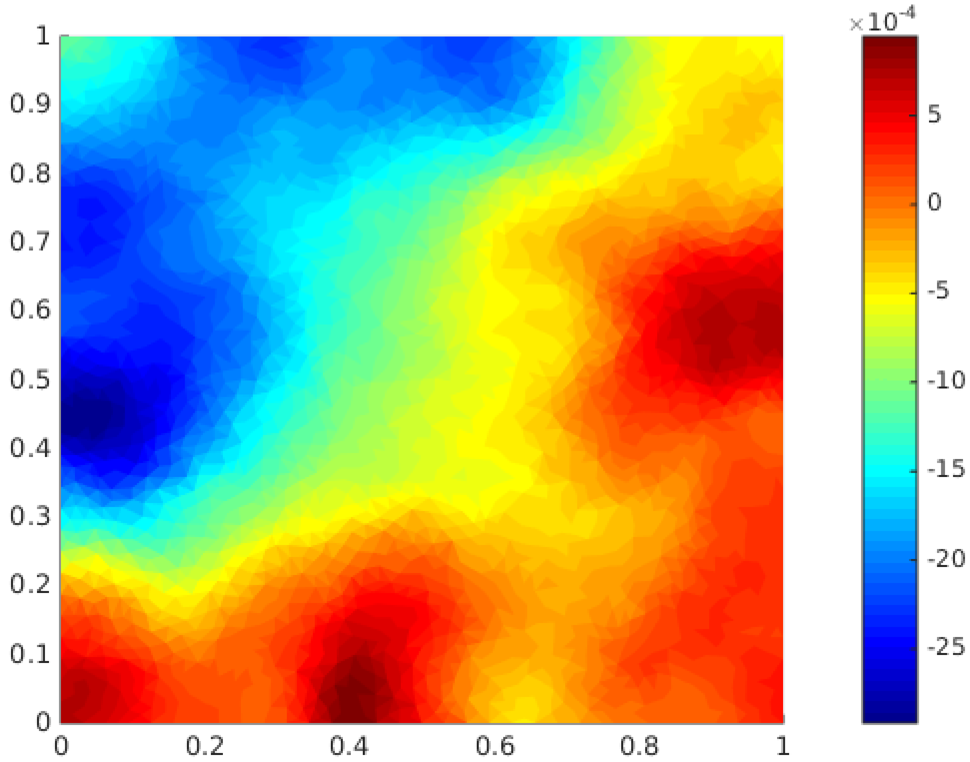

Figure 7: absolute error of regularized case

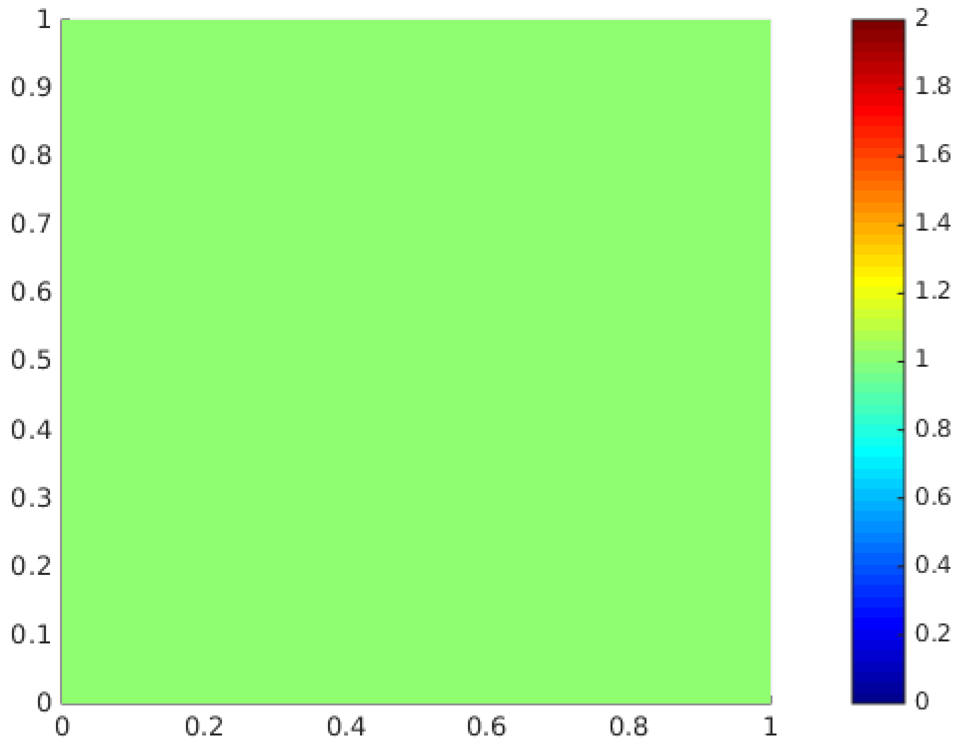

Figure 8: true constant σ_a

