Economics Project Report.

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Contents

1 Monthly, daily returns and statistical matrices.

This question involves analyzing the returns of a subset of 88 securities that are spreaded through a time series from January 1, 2015, to January 16, 2024. The dataset comprises 2359 daily returns derived from the closing prices of these securities. The calculation of the returns are done through this formula, that evaluate the logaritm change, here the ln operator is used to scale the results.

$$Return = ln\frac{P_t}{P_{t-1}} \tag{1}$$

where $Price_t$ denotes the the price ”today”, and $Price_{t-1}$ the price ”yesterday”, more formally the time series is lagged by one period, one day in the case of the daily return and one month in the monthly dataset and then compared with the list of the original observations. By giving a look at table A.1 and A.3, in appendix A, we can see that the returns are not normally distribuited, even if the non-normality in the monthly data it is slightly less pronounced, the two indicators: skewness, that indicates the simmetry with respect to the mean, and kurtosis, that measure the tailedness of the curve, are very different from the one that a normal distribuition would originate, 0 for the first and 3 for the second, assuming a mesokurtic distribution, thus meaning that some model like the Markowitz (mean-variance) could be unreliable, this meaning that the datas may need some additional manipulations.

2 Covariance, Correlation Matrices and stocks-picking.

Now for the second and third part of Question 1 we are asked to compute the covariance and the correlation matrices so we choose to portray them through two heatmaps respectively in table A.5 and A.6 in the appendix. Both for daily and monthly returns. To pick the stocks we choose to use the PCA (Principal Components Analysis) technique. Since, due to the volatile nature of financial data, it can often be overwhelming to make sense of it all. That's where PCA steps in. This powerful tool takes the convoluted data from various stocks and transforms it into a manageable set of factors. These factors, known as principal components, allow us to focus on the key aspects of the stock market without getting lost in the intricate details. Moreover, it illuminates the correlations between stocks, allowing us to understand why some move in unison while others chart their own courses. This knowledge is invaluable in determining which stocks complement each other in our portfolio. As we carefully select our portfolio of 10 stocks from a vast pool of 88, PCA plays a crucial role in our decision-making process. PCA helps us pinpoint the stocks that make the biggest impact on our portfolio in term of the variations. With this powerful tool, our portfolio gains laser-like precision, increased efficiency, and optimum readiness for market fluctuations. We finally choose the stocks, that are far right and more high, and with the second criterion of being in the same cluster, so as close as possible to each other.

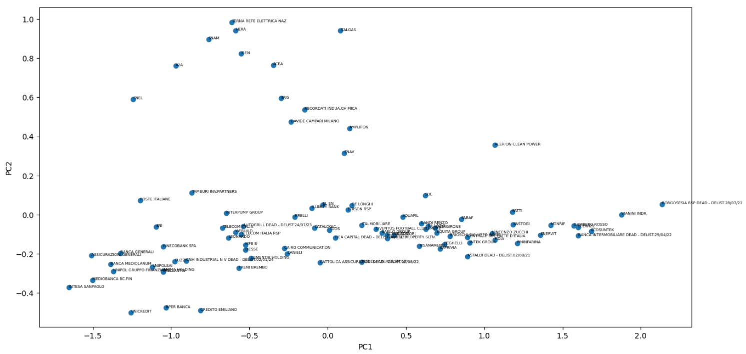

Figure 1: PCA Plot

- Ratti (Textile)
- Bastogi (Various)
- Monrif (Holfing)
- Gambero Rosso (Media)
- Beewize (Marketing)
- Ecosuntek (Construction)
- Vianini Indr. (Construction)
- Enervit (Food and Beverages)
- Sabaf (Interio Design)
- Alerion Clean Power (Utilities)

We show the plot of the prices, both for daily and monthly data:

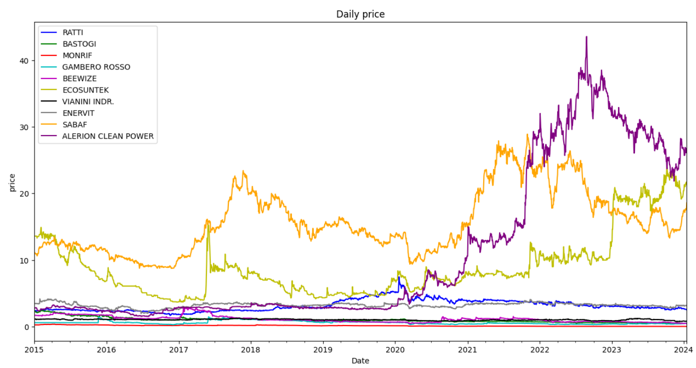

Figure 2: Daily Prices Trend.

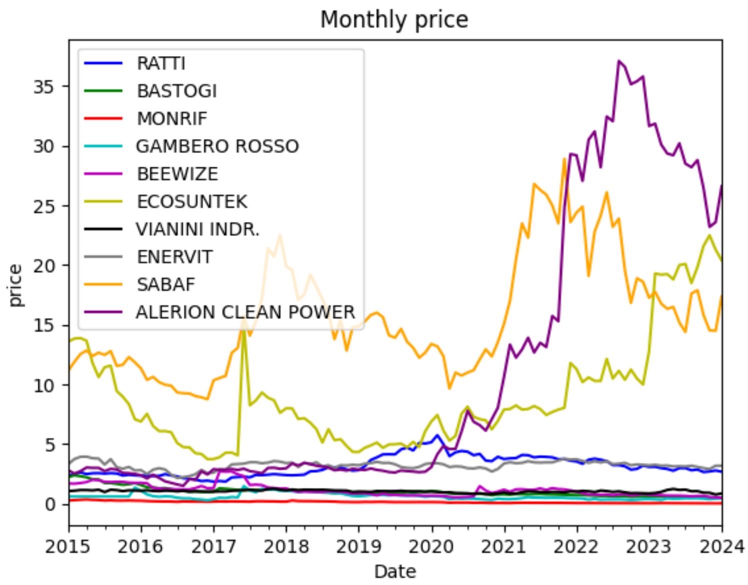

Figure 3: Monthly Prices Trend.

The flat lines that may appears at the beginning of some of the picked stocks are explained by the method we choose to treat with missing values in the dataset, we substituted them with the average value of all the available observations of the same stock. We also provide the graphs of the daily and monthly log returns of the selected portfolio stocks:

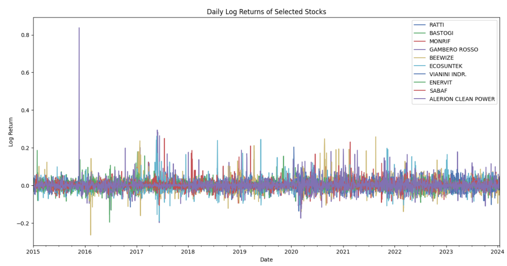

Figure 4: Plot of the ln returns with daily data.

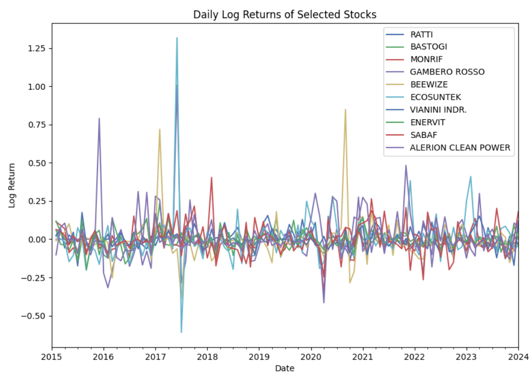

Figure 5: Plot of the ln returns with monthly data.

The persistent nature of natural logarithm returns in specific stocks presents a vital aspect in the field of financial time series analysis. Stationarity, the key concept at play, indicates that fundamental statistical measures like mean and variance do not exhibit significant changes over time. This provides a solid basis for creating more accurate models and predicting future outcomes. When we observe stationary behavior in ln returns, we can conclude that the instrument's performance is not influenced by prolonged trends or underlying shifts. With this consistency, we can effectively utilize statistical tools like autoregressive integrated moving average (ARIMA) models to capture past trends and make forecast predictions. The idea that returns tend to center around the mean supports the efficient market hypothesis, indicating that asset prices accurately reflect all available information. Within this stationary regime, any deviations from the mean are temporary, as the market efficiently incorporates new information. This concept is reinforced by the observation that logarithmic returns of select stocks remain stable around the mean, highlighting the importance of statistical stability in conducting financial time series analysis. It also serves as the foundation for creating models that utilize past data to better understand potential shifts in the market, while recognizing the fluid nature of financial markets.

3 Mean-Variance optimization

Mean-variance optimization is a quantitative framework widely employed in financial portfolio theory for the purpose of constructing portfolios that offer an optimal compromise between anticipated returns and risk. Originating from the seminal work of Harry Markowitz in the 1952[1], this methodology incorporates the concepts of mean return and variance (or standard deviation) of returns as pivotal considerations in the portfolio construction process.

In the context of mean-variance optimization, the assumption of a zero risk-free rate holds particular significance. This assumption implies an environment where investors can lend or borrow funds at no cost, simplifying the optimization process and rendering it both practical and broadly applicable.

