# Measurement of normal mode characteristics of a 1-D system of (N+1) equal springs and (N) equal masses for (N = 1, 2, 3, and 5)

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## Abstract

This experiment was carried out to measure the frequencies of the normal modes and associated amplitudes ratios of an 1-D system consisting of (N) gliders of equal mass (m) and (N+1) springs with spring constant (k). The majority of our measured frequencies and amplitude ratios matched the expected values of the theoretical model within one standard error. The final value calculated for $k/m$ using the measured frequencies of the normal modes was $(10.23 \pm 0.46)s^-2$ which matches the measured value of $(10.37 \pm 0.18)s^{-2}$ to one standard error.


## 1 Introduction

### 1.1 Intention

Any possible oscillatory motion of a system can be uniquely described by a linear combination of the normal modes of oscillation of the system [1]. Thus, these modes have an unique importance in these systems; theoretical work in this area has many applications in different fields including: molecular dynamics, structural analysis, and seismology. So it is clear that this experiment intends to replicate results that are in accordance with the standard theoretical model and explore ways in which the experiment diverged from the expected results. The following experiment is mostly based on the laboratory script provided by the Oxford physics department [2]. This report is laid out as follows: subsection 1.2 lays the theory needed to understand the experiment, section 2 describes the method and equipment used, section 3 covers the results and contextualizes them to the theory, and section 4 outlines the most important results of this report and how this experiment could be improved.

### 1.2 Theory

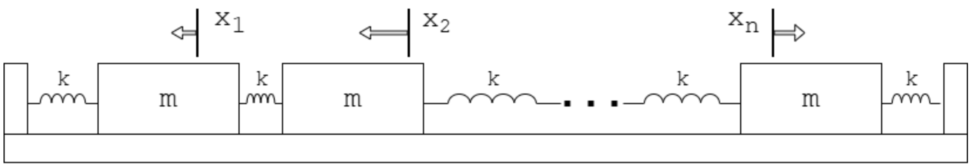


Figure 1: 1-D System of (N) masses each with displacement $x_n$. Figure from [3]

Figure (1) represents our experiment consisting of (N) equal mass gliders and (N+1) equal springs, where $x_r$ in the displacement of the rth glider from its equilibrium position, k is the spring constant and m is the mass of the gliders. Ignoring resistive forces, since they will be mostly accounted for in the experiment. It is possible to write the equation of motion for each mass depending on the position of the other neighbouring masses:

$$m\frac{\mathrm{d}^2 x_r}{\mathrm{d}t^2} = -k(x_r - x_{r-1}) - k(x_r - x_{r+1}) \text{ for r} \neq \text{(1,n)} \tag{1}$$

Given this, one can derive the formula for the angular frequency of the normal modes [1]:

$$\omega_j = 2\sin\left[\frac{j\pi}{2(N+1)}\right]\sqrt{\frac{k}{m}} \text{ (for j = 1...N)} \tag{2}$$

And the amplitude of the motion of the rth mass [1]:

$$a_r = A_j \sin\left(\frac{rj\pi}{N+1}\right) \tag{3}$$

Where $j$ is number of the normal mode and $A_j$ is the amplitude of the normal mode.

# 2 Method

## 2.1 Measuring the spring constant

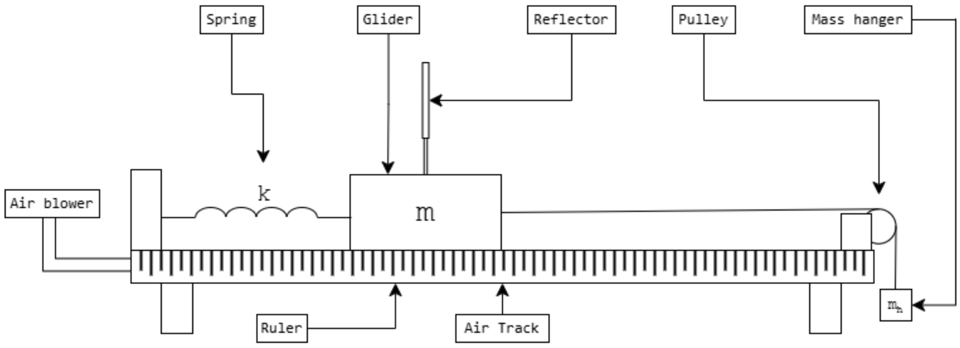


Figure 2: Experimental Setup for measuring the spring constant. Figure from [3]

