# Simulation of Vehicle on Ground

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# 1 Introduction

My project is to simulate a 4-wheel vehicle on different grounds. The application of the simulation is very useful. We can imagine the model be used in a motor-sports video game. It can also be seen as a simple task in reinforcement learning. I simulate the objects with both rigid body approach and non-rigid body approach. By combining these two approaches, it's possible to make the simulation accurate while keep the simplicity of the computation at the same time.

# 2 Equations

## 2.1 Dynamics in Non-rigid Body

We start from the dynamics in non-rigid body, more specifically, springs and dashpots between points. We consider an object as a group of points, where there is a spring and a dashpots connecting each points. Suppose we have a group of points index by $1, \cdots, n$. Then, the motion of the objects is governed by the following equations:

$$T_{jk} = S_{jk}\left(\left\|\mathbf{X}_j - \mathbf{X}_k\right\| - R^0_{jk}\right) + D_{jk}\frac{d}{dt}\left\|\mathbf{X}_j - \mathbf{X}_k\right\| \tag{1}$$

$$\frac{d\mathbf{X}_k}{dt} = \mathbf{U}_k \tag{2}$$

In (1) $T_{jk}$ is the scalar quantity of tension between point $j$ and point $k$, $S_{jk}$ is coefficient of elasticity of the spring, $\mathbf{X}_j$ and $\mathbf{X}_k$ are two 3-dimensional vector representing the position of the two points, $R^0_{jk}$ is the rest length of the spring, and $D_{jk}$ is the coefficient of resistance of the dashpot. While in (2) $\mathbf{U}_k$ is a 3-dimensional vector representing the velocity of point index by $k$.

This approach of modeling objects is very straightforward when we are only interested in the motion of each points in the objects. However, when we want to investigate more general property of the body itself, such as rotation, though this approach still works in some sense, it is more helpful if we can treat our objects as rigid bodies.

## 2.2 Dynamics in Rigid Body

In the approach of rigid body dynamics, we assume the deformation of the body at any time is zero. Under this assumption, we will treat an object as a group of points plus a point of center of mass. The center of mass is an artificially imagined point, which doesn't exist in reality, but makes it extraordinarily simple for our analysis. The property of the center of mass is determined by the following equations:

$$M = \sum_{k=1}^{n} M_k \tag{3}$$

$$M\mathbf{X}_{cm} = \sum_{k=1}^{n} M_k \mathbf{X}_k \tag{4}$$

$$M\mathbf{U}_{cm} = \sum_{k=1}^{n} M_k \mathbf{U}_k \tag{5}$$

$$\frac{d\mathbf{X}_{cm}}{dt} = \mathbf{U}_{cm} \tag{6}$$

In (3) $M$ is the mass we assign to the center of mass, and $M_k$ is the mass of points $k = 1, \cdots, n$ on the rigid body. In (4) $\mathbf{X}_{cm}$ is a 3-dimensional vector representing the position of the center of mass in the 3-dimensional space, while $\mathbf{X}_k$ is the position of each points on the rigid body. This equation can be interpreted as a weighted sum of the mass of each points on the body. Similarly, (5) represents the velocity of the center of mass and each points. Finally, (5) is a relation between the position and the velocity.

Now that we have the rigid body, if we consider the dynamics. We still apply equation 1, and we will find out that the dynamics inside the body is zero because they all cancels out. Hence, the motion of the body is governed by:

$$M\frac{d\mathbf{U}_{cm}}{dt} = \mathbf{F} = \sum_{k=1} n\mathbf{F}_k \tag{7}$$

Note that here when we want to use the Newton's second law, we think all forces that exerts on each points of the body to exert on the center of mass. We sum them up and get the accretion of the center of mass. This is intuitively correct, but also a consequence of the formula derivation.