Central to mean-variance optimization is the objective of identifying a portfolio allocation that maximizes the Sharpe ratio. The Sharpe ratio, a metric denoting risk-adjusted return, is computed by dividing the excess return of a portfolio (i.e., return above the risk-free rate) by its standard deviation. The mathematical representation of the Sharpe ratio is as follows:

$$SharpeRatio = \frac{E(R_p) - R_f}{\sigma_p} \tag{2}$$

Where $E(R_p)$ is the expected return, R_f is the risk free rate, in this case made equal to 0 and σ_p represent the Standard Deviation of the portfolio returns

3.1 Statistical Table of the portfolio.

As requested we give the statistical table of the two portfolio, with constraint to the non negativity of the weight and without constraint, formally this means that the first portfolio foresees the use of only capital, the latter however enable the use also of debt, but both are linked by the share condition that:

$$\sum_{i=0}^{n} w_i = 1 \tag{3}$$

We plot the two tables:

Table 1: Constrained Portfolio.

	Portfolio Stats Daily	Portfolio Stats Monthly
Mean	0.196912	0.194464
Variance	0.003551	0.003486
Standard Deviation	0.059593	0.059042
Skewness	3.156180	3.158544
Kurtosis	9.971121	9.982495

[1] Markowitz, 1952, https://www.jstor.org/stable/2975974

Table 2: Unconstrained portfolio.

	Portfolio Stats Daily	Portfolio Stats Monthly
Mean	0.470874	0.431060
Variance	0.005154	0.004902
Standard Deviation	0.071793	0.070014
Skewness	1.964293	2.294074
Kurtosis	3.976118	5.673344

The statistics are compute as follows[2]:

$$PortfolioMean = \sum_{i=1}^{n} (w_i \times R_i) \tag{4}$$

$$PortfolioVariance = E[\sum_{i}^{n} (w_i \times (\tilde{R}_i - \mu_i)]^2 \tag{5}$$

$$R_P ortfolioSkewness = \frac{\sum_{i=1}^{n} (w_i^3 \times Skew(R_i))}{(\sum_{i=1}^{n} w_i^2)^{3/2}} \tag{6}$$

$$PortfolioKurtosis = \frac{\sum_{i=1}^{n} (w_i^4 \times Kurt(R_i))}{(\sum_{i=1}^{n} w_i^2)^2} \tag{7}$$

We also provide a Table with the weight and the performance of the portfolios in the various settings:

Table 3: Constrained portfolio.

	weights daily	weights monthly	perf daily	perf monthly
RATTI	0.000000	0.033630		
BASTOGI	0.000000	0.000000		
MONRIF	0.000000	0.000000		
GAMBERO ROSSO	0.000000	0.000000		
BEEWIZE	0.000000	0.000000		
ECOSUNTEK	0.066970	0.112880		
VIANINI INDR.	0.000000	0.000000		
ENERVIT	0.000000	0.000000		
SABAF	0.177090	0.127680		
ALERION CLEAN POWER	0.755950	0.725810		
Expected Return			0.217664	0.217749
Volatility			0.318877	0.380280
Sharpe Ratio			0.682597	0.572600

Table 4: Unconstrained portfolio.

	weights daily	weights monthly	perf daily	perf monthly
RATTI	0.205830	0.205210		
BASTOGI	-0.490990	-0.381610		
MONRIF	-0.533590	-0.453630		
GAMBERO ROSSO	0.017170	0.048020		
BEEWIZE	-0.224610	-0.276530		
ECOSUNTEK	0.218070	0.307000		
VIANINI INDR.	0.075230	0.076580		
ENERVIT	0.235480	0.140650		
SABAF	0.497400	0.334300		
ALERION CLEAN POWER	1.000000	1.000000		
Expected Return			0.498750	0.483738
Volatility			0.552381	0.669469
Sharpe Ratio			0.902910	0.722570

Examining the statistical characteristics of constrained and unconstrained portfolios unveils interesting insights into their respective distributions. When constraints are imposed on a portfolio, such as limitations on weights, the resulting statistical data often

[2]Mean-Variance slides, Massimiliano Marzo.

exhibit a tendency towards a more normally distributed shape compared to their unconstrained counterparts.

Constrained portfolios, by design, enforce specific rules or limits on the allocation of assets. These constraints introduce a level of regularization and structure that can lead to a more balanced distribution of returns. In contrast, unconstrained portfolios have the flexibility to take on a broader range of allocations, potentially resulting in a distribution that is less centered or more skewed.

The imposition of constraints acts as a stabilizing force, mitigating extreme outcomes and encouraging a more even distribution of returns. This can lead to a portfolio that is less prone to outliers or extreme events, contributing to the normality observed.

However, it's essential to note that the normality assumption is a simplification and may not hold in all cases.

We also provide for better explaining the portfolio the relative variance-covariance and correlation heatmaps:

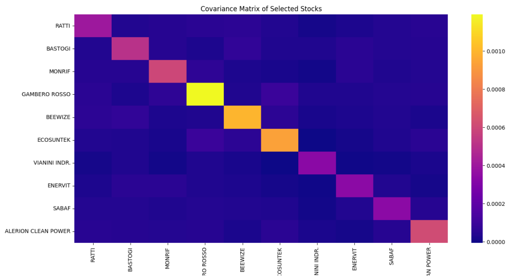

Figure 6: Covariance Heatmap of the selected stocks.

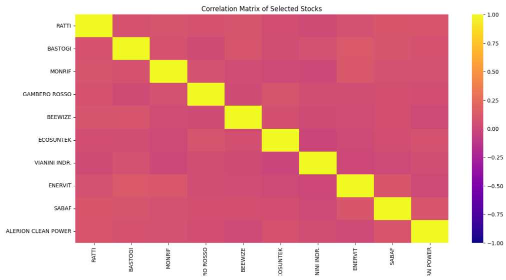

Figure 7: Correlation Heatmap of the selected stocks.

3.2 Efficient Frontier.

At its core, the efficient frontier is a graphical representation of different portfolios that offer the maximum expected return for a given level of risk or the minimum risk for a targeted level of return. The term "efficient" in this context implies that portfolios lying on the frontier provide the best possible trade-off between risk and return, making them optimal choices for investors seeking to maximize their investment outcomes.

The construction of the efficient frontier begins with the identification of various asset combinations and their associated risk-return profiles. Each point on the efficient frontier represents a unique portfolio, with its risk and return characteristics determined by the allocation of assets within that portfolio. By systematically adjusting the weights of different assets, investors can navigate along the efficient frontier to find the portfolio that aligns with their specific risk tolerance and return objectives.

Diversification plays a key role in the efficiency of the portfolios on the frontier. Markowitz's groundbreaking work emphasized the importance of spreading investments across different asset classes to achieve optimal diversification. The correlation and covariance between asset returns are crucial considerations, as they impact the overall risk of the portfolio. Well-diversified portfolios can, in theory, achieve a higher level of return for a given level of risk or a lower level of risk for a targeted return.

We give the efficient frontier of the selected stocks, with regard to monthly and daily data and in both scenarios:

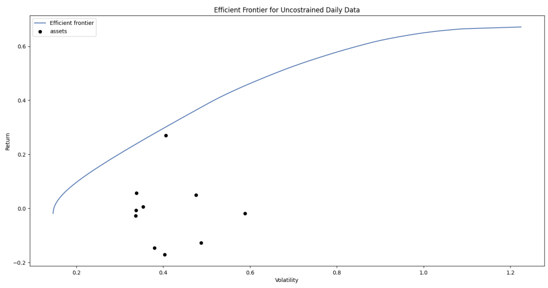

Figure 8: Efficient frontier for the Unconstrained Portfolio with daily data.

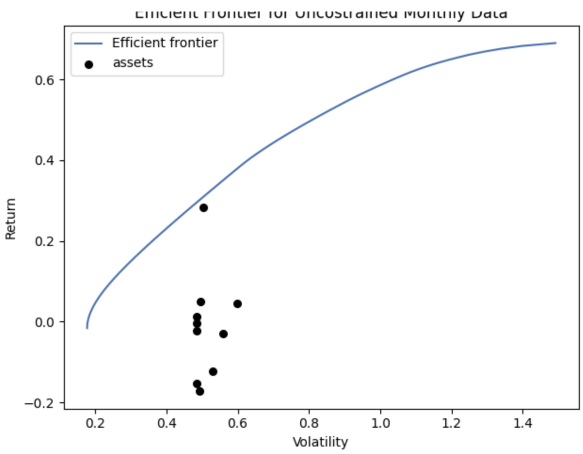

Figure 9: Efficient frontier for the Unconstrained Portfolio with monthly data

We also provide a random generated sample of 1000 portfolios given the maximization of the Sharpe ratio:

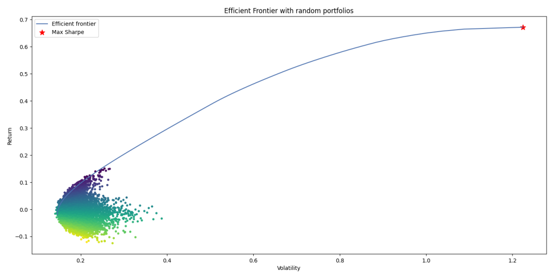

Figure 10: Plot of 1000 randomly sampled portfolios, with MV optimization and the Maximum Sharpe Ratio criterion.

4 Index FTSEMIB(PI)

When it comes to analyzing finances, our portfolio takes center stage with its great daily and monthly average of returns. With a roller-coaster-like character, this dynamic portfolio promises investors an exhilarating ride filled with unpredictable twists and turns. But let's not forget the potential for volatility, as seen in its daily and monthly standard deviations of 0.059593 and 0.059042. This portfolio isn't for the faint of heart and requires a strong and risk-tolerant investor to ride along. The plot thickens with positive skewness values of 3.156180 and 3.158544 for both daily and monthly reports, signaling that our portfolio is poised for success and growth.