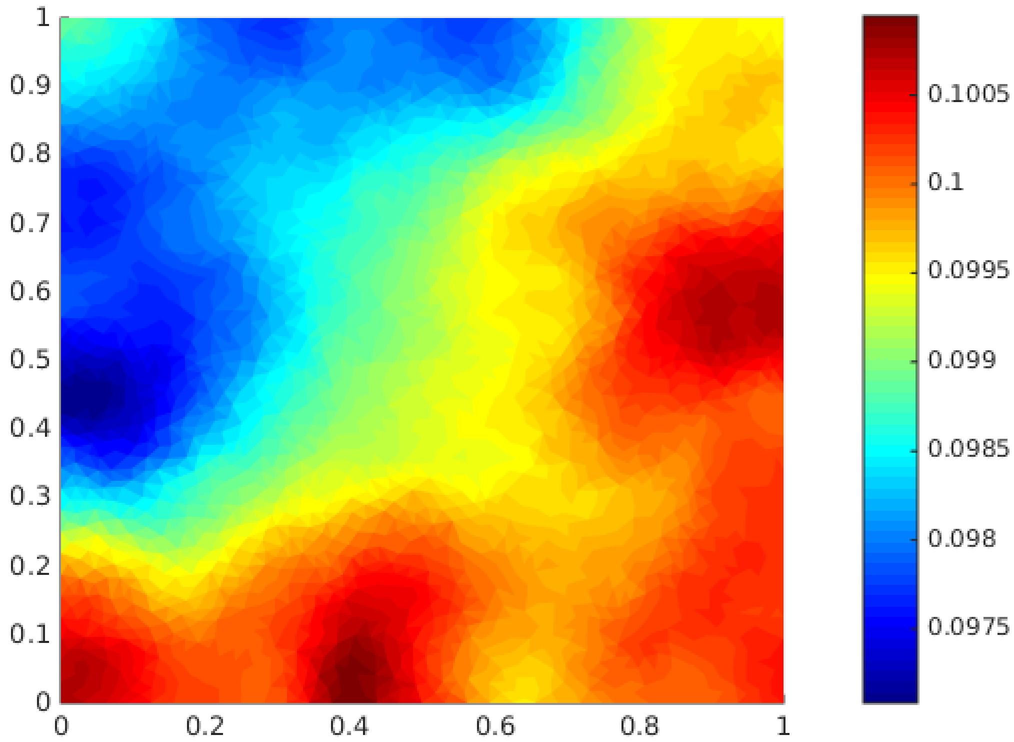

Figure 9: recovered σ_a

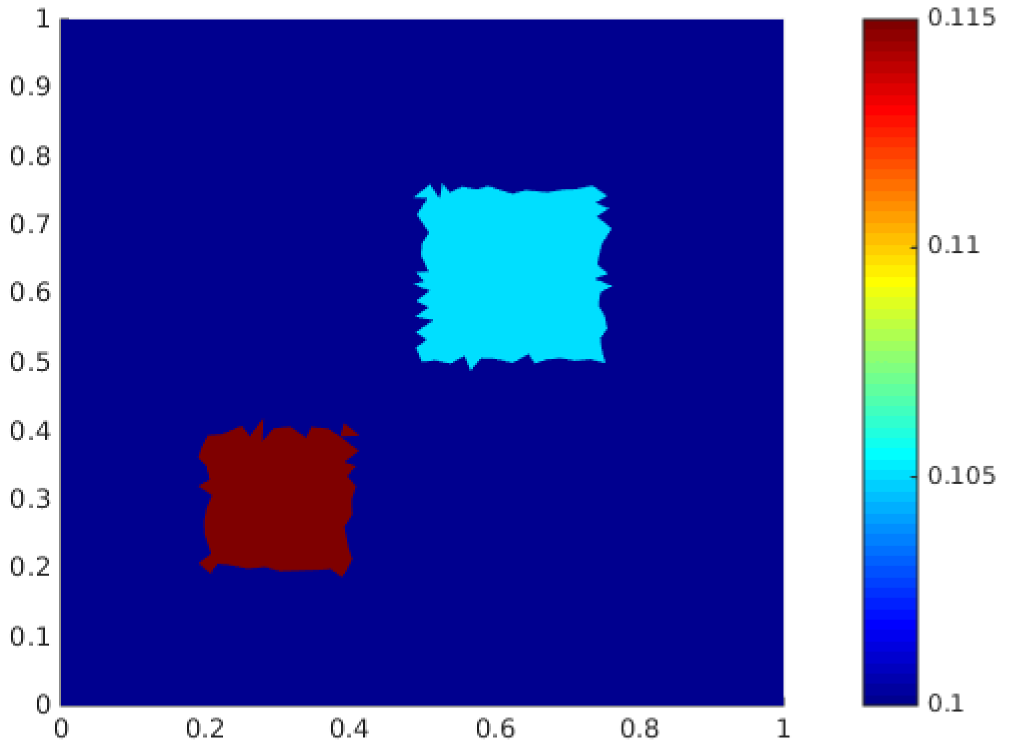

Figure 10: Two square blocks, true σ_a