The spring constant is an integral part of the experiment, since it will be used to calculate the expected normal mode frequencies. So, it was necessary to determine the value by attaching a spring to one end of the air track and attaching a glider to it. Then a string was used to connect a mass hanger to the glider, which provides a force to extend the spring. This is depicted in Figure (2).

Initially, the air track was turned on with the hanger not attached and let oscillating until it came to rest. The equilibrium position was recorded using the ruler. Then the mass hanger is attached and the glider is allowed to oscillate until coming to rest, then the extension is measured using the ruler, with the mass of the mass hanger being recorded using a digital scale. Repeated measurements were taken by varying the mass hanger's mass by adding weights. Finally, a plot of the extension against the mass was used to determine the spring constant k, with g being the gravitational field strength.

$$k = \frac{g}{\text{gradient}} \tag{4}$$

Two additional springs were tested to investigate if they had the same spring constant. This was done by measuring the extension for a single mass and comparing it with the predicted result if both had the spring constant of the first spring.

## 2.2 Measuring the normal mode characteristics

As depicted in Figure (3), to measure the frequency and amplitude each mass was connected to each of its neighbours by one spring. And the boundary masses were connected by a single spring to the stationery apparatus. An air track was used to decrease the amount of friction experienced by the gliders and an ultrasonic motion sensor was used in combination with a computer to record the positions of the masses. The mass of each glider was measured with a digital scale to ensure they had similar masses: $(283.5 \pm 0.5)$g

The air track was then turned on and the masses were allowed to reach equilibrium, the position of equilibrium of each mass was measured using the ruler. The reflectors were strategically placed to avoid symmetrical pairs of masses thus minimising the amount of times the experiment had to be repeated and then the masses were displaced by the same amount: $(4.5 \pm 0.1)$cm in the directions needed to produce normal modes of oscillation. This was repeated for N = 1,2,3, and 5 gliders.

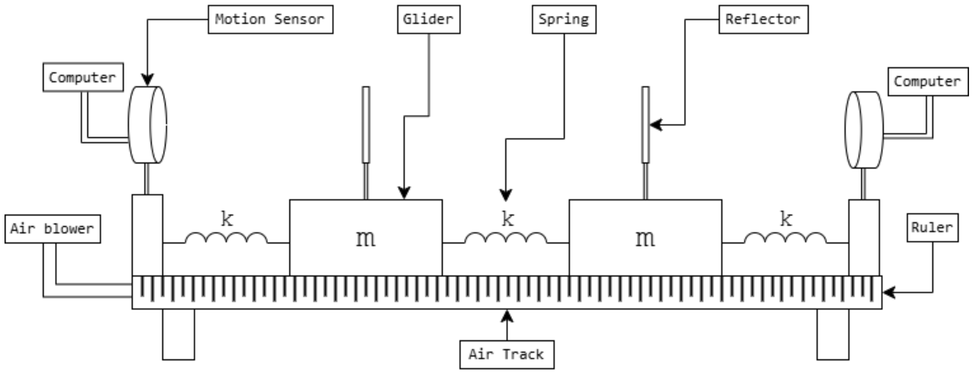


Figure 3: Experimental Setup for the case of N = 2. Figure from [3]

The data was then analysed using PASCO Capstone software, the dominating frequency was extracted from the fast Fourier transform (FFT) with a setting of 40Hz and amplitudes were obtained from fitting the data acquired by the motion sensors using a damped sine model.