Table 5: FTSEMIB.

Statistic	Daily Data	Monthly Data
Mean	0.000198	0.004331
Variance	0.000203	0.003471
Standard Deviation	0.014245	0.058917
Skewness	-1.648800	-0.935326
Kurtosis	19.324204	4.082020

5 Beta

$$\beta_{pf} = \sum_{i=1}^{n} w_i \beta_i \tag{8}$$

where the w_i is the portfolio weights and β_i is the beta of the coefficient which is:

$$\beta_i = \frac{\sigma_{m,i}}{\sigma_m^2} \tag{9}$$

where m is the market.
When navigating the complex realm of finance, the beta of a portfolio plays a pivotal role by providing a holistic assessment of the overall risk and volatility embedded in a combination of investments. Considered as a guiding light for investors, this metric integrates the unique features of individual securities to create a unified risk profile for

the entire portfolio.The beta of a portfolio is essentially a combination of the betas of all the securities that make it up. This calculation involves taking each security's beta and multiplying it by its proportionate weight in the portfolio. By doing so, we not only account for the riskiness of each individual component, but also take into consideration their significance within the portfolio. The meaning behind the numerical value of the portfolio's beta carries great weight. A beta of 1 signifies a general alignment between the portfolio's movements and the overall market. Should the market experience a rise of, say, 10%, the portfolio is likely to follow suit. A beta higher than 1 serves as a warning sign for increased volatility, suggesting that the portfolio may endure larger fluctuations compared to the market. On the other hand, a beta lower than 1 reveals a more steady performance, as the portfolio showcases less volatility than the market.Think of the portfolio beta as the conductor of a musical masterpiece, harmonizing risk and return. Each security acts as an instrument, playing a crucial part in creating the portfolio's overall tune. With a delicate balance of risk and return, savvy investors can expertly craft a portfolio that aligns with their financial goals. It becomes an essential tool for refining their investments, enabling them to fine-tune the mix of securities to achieve the desired level of risk exposure.By harnessing the powerful perspectives of the portfolio beta, investors gain the knowledge and foresight needed to make sound choices. This influential tool directs the creation of a portfolio that harmonizes with their desired level of risk and expected returns. Whether favoring stability or embracing volatility for the possibility of greater gains, the portfolio beta serves as a dependable guide through the intricacies of the financial world. As risk and return engage in a delicate dance, the portfolio beta takes center stage, providing investors with a steadfast source of guidance as they navigate the unpredictable tides of the market.

Table 6: Beta.

Stock	Beta (Daily)	Beta (Monthly)	Weight (Daily)	Weight (Monthly)
RATTI	0.459421	0.459421	0.205830	0.205210
BASTOGI	0.520028	0.520028	-0.490990	-0.381610
MONRIF	0.849984	0.849984	-0.533500	-0.453630
GAMBERO ROSSO	2.156418	2.156418	0.017170	0.048020
BEEWIZE	1.292401	1.292401	-0.224610	-0.276530
ECOSUNTEK	1.969926	1.969926	0.218070	0.307000
VIANINI INDR.	0.355095	0.355095	0.075230	0.076580
ENERVIT	0.551147	0.551147	0.235480	0.140650
SABAF	1.039551	1.039551	0.497400	0.334300
ALERION CLEAN POWER	0.806029	0.806029	1.000000	1.000000
Portfolio	0.100586	0.080602		

6 Security Market Line.

The Security Market Line (SML) is a pivotal concept in finance that delineates the correlation between the anticipated return of an investment and its systematic risk, denoted by the beta coefficient. This line graphically portrays the Capital Asset Pricing Model (CAPM), a widely employed framework for estimating the expected return on an asset. The SML formula is articulated as follows:

$$E(R_i) = R_f + \beta_i \times (E(R_m) - R_f) \tag{10}$$

where $E(R_i)$ is the represents the expected return of the investment, R_f signifies the risk-free rate, $beta_i$ stands for the beta coefficient of the investment, $E(R_m)$ denotes the expected return of the market. Discerning the stock's position relative to the SML holds significant implications for investors and analysts: if a stock's expected return surpasses the SML, it implies potential undervaluation. Investors may view it as an appealing investment opportunity, expecting returns that exceed the justified level of systematic risk. Conversely, if a stock's expected return falls below the SML, it suggests potential overvaluation. Investors might perceive the stock as offering returns insufficient for the systematic risk it bears, making it less desirable for their portfolio. When a stock's expected return aligns precisely with the SML, it signifies that the stock is priced commensurate with its systematic risk. According to CAPM, this represents the fair market value, and investors should anticipate a return proportional to the systematic risk. We plot the graph of the SML both with, daily and monthly frequencies:

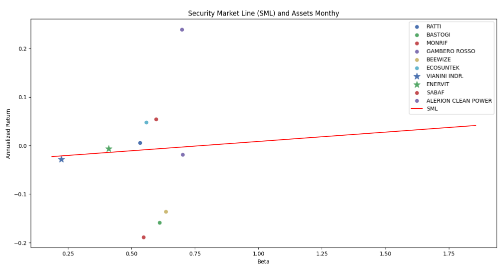

Figure 11: SML with Daily frequencies.

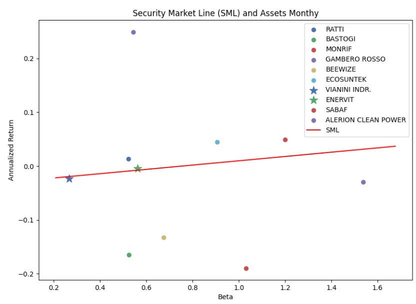

Figure 12: SML with Monthly frequencies.

Vianini and Enervit, both positioned along the Security Market Line (SML), exhibit interesting dynamics in their daily and monthly stock performances.

On a daily frequency, these stocks reflect the short-term fluctuations and market sentiment. Vianini, being on the SML, suggests that its daily returns are aligned with its systematic risk, providing a real-time insight into how market dynamics impact its valuation, and it's the same for Enervit.

Zooming out to a monthly frequency provides a broader perspective. Monthly data smoothens the characteristic noise of daily fluctuations and gives a clearer view of the stocks' performance over a more extended period.Again, Vianini's placement on the SML at this frequency reinforces the notion that its monthly returns are consistent with its systematic risk, offering investors a stable outlook. Enervit, similarly, maintains its SML position on a monthly basis. For clarification purposes we saw that the SML is verified, since Alerion Clean Power, was undervalued by the regression, in fact it is the higher asset in the plot, and it is above the SML and if we look at the table in the appendix, (A1) Alerion exhibit the greatest returns of all the portfolio assets, instead for Monrif, the regression overvalue the stock, in fact it lies below the line, and if we look at the same appendix (A1) the daily returns are not even positive for the latter.

7 Black-Litterman Approach.

The Black-Litterman model stands as a sophisticated and influential approach in the realm of modern portfolio theory, addressing the limitations and challenges posed by the traditional mean-variance optimization framework. Coined by Fischer Black and Robert Litterman in 1990 and published in 1992[3], this model offers a novel perspective on the asset allocation process by integrating subjective views of investors with equilibrium market expectations. The Black-Litterman asset allocation model, is a portfolio construction method that overcomes the problem of unintuitive, highly-concentrated portfolios, input-sensitivity, and estimation error maximization. These three related and well-documented problems with mean-variance optimization are the most likely reasons that more practitioners do not use the Markowitz paradigm, in which return is maximized for a given level of risk. The Black-Litterman model uses a Bayesian approach to combine the subjective views of an investor regarding the expected returns of one or more assets with the market equilibrium vector of expected returns (the prior distribution) to form a new, mixed estimate of expected returns. The resulting new vector of returns (the posterior distribution), leads to intuitive portfolios with sensible portfolio weights. [4] For the Ceteris Paribus condition we assume the same selected stocks to make the Black-Litterman model, we choose the views to use:

- Positive Absolute View on ENERVIT: Hypothetical Percentage: 0.15 Justification: A moderately positive view is assigned, indicating a degree of confidence in the growth potential of Enervit given favorable industry trends.
- Negative Absolute View on BEEWIZE: Hypothetical Percentage: 0.1 Justification: A relatively lower negative view is assigned, reflecting concerns but acknowledging potential mitigating factors or uncertainties in the renewable energy sector.
- Neutral Absolute View on SABAF: Hypothetical Percentage: 0.05 Justification: A low percentage is assigned due to a neutral stance, indicating a cautious approach until more concrete information on growth prospects becomes available.
- Relative Positive View on RATTI compared to MONRIF: Hypothetical Percentage: 0.12 Justification: A moderate allocation reflects a relatively higher confidence in Ratti's potential compared to Monrif based on sector-specific considerations.
- Relative Negative View on BASTOGI compared to GAMBERO ROSSO: Hypothetical Percentage: 0.08 Justification: A lower allocation indicates a relatively weaker conviction in the negative outlook for Bastogi compared to the positive outlook for Gambero Rosso.

[3]Black and Litterman, 1992, https://www.jstor.org/stable/4479577

[4]Idzorek, T.M. (no date) A step-by-step guide to the black-litterman model. Available at: https://people.duke.edu/ charvey/Teaching/BA453$_2$006/$Idzorek_onBL.pdf(Accessed$: $21January$2024).

- Relative Positive View on VIANINI INDR. compared to ECOSUNTEK: Hypothetical Percentage: 0.1 Justification: A moderate allocation signifies a relatively stronger positive outlook for Vianini compared to the challenges faced by Ecosuntek in the renewable energy sector.