Now we want to study the rotation of the body. We are mostly interested in the angular momentum, which is given by:

$$\mathbf{L} = \sum_{k=1}^{n} M_k(\mathbf{X}_k - \mathbf{X_{cm}}) \times \mathbf{U}_k \tag{8}$$

$$\frac{d\mathbf{L}}{dt} = \sum_{k=1}^{n} (\mathbf{X}_k - \mathbf{X_{cm}}) \times \mathbf{F}_k = \tau \tag{9}$$

where $\mathbf{L}$ is a 3-dimensional vector denoting the angular momentum of the body and $\tau$ is the total torque exerts on the body. Moreover, since we essentially need to know the angular velocity of the body in the simulation, we use $\mathbf{L} = I\Omega$ (where $\Omega$ is a 3-dimensional vector denoting the angular velocity of the body, and $I$ is the non-singular 3-by-3 matrix denoting the moment of inertia tensor of the body) to convert between angular momentum and angular velocity.

## 2.3 Interaction with the Ground

Finally, since we want our vehicle to drive on the ground, we need to find a way to represent the ground. Neither non-rigid body or rigid body seems to be a good choice, but some very similar to a spring might be a practical idea. First, we use an function of $x, y, z$ to represent the ground, and set all points where the function retains zero to be the ground. For example, the function $H(x, y, z) = z$ represents the plain ground with $z = 0$. In general, the function can be any functions that represents a manifold in the 3-dimensional space with $H(x, y, z) = 0$. I'll show more examples in the experiment section later.

After determining the equation, we decide all the points with $H(x, y, z) > 0$ to be above the ground and with $H(x, y, z) < 0$ to be below the ground. When a point is below the ground, we will give it a force determined by:

$$\mathbf{F}_{ground} = S\frac{-H(\mathbf{X})}{\|\nabla H(\mathbf{X})\|}\frac{\nabla H(\mathbf{X})}{\|\nabla H(\mathbf{X})\|} - \mu S\frac{-H(\mathbf{X})}{\|\nabla H(\mathbf{X})\|}\frac{\mathbf{U}^{\tan}}{\|\mathbf{U}^{\tan}\|} \tag{10}$$

where $S$ is the coefficient of elasticity of the ground, $\mathbf{X}$ is the position of the point, $\mu$ is the coefficient of friction of the ground, and $\mathbf{U}^{\tan}$ denotes the velocity of the point in the tangent direction of the ground, in other words, the projection of $\mathbf{U}$ onto the tangent space of the ground. Of course, when $H(\mathbf{X}) > 0$, we will just simply set $\mathbf{F}_{ground} = 0$ because the point doesn't touches the ground in this condition.

With this mechanism, we actually allow the point to go a little below under the ground. This wouldn't be a big problem since we can set the coefficient of elasticity very large so that a very large force will exert on the point if it goes too deep into the ground.

# 3 Vehicle Setup

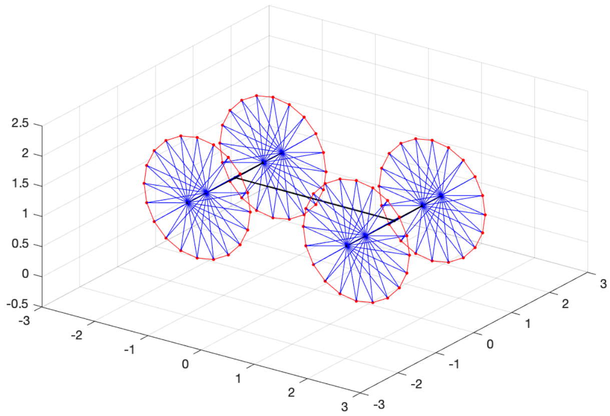



Figure 1: Plot of the vehicle

There are several different parts in my vehicle. I made the four wheels and the body of the vehicle to be rigid bodies. For example, each wheel is a rigid body with 20 points on a circle, and a body consists 8 points, with 2 points on each corner of the body, which are the axles of the wheels. Then, I use springs and dashpots to connect these rigid bodies. Each wheel is connected to the corresponding axle with springs and dashpots connected to each point on the wheel. A plot of the vehicle will look like figure 1 (red lines connect the points on the wheel, black lines connect the points on the body, and blue lines show the springs and dashpots combined). This setup makes the numerical update more easier and hopefully the simulation more accurate.

# 4 Numerical Methods

Now, suppose I have built the vehicle, and have all the initial parameters configured, we need to follow the following steps to do the simulation numerically.