# 3 Results and analysis

### 3.0.1 Measurement of spring constant

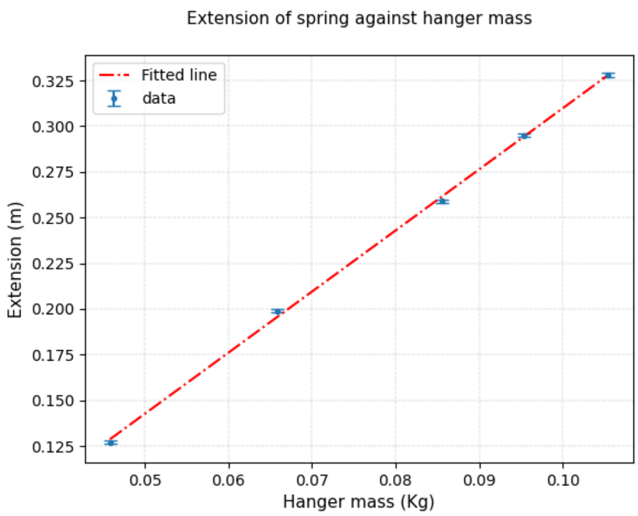


Figure 4: Graph of the extension of the spring vs the hanger mass.

Figure (4) shows the graph of the extension vs hanger mass for a particular spring, the gradient was found to be $(3.34 \pm 0.056) mkg^{-1}$. Using the previously derived formula for the spring constant (4) yields a value of $(2.94 \pm 0.05) Nm^{-1}$. It is clear from the graph that the points do fit a linear model as predicted.

Table 1: Extra measurements for 2 other springs.

| Expected extension / cm | Measured extension / cm | Spring position |
|---|---|---|
| $29.5 \pm 0.5$ | $29.4 \pm 0.2$ | 2nd |
| $29.5 \pm 0.5$ | $28.7 \pm 0.2$ | 3rd |

The values in Table (1) were obtained by using a mass hanger of mass $(95.40 \pm 0.05)g$. It is clear that spring 2 is well within one standard error of the expected result, however spring 3 seems to have a slightly different spring constant, though this can be argued to be random error since the difference is in the order of 1.5 standard errors.

### 3.0.2 Measurements of normal mode frequencies

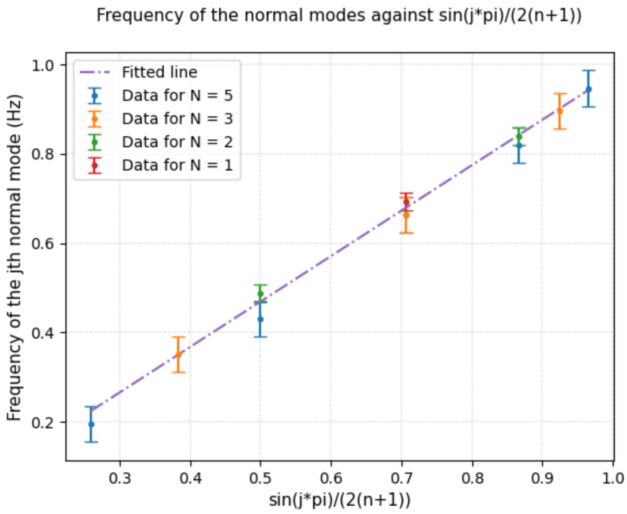


Figure 5: Frequencies of the normal modes for N = 1,2,3, and 5 against $\sin\left(\frac{j\pi}{2(N+1)}\right)$ where j is the number of the normal mode.