We also give the plot of the market implied prior returns, that are calculated as follows:

$$\Pi = \delta \Sigma w_{mkt} \tag{11}$$

Where δ is computed in this way:

$$\delta = \frac{R - R_f}{\sigma^2} \tag{12}$$

Here, w_{mkt} denotes the market-cap weights. This formula is calculating the total amount of risk contributed by an asset and multiplying it with the market price of risk, resulting in the market-implied returns vector Π.

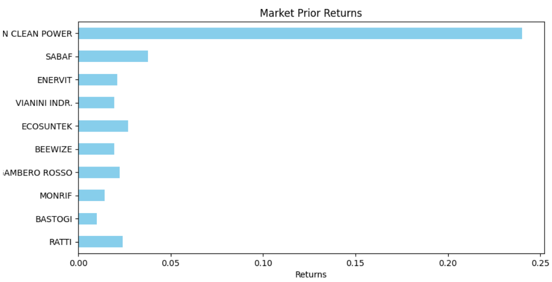

Figure 13: Priors Daily frequencies.

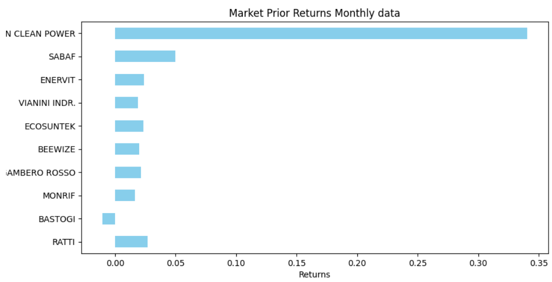

Figure 14: Priors Monthly frequencies.

Then the views are represented as a vector and a matrix, we have the views as a vector:

$$Q = [v_1, v_2, ...v_n] \tag{13}$$

and the picking matrix, which is a matrix nxm where the each columns is associated with a stock of our portfolio, and the n-th rows is relative to the n-th view, the the entries are weighted to make the relative and absolute views. We now give the table with the returns, both daily and monthly:

Table 7: Daily BL returns table.

	Prior	Posterior
RATTI	0.024056	0.051405
BASTOGI	0.010170	0.034856
MONRIF	0.014314	-0.012096
GAMBERO ROSSO	0.022405	-0.019219
BEEWIZE	0.019551	0.053588
ECOSUNTEK	0.026932	-0.022114
VIANINI INDR.	0.019370	0.045345
ENERVIT	0.021042	0.098285
SABAF	0.037674	0.043617
ALERION CLEAN POWER	0.240213	0.240675

Table 8: Monthly BL returns table.

	Prior	Posterior
RATTI	0.027132	0.054462
BASTOGI	-0.010321	0.022174
MONRIF	0.016194	-0.010396
GAMBERO ROSSO	0.021582	-0.026584
BEEWIZE	0.020024	0.052459
ECOSUNTEK	0.023475	-0.025443
VIANINI INDR.	0.019069	0.045837
ENERVIT	0.023806	0.099612
SABAF	0.049594	0.049005
ALERION CLEAN POWER	0.340336	0.340696

We now compute and plot the table relative to the statistics of the two portfolios:

Table 9: Stats table BL Daily.

	Mean	SD	Variance	Skewness	Kurtosis	Sharpe Ratio
0	0.051433	0.076608	0.005869	1.812469	4.264149	0.279775

Table 10: Stats table BL Monthly.

Mean	SD	Variance	Skewness	Kurtosis	Sharpe Ratio
0.060183	0.106434	0.011328	2.354864	6.429711	0.283581

Maximizing the Sharpe ratio is a fundamental objective in investment optimization. When applied in the context of the Black-Litterman model, which enhances the traditional mean-variance optimization by incorporating subjective investor views, it provides a powerful framework for constructing a well-balanced portfolio.

In the Black-Litterman model, this optimization process is augmented by blending market equilibrium returns with investor views, allowing for a more nuanced and customized portfolio construction.

Assuming a risk-free rate of 0.03, the Max Sharpe optimization within the Black-Litterman framework aims to achieve the highest possible Sharpe ratio by adjusting the weights of assets in the portfolio. This process involves finding the optimal allocation that balances the trade-off between expected returns and portfolio volatility, while considering both the market's equilibrium and investor-specific views. We give the resulting weights both in daily and monthly frequencies:

Table 11: Weights BL, no shorting allowed.

Asset	Weights Daily	Weights Monthly
RATTI	0.071390	0.066440
BASTOGI	0.029540	0.003310
MONRIF	0.000000	0.000000
GAMBERO ROSSO	0.000000	0.000000
BEEWIZE	0.072060	0.060100
ECOSUNTEK	0.000000	0.000000
VIANINI INDR.	0.060730	0.049810
ENERVIT	0.192910	0.153450
SABAF	0.052870	0.053070
ALERION CLEAN POWER	0.520480	0.613830

8 Bayesian approach.

In this exercise, we are implementing a Bayesian Asset Allocation approach with specific assumptions regarding prior distributions. The prior distribution chosen is a conjugate prior, specifically a normal distribution. Conjugate priors simplify calculations as they result in a posterior distribution of the same form as the prior when combined with the likelihood function.

For the mean of the prior distribution, we assume a normal distribution with a mean equal to the mean of the vector of returns plus one standard deviation. This choice integrates historical returns information while allowing for some flexibility to account for uncertainties or potential future changes. By perturbing the original variance-covariance matrix, the covariance matrix of the prior distribution is determined. This perturbation, achieved by multiplying the original matrix by a factor of 2, introduces a sense of caution into prior beliefs. It acknowledges the presence of uncertainty and broadens the spectrum of potential outcomes. The process of Bayesian Asset Allocation entails utilizing observed data (returns) to update the prior distribution and obtain the posterior distribution. By applying Bayes' theorem, the prior distribution is combined with the likelihood function calculated from the data. The resulting posterior distribution reflects a refined perspective on the parameters of interest: the mean and covariance matrix of returns. The adoption of the Bayesian methodology proves valuable in cases where data from the past is sparse or there is a need to integrate personal convictions into the determination of asset allocation. By merging preexisting beliefs with observable information, the Bayesian Asset Allocation strategy offers a stronger and more tailored approach to building portfolios, taking into account both historical trends and individual viewpoints. The selection of prior distributions and perturbations can greatly influence the results, making sensitivity analysis a crucial step in evaluating the reliability of the outcomes. We give the two tables:

Table 12: Bayesian statistical table, both frequencies.

	Mean	Standard Deviation	Variance	Skewness	Kurtosis	Sharpe Ratio
Daily	0.236638	0.283611	0.080435	1.560581	3.500346	0.763858
Monthly	0.243531	0.350757	0.123031	1.687607	4.022484	0.637282

For completness purpose, we also coded a samle of 1000 random portfolios, given the Bayesian allocation, and the max sharpe criterion:

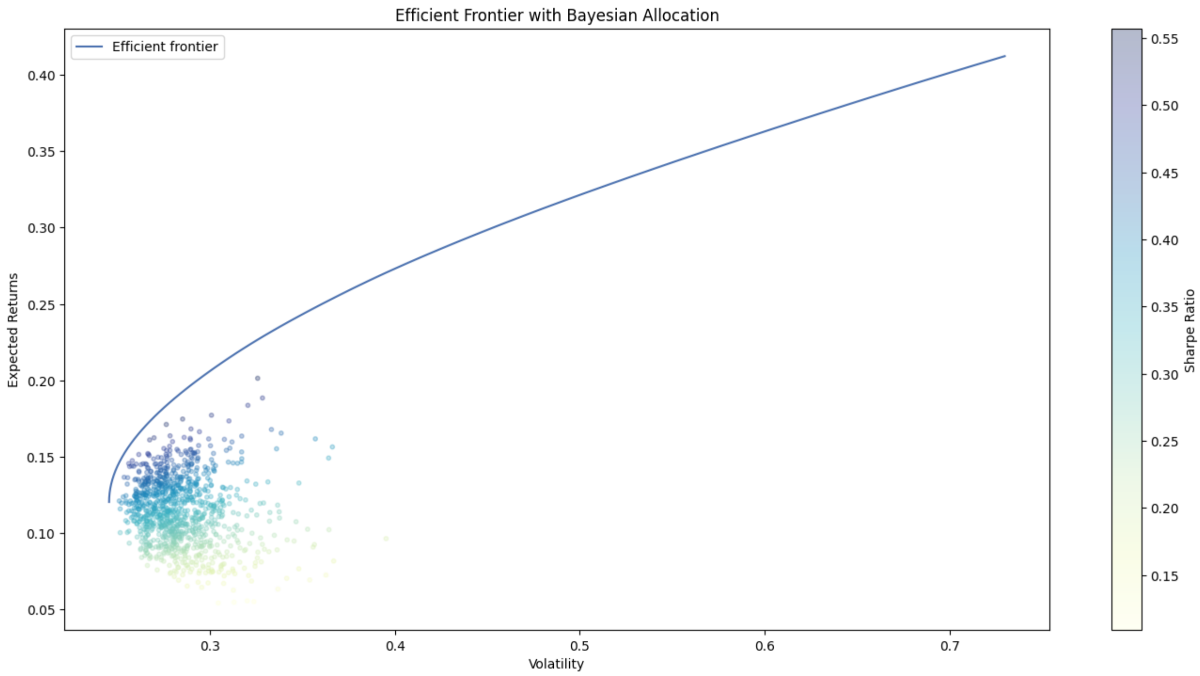

Figure 15: Sample of 1000 randomly generated portfolios.