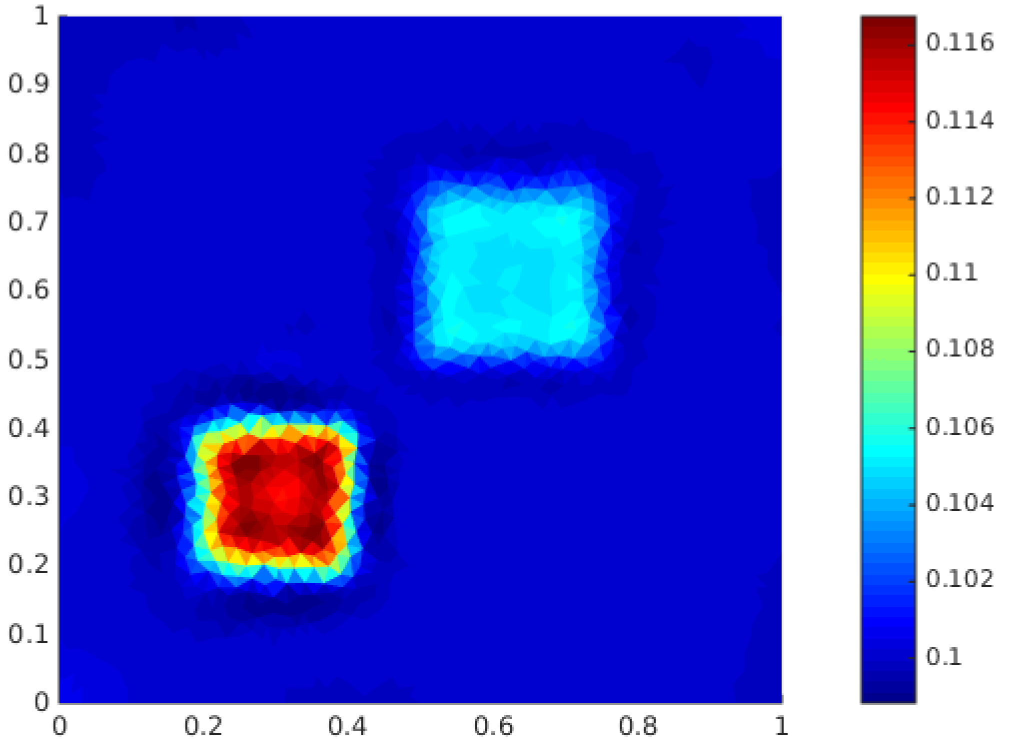

Figure 11: Recovered two square blocks σ_a, error less than 0.94%

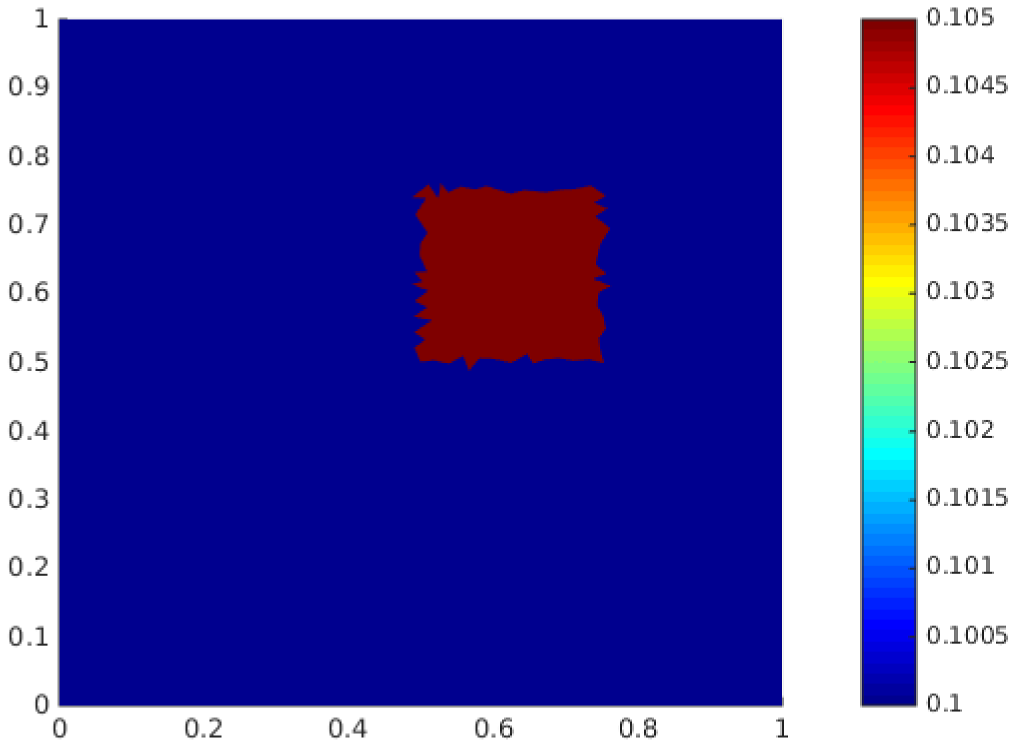