Figure (5) shows the graph of the frequency of the normal modes measured in the experiment, the gradient was found to be $(1.018 \pm 0.046)s^{-1}$. Using equation (2) is is possible to calculate the expected value for $k/m$ where k is the spring constant and m is the mass of the gliders, using:

$$\text{gradient} = \frac{f}{\sin\left(\frac{j\pi}{2(n+1)}\right)} = \frac{\omega_j}{2\pi \sin\left(\frac{j\pi}{2(n+1)}\right)} = \frac{2\sin\left(\frac{j\pi}{2(n+1)}\right)\sqrt{\frac{k}{m}}}{2\pi \sin\left(\frac{j\pi}{2(n+1)}\right)} = \frac{1}{\pi}\sqrt{\frac{k}{m}} \tag{5}$$

So $k/m$ was measured as $(10.23 \pm 0.46)s^-2$ this is less than one standard error away from the theoretical value of $(10.37 \pm 0.18)s^-2$.

### 3.0.3 Measurements of normal mode amplitudes

Table 2: Ratio of amplitudes for N = 3.

| - | Theoretical | Theoretical | Measured | Measured |
|---|---|---|---|---|
| Normal mode number | $a_1/a_2$ | $a_2/a_3$ | $a_1/a_2$ | $a_2/a_3$ |
| 1 | 0.707 | 1.414 | $0.511 \pm 0.033$ | n/a |
| 2 | indeterminate | 0.000 | indeterminate | $0.000 \pm 0.002$ |
| 3 | 0.707 | 1.414 | $0.619 \pm 0.038$ | n/a |

Presented in Table (2) are the measured amplitude ratios of the rth masses in the system, and the theoretical prediction given by equation (2). It is clear that two values are many standard errors apart from the theoretical predictions, this could be due an underestimate of the standard errors or an systematic error in the experiment, the former seems more likely since the amplitude ratios for N = 5 were in much closer agreement to the theory.

Table 3: Ratio of amplitudes for N = 5.

| - | Theoretical | Theoretical | Measured | Measured |
|---|---|---|---|---|
| Normal mode number | $a_1/a_2$ | $a_3/a_1$ | $a_1/a_2$ | $a_3/a_1$ |
| 1 | 0.577 | 2.000 | n/a | $2.000 \pm 0.188$ |
| 2 | 1.000 | 0.000 | $0.943 \pm 0.057$ | n/a |
| 3 | indeterminate | 1.000 | n/a | $1.154 \pm 0.057$ |
| 4 | 1.000 | 0.000 | $1.029 \pm 0.062$ | n/a |
| 5 | 0.577 | 2.000 | n/a | $0.224 \pm 0.032$ |

Similarly, Table (3) has the measured amplitude ratios for N = 5. These value are much more aligned with the theory, since most are less than one standard error away from the predicted results. However, there is one clear outlier which is the value of $a_3/a_1$ for normal mode 5, which is in the order of 50 standard errors from the prediction. This single error is likely to be due to the difficulty in obtaining a single normal mode of oscillation.

# 4 Conclusion

The experiment measured normal mode characteristic for the N = 1,2,3, and 5 gliders cases and achieved a value of the ratio between the sprigs constant and glider mass of $(10.23 \pm 0.46)s^-2$ which is less than one standard error away from the theoretical calculation of $(10.37 \pm 0.18)s^-2$. It also correctly measured the associated amplitudes of the gliders and the majority of measurement agreed with the theory, however improvement could be made by repeating the amplitude measurements multiple times to reduce the random error encountered.

# References


[1] Matt Jarvis. *Waves & Normal Modes*. 2017. URL: https://www2.physics.ox.ac.uk/sites/default/files/2012-09-04/fullnotes2016_pdf_91657.pdf.

[2] University of Oxford. *Harmonic motion and normal modes*. 2022. URL: https://www-teaching.physics.ox.ac.uk/practical_course/scripts/srv/local/rscripts/trunk/General/GP06/GP06.pdf.

[3] Gabriel Bristot. *Note: All figures were made by me with the aid of the 'draw.io' website and assets*. 2024. URL: https://app.diagrams.net/.