8.1 Mathematical Bayesian formulation.

We provide the necessary mathematical computation behind the theory of the Bayesian allocation optimal model, the priors are computed as follows[5]

$$f_{pr}(\mu) = N(\mu_i, \lambda_i^2) \tag{14}$$

where: μ_i is computed as: $\mu_i = \hat{\mu} + \hat{\sigma}$ and λ_i^2 is $2 \times \hat{\Sigma}$, and then after the optimization the resulting distribution is $N(\mu_p, \Sigma_p)$ The resulting covariance table are the following:

Table 13: Bayesian Posterior Covariance daily Table.

	RATTI	BASTOGI	MONRIF	GAMBERO ROSSO	BEEWIZE	ECOSUNTEK	VIANINI INDR.	ENERVIT	SABAF	ALERION
RATTI	0.259529	0.011585	0.018127	0.018723	0.019243	0.010608	0.003879	0.008522	0.013901	0.016562
BASTOGI	0.011585	0.299812	0.014692	0.013704	0.024770	0.010272	0.009569	0.017359	0.012557	0.014018
MONRIF	0.018127	0.014692	0.336255	0.019172	0.007592	0.004436	0.001400	0.017259	0.010275	0.013081
GAMBERO ROSSO	0.018723	0.013704	0.019172	0.710567	0.009495	0.035331	0.009088	0.010602	0.013176	0.014355
BEEWIZE	0.019243	0.024770	0.007592	0.009495	0.488546	0.018095	0.004616	0.007021	0.012125	0.005169
ECOSUNTEK	0.010608	0.010272	0.004436	0.035331	0.018095	0.465010	0.002480	0.003593	0.009739	0.018993
VIANINI INDR.	0.003879	0.009569	0.001400	0.009088	0.004616	0.002480	0.234632	0.000058	0.002675	0.007474
ENERVIT	0.008522	0.017359	0.017259	0.010602	0.007021	0.003593	0.000058	0.232523	0.012002	0.002867
SABAF	0.013901	0.012557	0.010275	0.013176	0.012125	0.009739	0.002675	0.012002	0.237480	0.015473
ALERION	0.016562	0.014018	0.013081	0.014355	0.005169	0.018993	0.007474	0.002867	0.015473	0.345605

[5]Bayesian Slides, Massimiliano Marzo.

Table 14: Bayesian Posterior Covariance monthly Table.

	RATTI	BASTOGI	MONRIF	GAMBERO ROSSO	BEEWIZE	ECOSUNTEK	VIANINI INDR.	ENERVIT	SABAF	ALERION
RATTI	0.489115	0.002942	0.010942	0.006798	0.004623	0.004213	0.001260	0.002376	0.003684	0.001632
BASTOGI	0.002942	0.487448	0.005354	0.019146	0.013459	0.001155	0.002149	0.004119	0.005338	0.000474
MONRIF	0.010942	0.005354	0.504592	0.006955	0.012199	0.004282	0.005163	0.005103	0.007830	0.001695
GAMBERO ROSSO	0.006798	0.019146	0.006955	0.646271	0.013630	0.139192	0.005279	0.002388	0.022543	0.004831
BEEWIZE	0.004623	0.013459	0.012199	0.013630	0.581515	0.001968	0.004299	0.006642	0.007083	0.000758
ECOSUNTEK	0.004213	0.001155	0.004282	0.139192	0.001968	0.736940	0.014638	0.004395	0.017809	0.011251
VIANINI INDR.	0.001260	0.002149	0.005163	0.005279	0.004299	0.014638	0.488204	0.002504	0.002446	0.001023
ENERVIT	0.002376	0.004119	0.005103	0.002388	0.006642	0.004395	0.002504	0.481596	0.006499	0.007530
SABAF	0.003684	0.005338	0.007830	0.022543	0.007083	0.017809	0.002446	0.006499	0.509636	0.009475
ALERION	0.001632	0.000474	0.001695	0.004831	0.000758	0.011251	0.001023	0.007530	0.009475	0.533076

9 GMV portfolio.

The Global Minimum Variance (GMV) Portfolio is a crucial concept in modern portfolio theory, emphasizing the importance of diversification in minimizing risk. It aims to find the perfect balance between different assets in order to achieve the highest expected return for a particular risk level, or alternatively, the lowest risk for a desired return. Essentially, this portfolio is known for having the lowest possible variance, which determines the spread of returns around the expected value. Diversification is a crucial principle in investing, as it involves spreading out investments across a variety of assets. This helps to reduce the impact of individual asset volatility on the overall portfolio. The GMV Portfolio is an essential tool in creating an efficient frontier, which showcases a collection of ideal portfolios that offer the highest expected return minimizing the level of risk. This involves taking into account the covariance matrix of asset returns and their expected returns, making it a highly rigorous and mathematical process. Although the GMV Portfolio is known for its ability to mitigate risk, it may not offer the most significant returns in comparison to more volatile portfolios. Therefore, investors must carefully strike a balance between their risk tolerance and desired returns before choosing this portfolio. Furthermore, the GMV Portfolio's composition is heavily influenced by various input factors, such as expected returns and the covariance matrix. Even minor adjustments to these inputs can result in significant fluctuations in the most optimal asset allocation. We give the performances of the GMV portfolios, both in daily and monthly settings:

Table 15: GMV Stats.

	Mean	Standard Deviation	Skewness	Kurtosis	Sharpe Ratio
Daily	-0.008113	0.140247	0.846857	0.548629	-0.200455
Monthly	-0.012628	0.169896	0.924566	0.732018	-0.192049

In this case the criterion is the following: $\Downarrow min(volatility) = \Downarrow \sigma_p$.

10 Portfolios Linear Combinations.

As requested we first provide the results from the output portfolio that is been created by equally splitting the weight between the already found portfolios: this portfolio is given

Table 16: Equally Weighted Portfolio Stats.

	Mean	SD	Var	Skewness	Kurtosis	Sharpe Ratio
Resulting Portfolio	0.119218	0.140015	0.027381	1.844022	4.571061	0.381444

by:

$$ResultingPortfolio = \alpha \times MV + \alpha \times BL + \alpha \times BY + \alpha \times GMV \tag{15}$$

where $\alpha = 0.25$. We propose another methodology, we reward the portfolios that maximize the ratio between the Expected Mean Returns and Volatility(standard deviation) so the weight are as follow:

- MV:0.71, β
- BL: 0.13, ε
- BY: 0.15, λ
- GMV: 0.01, τ

Table 17: Ponderated Portfolio.

	Mean	SD	Var	Skewness	Kurtosis	Sharpe Ratio
Resulting Portfolio	0.181908	0.096214	0.015546	2.719064	8.164373	0.633589

the Optimized Portfolio is made as follows:

$$OptimizedPortfolio = \beta MV + \varepsilon \times BL + \lambda \times BY + \tau \times GMV \tag{16}$$

and the singular weight$(\beta, \varepsilon, \lambda, \tau)$ are computed as:

$$Weights_{(\beta,\varepsilon,\lambda,\tau)} = \frac{Ratio_i}{\sum_i^n PortfolioRatio} \tag{17}$$

where $Ratio_i$ is given by $Ratio_i = \frac{\mu_i}{\sigma_i}$ We see that the results are far more interesting and rewarding more worthy allocations give us a portfolio with an annualized expected return of the 18.2% and a volatility(SD) of the 9.6%.

We now forecast a 10 year investment with a starting capital of 100'000\$:

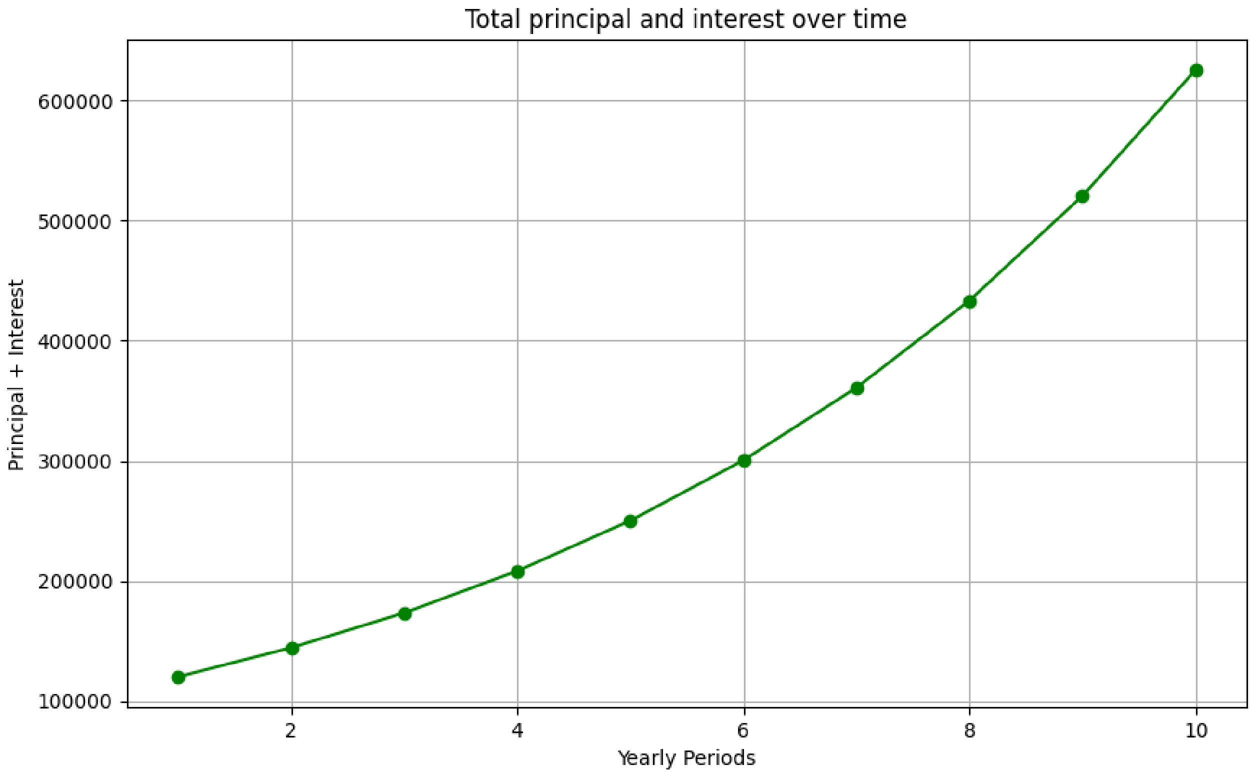

Figure 16: Investment forecast.