Figure 12: Single block discontinuous, true σ_a

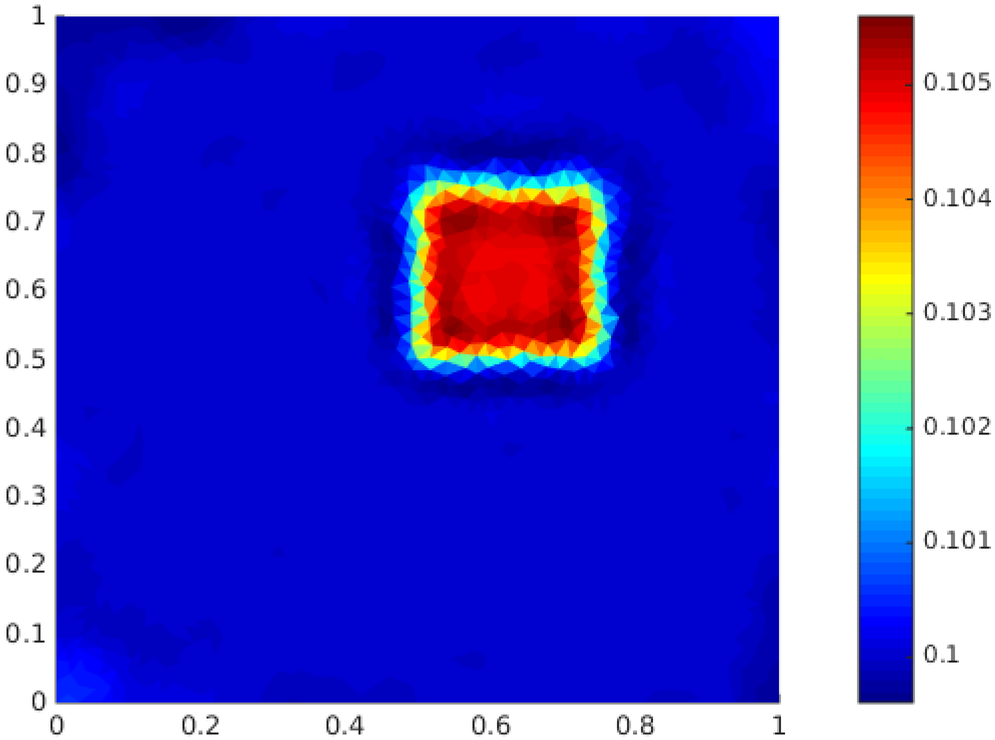

Figure 13: Recovered single block σ_a, error less than 0.35%