## 4.1 Rotate

In this step, we need to apply Rodrigues' rotation formula to the wheels and the body:

$$\omega(t) = \Omega(t) \cdot \Delta t \tag{11}$$

$$\tilde{\mathbf{X}}(t+1) = \text{Rodrigues}\left(\tilde{\mathbf{X}}(t), \omega(t)\right) \tag{12}$$

Here $\Omega$ is the angular velocity of the wheels and the body, $\Delta t$ is the time step we choose, $\tilde{\mathbf{X}}$ is the difference of position from each point to the center of mass, and "Rodrigues" is a function that applies the rotation $\omega$ to the differences of positions, $\tilde{\mathbf{X}}$.

### 4.2 Calculate Forces

In this step, we calculate all the forces that exerts on the vehicle:

$$T_{jk}(t) = S_{jk}\left(\left\|\mathbf{X}_j(t) - \mathbf{X}_k(t)\right\| - R^0_{jk}\right) + D_{jk}\left\|\mathbf{U}_j(t) - \mathbf{U}_k(t)\right\| \tag{13}$$

$$\mathbf{F}_{jk}(t) = \sum_{j \in \textbf{neighbors of k}} T_{jk}(t)\frac{\mathbf{X}_j(t) - \mathbf{X}_k(t)}{\left|\left|\mathbf{X}_j(t) - \mathbf{X}_k(t)\right|\right|} \tag{14}$$

$$\mathbf{F}_{ground}(t) = \mathbb{1}_{H(\mathbf{X})<0}\left(S\frac{-H(\mathbf{X})(t)}{\|\nabla H(\mathbf{X})(t)\|}\frac{\nabla H(\mathbf{X})(t)}{\|\nabla H(\mathbf{X})(t)\|} - \mu S\frac{-H(\mathbf{X})(t)}{\|\nabla H(\mathbf{X})(t)\|}\frac{\mathbf{U}^{\tan}(t)}{\|\mathbf{U}^{\tan}(t)\|}\right) \tag{15}$$

$$\mathbf{G}(t) = M\mathbf{g} \tag{16}$$

$$\mathbf{F}(t) = \mathbf{G}(t) + \mathbf{F}_{ground}(t) + \mathbf{F}_{jk}(t) \tag{17}$$

Here $g$ is the gravitational acceleration, and other parameters have the same meaning in the previous section. Note that in the previous section we said when $H(\mathbf{X}) > 0$, we should have zero force from the ground. Hence, we use an indicator function to reflect this relation in the numerical method. Also note that our setting of the vehicle, each point on the wheels has two neighbor points (the two axle points), while the neighbors of each axle point are all the points on the wheels.

### 4.3 Update Parameters

$$\mathbf{U}_{cm}(t+1) = \mathbf{U}_{cm}(t) + \frac{\mathbf{F}(t)}{M}\cdot\Delta t \tag{18}$$

$$\mathbf{X}_{cm}(t+1) = \mathbf{X}_{cm}(t) + \mathbf{U}_{cm}(t+1)\cdot\Delta t \tag{19}$$

$$\mathbf{L}(t+1) = \mathbf{L}(t) + \sum_{k=1}^{n}\tilde{\mathbf{X}}_k(t)\times\mathbf{F}_k(t) \tag{20}$$

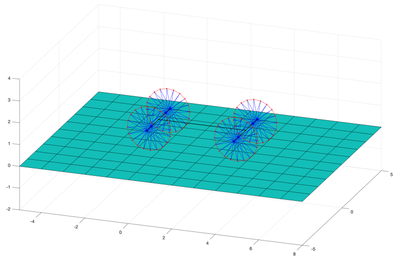
$$\Omega(t+1) = \Omega(t) + I^{-1}\mathbf{L}(t+1) \tag{21}$$

Again, the parameters have the same meaning as that in the previous section. Then, by doing these three steps iteratively, we can simulate the whole process

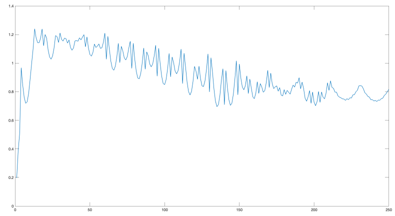
## 5 Experiments

I have tried to run experiments on different types of grounds. Here I would like show four most interesting ones of them.