Here we taken into account the volatility computing the effective return as $r_{eff} = \sqrt{r^2 + \sigma^2}$ and then the capital at time t is $Capital_{(t)} = Capital_{(t=0)} \times (1 + r_{eff})^t$.

11 References.

1 Markowitz, 1952, https://www.jstor.org/stable/2975974

2 Mean-Variance, Massimiliano Marzo

3 Black and Litterman, 1992, https://www.jstor.org/stable/4479577

4 Idrozek, T.M. A step-by-step guide to the black-litterman model. https://people.duke.edu/charvey/Teaching/BA4532006/IdzorekonBL.pdf

5 Bayesian, Massimiliano Marzo.

A Appendix A

Table 18: Daily returns table. A1.

	MeanRtr	SDRtr	VarRtr	SkewRtr	KurtRtr
LEONARDO	0.000336	0.023145	0.000536	-0.862729	15.202548
ECOSUNTEK	0.000190	0.030539	0.000933	1.869836	14.491404
LANDI RENZO	-0.000339	0.029825	0.000890	0.448120	12.188471
PIRELLI	0.000001	0.018895	0.000357	0.770789	25.172626
STELLANTIS	0.000588	0.024071	0.000579	-0.731037	6.861303
PININFARINA	-0.000435	0.038515	0.001483	-10.650306	366.724951
FRENI BREMBO	0.000297	0.019110	0.000365	-0.023706	3.832859
INTESA SANPAOLO	0.000049	0.021024	0.000442	-1.287437	16.753749
ILLIMITY BANK	-0.000230	0.014474	0.000210	-1.237996	25.653520
UNICREDIT	0.000012	0.026811	0.000719	-0.452296	9.093765
BANCA GENERALI	0.000173	0.019131	0.000366	-0.294873	8.623387
BPER BANCA	-0.000067	0.028506	0.000813	-0.292492	8.751137
FINECOBANK SPA	0.000467	0.020442	0.000418	-0.318016	3.791565
DAVIDE CAMPARI MILANO	0.000541	0.015969	0.000255	-0.557945	9.798094
AQUAFIL	-0.000338	0.022525	0.000507	0.860736	17.721856
CALTAGIRONE	0.000331	0.017265	0.000298	-0.039772	4.574205
ASTALDI DEAD - DELIST.02/08/21	-0.001012	0.035685	0.001273	-1.486711	35.771876
ENEL	0.000255	0.015652	0.000245	-1.525561	19.822438
ALERION CLEAN POWER	0.000950	0.024733	0.000612	1.237411	10.174039
A2A	0.000332	0.016360	0.000268	-1.198036	14.417751
TERNA RETE ELETTRICA NAZ	0.000312	0.013784	0.000190	-0.859326	10.385374
ACEA	0.000194	0.015748	0.000248	-0.458037	7.274219
DEA CAPITAL DEAD - DELIST.08/03/23	-0.000019	0.015726	0.000247	0.696119	39.922044
BANCA MEDIOLANUM	0.000234	0.019489	0.000380	-0.683798	6.526864
BANCA INTERMOBILIARE DEAD - DELIST.29/04/22	-0.001464	0.044336	0.001966	-0.315188	22.834253
TAMBURI INV.PARTNERS	0.000525	0.015979	0.000255	-0.068803	9.886341
MEDIOBANCA BC.FIN	0.000226	0.020749	0.000431	-1.348905	16.265994
EQUITA GROUP	0.000065	0.012835	0.000165	0.696205	17.224065
ANIMA HOLDING	0.000012	0.023395	0.000547	-0.133498	5.325789
TELECOM ITALIA RSP	-0.000361	0.023829	0.000568	-0.483936	14.966206
TELECOM ITALIA	-0.000481	0.024029	0.000577	-0.046554	13.225784
ENERVIT	-0.000026	0.018337	0.000336	0.367751	13.861072
VALSOIA	-0.000198	0.017236	0.000297	0.474334	6.557030
CENTRALE DEL LATTE D'ITALIA	0.000042	0.019230	0.000370	2.248971	21.657237
HERA	0.000187	0.015644	0.000245	-0.844740	14.108009
IREN	0.000332	0.016225	0.000263	-0.695990	7.231571
ITALGAS	0.000010	0.014429	0.000208	-3.344295	57.656822
EL EN	0.000735	0.023390	0.000547	0.073263	3.571788
AMPLIFON	0.000775	0.020093	0.000404	-0.622594	7.627547
DE LONGHI	0.000290	0.020891	0.000436	0.062977	4.006922
BORGOSESIA RSP DEAD - DELIST.28/07/21	0.000323	0.025232	0.000637	-1.222396	195.776883
CNH INDUSTRIAL N V DEAD - DELIST.02/01/24	0.000272	0.022415	0.000502	-0.628919	5.773349
FIDIA	-0.000785	0.030714	0.000943	1.782622	15.773675
INTERPUMP GROUP	0.000578	0.019160	0.000367	-0.249510	2.769530
INTEK GROUP	0.000459	0.021496	0.000462	0.708224	8.912695
ENAV	-0.000085	0.013920	0.000194	0.530435	17.842316
POSTE ITALIANE	0.000079	0.017332	0.000300	-2.620010	35.000160
CATTOLICA ASSICURAZIONI DEAD - DELIST.12/08/22	0.000069	0.019371	0.000375	2.058810	43.651911
RIZZOLI CRER.DLSM.GP.	-0.000092	0.024446	0.000598	0.647454	9.271796
CAIRO COMMUNICATION	-0.000413	0.020412	0.000417	0.231981	5.182349
MONRIF	-0.000748	0.024356	0.000593	1.681480	17.021993
GAMBERO ROSSO	-0.000075	0.034522	0.001192	7.542303	157.549582
UNIPOL GRUPPO FINANZIARIO	0.000112	0.020908	0.000437	-0.662194	10.899449
ASSICURAZIONI GENERALI	0.000070	0.015472	0.000239	-1.093862	15.233845
SNAM	0.000154	0.014568	0.000212	-1.613897	22.940211
ENI	0.000012	0.017925	0.000321	-1.523961	22.582868
TODS	-0.000357	0.022256	0.000495	0.322251	13.067158
RECORDATI INDUA.CHIMICA	0.000578	0.017123	0.000293	-0.331051	12.574490
RISANAMENTO	-0.000487	0.033513	0.001123	0.877761	8.671781
BRIOSCHI SVILUPPO IMMBL	-0.000166	0.022986	0.000528	0.245687	6.731404
BEEWIZE	-0.000540	0.031611	0.000999	1.543177	15.104507
EXPRIVIA	0.000360	0.028230	0.000797	1.362706	12.089612
AUTOGRILL DEAD - DELIST.24/07/23	0.000104	0.021855	0.000478	0.370147	25.323539
JUVENTUS FOOTBALL CLUB	0.000182	0.025826	0.000667	-0.142470	8.478797
SS LAZIO	0.000196	0.024361	0.000593	-0.177783	11.713687
CLASS EDITORI	-0.001100	0.029635	0.000878	0.926912	11.587723
BASTOGI	-0.000632	0.022487	0.000506	0.767337	9.187105
CEMENTIR HOLDING	0.000274	0.020466	0.000419	0.033091	2.756923
UNIPOLSAI	0.000019	0.015484	0.000240	-0.345252	5.375956
BUZZI	0.000439	0.019640	0.000386	-0.199313	4.581073
CREDITO EMILIANO	0.000116	0.018032	0.000325	-0.226748	3.288480
DANIELI	0.000160	0.020337	0.000414	0.256890	8.197819
ITALMOBILIARE	0.000458	0.016748	0.000281	1.557120	23.220847
VINCENZO ZUCCHI	-0.000363	0.034368	0.001181	0.716303	44.941626
WEBUILD	-0.000205	0.024308	0.000591	-0.714217	14.889785
VIANINI INDR.	-0.000114	0.018245	0.000333	0.109654	3.929309
EDISON RSP	0.000246	0.013952	0.000195	-0.708311	15.773588
RATTI	0.000024	0.019935	0.000397	0.515790	10.321868
GABETTI PROPERTY SLTN.	-0.000025	0.027578	0.000761	0.903676	7.208720
MFE B	-0.000453	0.023520	0.000553	0.467786	15.762740
ERG	0.000454	0.017241	0.000297	-0.535803	14.559985
CEMBRE	0.000556	0.017665	0.000312	-0.010500	3.862601
SABAF	0.000216	0.018399	0.000339	0.255757	4.777694
BEGHELLI	-0.000196	0.022620	0.000512	1.551205	11.890731
SOL	0.000584	0.017349	0.000301	0.254016	1.680648
DATALOGIC	-0.000154	0.024080	0.000580	-0.034608	7.487679
BIESSE	0.000100	0.026998	0.000729	-0.493401	7.249355
SAFILO GROUP	-0.000777	0.028387	0.000806	-0.082859	11.434655

Table 19: Daily prices table. A2.

	MeanPrc	SDPrc	VarPrc	SkewPrc	KurtPrc
LEONARDO	9.954278	2.673585	7.148056	0.149908	-0.578113
ECOSUNTEK	8.942674	4.667347	21.784123	1.437510	1.329069
LANDI RENZO	0.778298	0.300597	0.090359	0.559551	-0.427172
PIRELLI	5.185271	0.952926	0.908068	0.694966	0.512315
STELLANTIS	11.206491	4.113018	16.916921	0.143244	-0.931467
PININFARINA	1.616563	0.777846	0.605044	0.938168	0.027243
FRENI BREMBO	10.556395	2.009052	4.036291	-0.082837	-0.613324
INTESA SANPAOLO	2.361081	0.456022	0.207956	0.307390	-0.399806
ILLIMITY BANK	8.862708	1.730320	2.994006	0.532004	1.181889
UNICREDIT	14.883504	6.195617	38.385673	1.016608	0.139768
BANCA GENERALI	27.659017	5.013543	25.135617	0.125182	-0.139783
BPER BANCA	2.723670	1.034090	1.069341	1.045074	0.768260
FINECOBANK SPA	10.297481	3.316052	10.996203	0.107858	-1.088062
DAVIDE CAMPARI MILANO	7.736804	2.768963	7.667157	-0.009466	-1.044682
AQUAFIL	7.261489	2.660390	7.077677	0.433102	-0.364069
CALTAGIRONE	2.951332	0.733216	0.537605	0.487928	-1.070657
ASTALDI DEAD - DELIST.02/08/21	2.323213	2.592483	6.720969	1.140204	0.054752
ENEL	5.591449	1.396330	1.949739	0.512623	-0.832531
ALERION CLEAN POWER	10.806458	11.623357	135.102426	1.078788	-0.465591
A2A	1.425360	0.246403	0.060715	-0.090253	-0.665470
TERNA RETE ELETTRICA NAZ	5.702808	1.164962	1.357137	0.349990	-1.073476
ACEA	14.515786	2.780108	7.729002	0.465720	-0.806991
DEA CAPITAL DEAD - DELIST.08/03/23	1.339123	0.175427	0.030775	0.527298	0.587242
BANCA MEDIOLANUM	7.157166	1.018882	1.038121	0.000806	-0.462146
BANCA INTERMOBILIARE DEAD - DELIST.29/04/22	0.351064	0.456109	0.208035	1.530616	1.165478
TAMBURI INV.PARTNERS	6.120011	1.930370	3.726328	-0.010195	-0.929521
MEDIOBANCA BC.FIN	8.725602	1.487101	2.211469	-0.300517	0.029003
EQUITA GROUP	3.172935	0.417287	0.174128	-0.366471	0.103350
ANIMA HOLDING	4.635314	1.324862	1.755260	1.142416	0.739541
TELECOM ITALIA RSP	0.512150	0.206841	0.042783	0.526148	-0.431457
TELECOM ITALIA	0.580053	0.279910	0.078349	0.586684	-0.729215
ENERVIT	3.235262	0.347497	0.120754	-0.591098	1.036169
VALSOIA	14.311437	4.087943	16.711280	0.840069	0.345595
CENTRALE DEL LATTE D'ITALIA	2.938593	0.386332	0.149253	0.580246	-0.096412
HERA	2.888909	0.490176	0.240273	0.467597	-0.387627
IREN	2.013896	0.464763	0.216004	-0.125721	-0.921510
ITALGAS	5.192563	0.510365	0.260472	-1.012005	2.126988
EL EN	7.334503	3.934121	15.477305	0.624372	-0.624208
AMPLIFON	21.295975	11.218277	125.849742	0.315856	-1.050880
DE LONGHI	23.928448	5.235312	27.408491	0.736524	0.443465
BORGOSESIA RSP DEAD - DELIST.28/07/21	1.150750	0.574431	0.329971	-0.195435	-1.436153
CNH INDUSTRIAL N V DEAD - DELIST.02/01/24	9.060666	2.970587	8.824388	0.478351	-0.832881
FIDIA	4.161915	2.216509	4.912910	0.157136	-1.365723
INTERPUMP GROUP	30.815214	13.481224	181.743397	0.492109	-0.595936
INTEK GROUP	0.382118	0.214711	0.046101	1.872204	2.524804
ENAV	4.082401	0.505650	0.255682	0.953004	1.792003
POSTE ITALIANE	8.405186	1.740284	3.028587	0.295780	-0.925689
CATTOLICA ASSICURAZIONI DEAD - DELIST.12/08/22	6.707554	1.219269	1.486617	0.055939	0.139536
RIZZOLI CRER.DLSM.GP.	0.875905	0.242836	0.058969	0.475637	-0.809819
CAIRO COMMUNICATION	2.889087	1.242692	1.544284	0.330472	-1.155623
MONRIF	0.154404	0.080422	0.006468	0.490765	-0.631311
GAMBERO ROSSO	0.614530	0.244138	0.059604	1.213505	0.721605
UNIPOL GRUPPO FINANZIARIO	4.202162	0.711859	0.506744	-0.548521	-0.166699
ASSICURAZIONI GENERALI	15.904419	2.223383	4.943433	-0.349210	-0.687351
SNAM	4.333511	0.484611	0.234848	0.055503	-0.928475
ENI	13.236396	2.320774	5.385994	-1.004446	0.477724
TOD'S	49.417762	17.075444	291.570785	0.597354	-0.247205
RECORDATI INDUA.CHIMICA	36.129373	9.705511	94.196947	-0.241142	-0.593473
RISANAMENTO	0.076653	0.044701	0.001998	0.459512	-0.957289
BRIOSCHI SVILUPPO IMMBL	0.074811	0.014895	0.000222	0.182952	-0.217493
BEEWIZE	1.140239	0.483208	0.233490	1.166722	1.153387
EXPRIVIA	1.158651	0.432639	0.187176	0.607320	-0.660577
AUTOGRILL DEAD - DELIST.24/07/23	7.035557	1.489644	2.219039	-0.238435	0.375166
JUVENTUS FOOTBALL CLUB	0.500868	0.282282	0.079683	0.756392	-0.463176
SS LAZIO	1.003743	0.338540	0.114609	-0.071209	-0.706251
CLASS EDITORI	0.282768	0.264381	0.069897	1.807875	2.895493
BASTOGI	1.013409	0.393987	0.155226	1.468696	2.225228
CEMENTIR HOLDING	6.444904	1.389135	1.929695	0.273139	-0.360050
UNIPOLSAI	2.207255	0.274522	0.075362	-0.571534	0.327419
BUZZI	19.461026	3.502610	12.268279	0.008606	-0.120281
CREDITO EMILIANO	6.022645	1.125950	1.267764	0.034627	-0.905835
DANIELI	19.618652	3.899917	15.209356	-0.079506	-0.152263
ITALMOBILIARE	23.593548	4.685768	21.956422	-0.279975	-0.085893
VINCENZO ZUCCHI	2.547481	0.958273	0.918287	1.373214	1.679400
WEBUILD	2.299083	0.865901	0.749785	0.645154	-0.666681
VIANINI INDR.	1.048234	0.099325	0.009865	-0.324578	-0.314781
EDISON RSP	1.032539	0.270676	0.073265	0.582039	-0.531075
RATTI	3.155549	0.876070	0.767498	0.871814	0.599970
GABETTI PROPERTY SLTN.	0.736010	0.463026	0.214393	1.146272	0.585473
MFE B	7.285551	2.839940	8.065258	0.054593	-0.794709
ERG	19.280078	6.829676	46.644469	0.294735	-1.095986
CEMBRE	21.644116	6.304075	39.741364	0.271313	-0.759071
SABAF	15.757821	4.734612	22.416548	0.657565	-0.442697
BEGHELLI	0.337087	0.087980	0.007741	-0.346664	-0.978548
SOL	13.213614	5.726615	32.794114	1.101032	0.191201
DATALOGIC	16.444574	7.292370	53.178665	0.596603	-0.482469
BIESSE	19.686744	9.594155	92.047819	1.312729	0.920477
SAFILO GROUP	2.435012	1.986016	3.944258	1.180428	0.305480

Table 20: Monthly Returns table. A3.