Note that all the graph of velocity in this section is based on frames. There are 50 frames in one second, i.e. the time interval between frames is 0.02 second.



(a) Plot



(b) Graph of velocity

Figure 2: Experiment on plain ground

### 5.1 Plain Ground

The most simple and straightforward ground of all would be a plain ground, with the ground function:

$$H(x,y,z) = z \tag{22}$$

Figure 2 shows one frame from the simulation and the graph of the velocity of the center of mass of the vehicle body in the simulation. We can see from the graph that in the beginning, the vehicle gains velocity very fast. This is because I give an initial angular momentum to each of the wheels, and the ground is rough ($\mu = 0.75$). Then, we can see the curve have lots of fluctuations because our mechanism of the ground. There are two interesting things to notice in the graph. First, the curve have a downward trend. Also, at the very right part of the graph, the fluctuation becomes more and more smooth. I think the reason behind this is the damping of the rims. So it i) looses kinetic energy as time goes by, and ii) absorbs the fluctuation and finally makes it smooth.

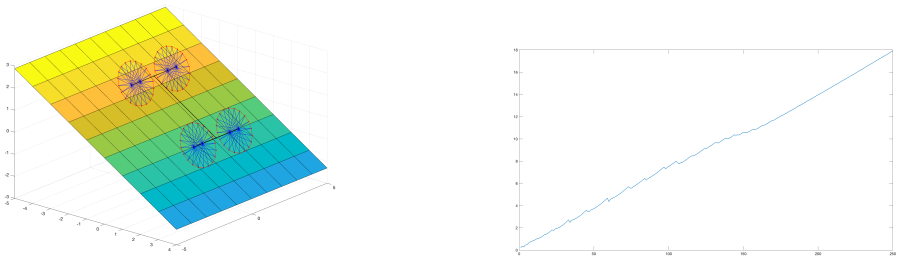
### 5.2 Downward Slope

Then, we consider the vehicle on a downward slope. The only thing we need to decide is how steep the slope should be. For the experiment, I use the ground governing by:

$$H(x,y,z) = z + \tan\left(\frac{\pi}{6}\right)x = z - \frac{\sqrt{3}}{3}x \tag{23}$$

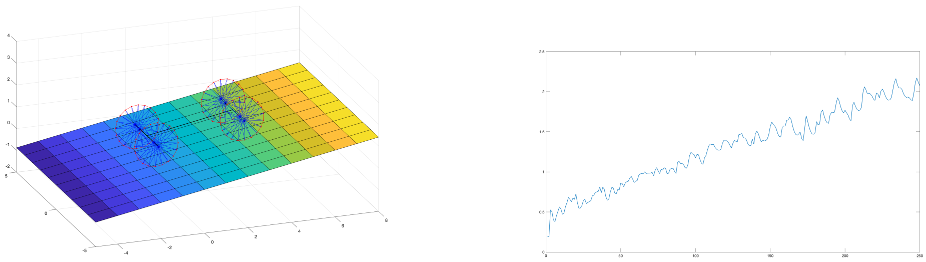
Because of the existence of gravity, it's intuitive to imagine the vehicle will drive downwards even without a initial angular momentum. Before the experiment, I also imagined the velocity of the vehicle will converge to a constant because of the existence of friction.

Figure 3 shows the result of the experiment. In the first half of the simulation, similar to that on plain ground, we can see the graph fluctuates a bit, with an approximately constant acceleration. However, in contrast to my expectation, the velocity didn't converge. After further analysis, I figured out it's actually a misunderstanding of sliding friction and rolling friction. In terms of sliding friction, it's true that the object will maintain a constant speed when the friction coefficient is equal to the slope ($\mu = \tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$ in the experiment). But it is not true in the case of rolling friction. If we consider the situation where the velocity points on the wheels is equal to the velocity of the center of mass of the vehicle body, then, the relative velocity of points on the wheel to the ground is zero! Therefore, the friction doesn't make any difference in this setting.



(a) Plot (b) Graph of velocity

Figure 3: Experiment on downward slope



(a) Plot (b) Graph of velocity

Figure 4: Experiment on upward slope

### 5.3 Upward Slope

Then, in stead of a downward slope, we consider an upward slope. The ground is given by:

$$H(x,y,z) = z - \tan\left(\frac{\pi}{18}\right)x \approx z - 0.176x \tag{24}$$

Of course, to get the vehicle driving on an upward slope, not only do we need to provide it with an initial angular momentum, the vehicle also need an extra torque applied to the wheels all the time. Though in real life, this is done by the torque provided by the engine of the vehicle, in the simulation, we can simply add an extra positive angular momentum to each wheel at each time step.

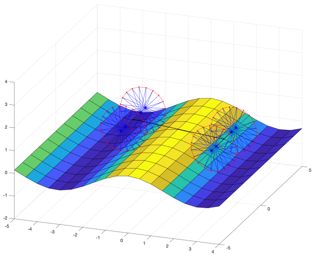
Result of the experiment is shown in figure 4. Because the positive angular momentum we add to the wheels is larger than the loss caused by gravity, the vehicle accelerates with a constant acceleration.

### 5.4 Cosine Wave

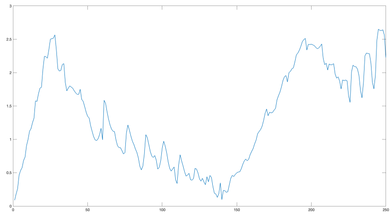
In the last experiment, we make the ground to be a scaled cos wave, given by:

$$H(x,y,z) = z - \frac{1}{2}\cos(x) \tag{25}$$

In this experiment, we still gives the vehicle an initial angular momentum. Then, in the graph, we can see the vehicle first gains velocity from the friction and initial angular momentum. Then, the velocity decreases because the vehicle is climbing up a hill on the cosine wave. Now, since the hill is too high



(a) Plot



(b) Graph of velocity

Figure 5: Experiment on scaled cosine wave

compared to the initial angular momentum we give, the vehicle fails to climb to the top. Instead, it falls down backwards. Since the graph of velocity only shows the absolute value of the velocity, so the curve goes up again in the graph. Finally, the vehicle ends up in the valley and move back and forth. We can also expect that if we give the vehicle enough initial angular momentum, and the cosine wave is scaled with a smaller coefficient, the vehicle can climb to the top and goes to the next valley.

## 6 Summary and Conclusions

In general, I formulated the numerical methods from the equations of dynamics. By combining several rigid bodies with springs, I'm able to make the simulations straightforward and reasonable. In the experiment, I mainly tried with different ground functions to see how the vehicle will behave on these different conditions. If we can ignore the fluctuations caused by our mechanism of the ground, the car behaves exactly what we would expect it to behave on each type of ground. Besides the fluctuation, this formulation of the ground also requires a very small time step to use in the simulation. Since we want to make the ground as stiff as reasonable (instead of something like a trampoline), we need to set the coefficient of elasticity of the ground to be very big (usually around 1000-2000). If we don't have a very small time step, the points will go very deep into the ground within one single time step, and be blown up in the next time step.

To deal with the huge computational power needed resulted from small time steps and huge number of iterations, I utilized the Virtual Computer Lab (VCL) provided by the university and run the simulation with a much faster speed compared to that on my laptop.

## References


[1] Peskin, Charles. *Dynamics of Structures: Networks of Springs and Dashpots with Point Masses at the Nodes*. Accessed `https://www.math.nyu.edu/~peskin/modsim_lecture_notes/structural_mechanics.pdf`

[2] Peskin, Charles. *Interaction of Dynamic Structures with the Ground: Non-Penetration and Sliding Frictional Forces*. Accessed `https://www.math.nyu.edu/~peskin/modsim_lecture_notes/interaction_with_ground.pdf`

[3] Peskin, Charles. *Dynamics of Rigid Bodies*. Accessed https://www.math.nyu.edu/~peskin/modsim_lecture_notes/rigid_body_motion.pdf

# Appendix A Relavent links

Codes of this project can be viewed on GitHub at:
https://github.com/Gaaaavin/vehicle-on-ground

Animations of this project can viewed in Google Drive at:https://drive.google.com/drive/folders/1t5nzPWrnk4rcjgR5YbNU1b_wFOdmFU9V?usp=sharing