	MeanRtr	SDRtr	VarRtr	SkewRtr	KurtRtr
LEONARDO	0.006092	0.110494	0.012209	-0.632756	3.579094
ECOSUNTEK	0.003754	0.176531	0.031163	3.734739	29.461470
LANDI RENZO	-0.006620	0.141909	0.020138	1.569183	9.399469
PIRELLI	-0.000389	0.084621	0.007161	-0.505831	4.086679
STELLANTIS	0.013401	0.120373	0.014490	-0.877924	3.127709
PININFARINA	-0.009536	0.113204	0.012815	0.314677	3.052899
FRENI BREMBO	0.006435	0.094249	0.008883	-0.225159	-0.172335
INTESA SANPAOLO	0.000810	0.097273	0.009462	-0.983875	3.074800
ILLIMITY BANK	-0.004485	0.071616	0.005129	-2.186431	11.956256
UNICREDIT	-0.000204	0.122850	0.015092	-0.950049	3.113329
BANCA GENERALI	0.003500	0.091964	0.008457	-1.039720	2.136643
BPER BANCA	-0.001860	0.125906	0.015852	-0.029819	0.097617
FINECOBANK SPA	0.009891	0.087797	0.007708	-0.335365	0.027980
DAVIDE CAMPARI MILANO	0.012741	0.067864	0.004606	-0.520289	0.695328
AQUAFIL	-0.006932	0.096803	0.009371	-0.171392	2.751490
CALTAGIRONE	0.007485	0.074123	0.005494	-0.483853	1.262911
ASTALDI DEAD - DELIST.02/08/21	-0.022087	0.170904	0.029208	-2.833284	16.590705
ENEL	0.005549	0.063444	0.004025	-0.348962	1.161178
ALERION CLEAN POWER	0.020793	0.111603	0.012455	1.431956	2.509441
A2A	0.007383	0.074228	0.005510	-1.297434	3.551378
TERNA RETE ELETTRICA NAZ	0.006460	0.047223	0.002230	-0.205422	-0.402852
ACEA	0.004040	0.078272	0.006127	-0.488859	0.364844
DEA CAPITAL DEAD - DELIST.08/03/23	-0.000409	0.078958	0.006234	0.264132	2.514739
BANCA MEDIOLANUM	0.004428	0.090659	0.008219	-1.135036	5.202474
BANCA INTERMOBILIARE DEAD - DELIST.29/04/22	-0.031974	0.162739	0.026484	-0.195691	2.372828
TAMBURI INV.PARTNERS	0.011716	0.064234	0.004126	-0.265258	-0.044715
MEDIOBANCA BC.FIN	0.004665	0.095586	0.009137	-1.321088	4.740088
EQUITA GROUP	0.001394	0.053246	0.002835	-0.346165	4.763525
ANIMA HOLDING	0.000141	0.108695	0.011815	-0.463196	1.526569
TELECOM ITALIA RSP	-0.007670	0.101010	0.010203	0.213689	0.579790
TELECOM ITALIA	-0.010166	0.103843	0.010783	0.553878	1.740168
ENERVIT	-0.000396	0.062013	0.003846	-0.213014	1.105577
VALSOIA	-0.004524	0.073619	0.005420	0.568067	2.087894
CENTRALE DEL LATTE D'ITALIA	0.001274	0.065934	0.004347	1.445001	4.674358
HERA	0.003926	0.065647	0.004310	-0.838392	1.363849
IREN	0.007211	0.077041	0.005935	-0.807829	1.101644
ITALGAS	-0.000006	0.068448	0.004685	-2.601563	15.436384
EL EN	0.016404	0.128936	0.016624	-0.310111	0.899773
AMPLIFON	0.017174	0.085846	0.007369	-1.009529	2.174621
DE LONGHI	0.006596	0.098474	0.009697	-0.207952	0.311214
BORGOSESIA RSP DEAD - DELIST.28/07/21	0.007050	0.134244	0.018021	1.774307	16.047862
CNH INDUSTRIAL N V DEAD - DELIST.02/01/24	0.005946	0.098153	0.009634	-0.794417	3.530871
FIDIA	-0.010918	0.151707	0.023015	2.667958	14.884037
INTERPUMP GROUP	0.012890	0.093520	0.008746	-0.751380	0.423681
INTEK GROUP	0.010255	0.092611	0.008577	0.520738	1.710542
ENAV	-0.001590	0.060938	0.003713	-0.546962	3.330092
POSTE ITALIANE	0.001880	0.069586	0.004842	-0.832577	2.044907
CATTOLICA ASSICURAZIONI DEAD - DELIST.12/08/22	0.001517	0.106791	0.011404	0.117422	3.987231
RIZZOLI CRER.DLSM.GP.	-0.002148	0.113189	0.012812	0.448800	2.169154
CAIRO COMMUNICATION	-0.009144	0.096247	0.009263	-0.033998	0.529375
MONRIF	-0.015778	0.088574	0.007845	0.918181	6.063926
GAMBERO ROSSO	-0.002433	0.170246	0.028984	2.830573	14.937734
UNIPOL GRUPPO FINANZIARIO	0.002088	0.097819	0.009569	-1.125053	3.693689
ASSICURAZIONI GENERALI	0.001081	0.072150	0.005206	-0.645014	2.810760
SNAM	0.002952	0.051794	0.002683	-0.341456	-0.276356
ENI	0.000520	0.073231	0.005363	0.138131	2.214879
TODS	-0.006909	0.104926	0.011009	0.505222	1.974513
RECORDATI INDUA.CHIMICA	0.012361	0.067989	0.004623	-0.240957	1.055244
RISANAMENTO	-0.009367	0.157633	0.024848	0.229986	2.705598
BRIOSCHI SVILUPPO IMMBL	-0.003724	0.090206	0.008137	-0.546842	2.536598
BEEWIZE	-0.011008	0.140185	0.019652	3.222259	19.152816
EXPRIVIA	0.007894	0.141200	0.019937	0.695584	6.120844
AUTOGRILL DEAD - DELIST.24/07/23	0.002275	0.103969	0.010810	-0.016599	7.668754
JUVENTUS FOOTBALL CLUB	0.004257	0.139800	0.019544	0.265995	3.036648
SS LAZIO	0.004281	0.126022	0.015882	-0.246661	6.009030
CLASS EDITORI	-0.025102	0.114262	0.013056	0.079541	2.324822
BASTOGI	-0.013752	0.057539	0.003311	0.760684	3.168830
CEMENTIR HOLDING	0.005954	0.094623	0.008953	0.302429	0.286119
UNIPOLSAI	0.000107	0.070461	0.004965	-0.411900	1.296289
BUZZI	0.008920	0.076735	0.005888	-0.316408	-0.247454
CREDITO EMILIANO	0.002362	0.077600	0.006022	-0.017265	2.001163
DANIELI	0.003287	0.086403	0.007465	-0.466582	1.211596
ITALMOBILIARE	0.010023	0.067600	0.004570	1.399133	5.952866
VINCENZO ZUCCHI	-0.008016	0.093071	0.008662	0.938655	4.035683
WEBUILD	-0.004641	0.104172	0.010852	0.308691	0.888675
VIANINI INDR.	-0.001944	0.059036	0.003485	-0.003429	1.129798
EDISON RSP	0.005447	0.055201	0.003047	-0.187464	3.183785
RATTI	0.001091	0.059316	0.003518	-0.090860	1.983343
GABETTI PROPERTY SLTN.	-0.000502	0.156185	0.024394	1.209581	3.344588
MFE B	-0.009978	0.108570	0.011787	1.666972	8.292313
ERG	0.010530	0.069607	0.004845	-0.223332	1.752500
CEMBRE	0.011972	0.081172	0.006589	-0.862733	6.101200
SABAF	0.004083	0.097734	0.009552	-0.048282	0.260679
BEGHELLI	-0.003387	0.087536	0.007663	0.874564	2.720945
SOL	0.013238	0.060978	0.003718	0.179079	-0.477107
DATALOGIC	-0.002547	0.108892	0.011858	0.007013	-0.314710
BIESSE	0.002826	0.129905	0.016875	-0.613868	0.615720
SAFILO GROUP	-0.017441	0.133994	0.017954	0.148269	2.760068

Table 21: Monthly prices table. A4

	MeanPrc	SDPrc	VarPrc	SkewPrc	KurtPrc
LEONARDO	9.934761	2.674632	7.153656	0.132952	-0.514021
ECOSUNTEK	9.033991	4.689388	21.990357	1.351072	1.081425
LANDI RENZO	0.786944	0.310350	0.096317	0.726381	0.115374
PIRELLI	5.138627	0.964950	0.931129	0.673211	0.574566
STELLANTIS	11.146297	4.109262	16.886031	0.166430	-0.896743
PININFARINA	1.623832	0.791483	0.626445	0.965196	0.083648
FRENI BREMBO	10.562560	2.059267	4.240581	-0.105504	-0.558551
INTESA SANPAOLO	2.353232	0.462950	0.214323	0.324755	-0.331002
ILLIMITY BANK	8.854638	1.740341	3.028787	0.577397	1.350138
UNICREDIT	14.867052	6.281218	39.453697	1.005507	0.085909
BANCA GENERALI	27.582936	5.046132	25.463447	0.091302	-0.181894
BPER BANCA	2.723757	1.042698	1.087218	1.011961	0.721930
FINECOBANK SPA	10.274633	3.301740	10.901489	0.091137	-1.102752
DAVIDE CAMPARI MILANO	7.701518	2.776170	7.707119	-0.012411	-1.029871
AQUAFIL	7.273052	2.658400	7.067092	0.441546	-0.296869
CALTAGIRONE	2.948174	0.737398	0.543756	0.506068	-1.070905
ASTALDI DEAD - DELIST.02/08/21	2.340642	2.603789	6.779719	1.089804	-0.143140
ENEL	5.572413	1.378511	1.900294	0.489489	-0.932304
ALERION CLEAN POWER	10.819174	11.682611	136.483399	1.069080	-0.545463
A2A	1.419564	0.245444	0.060243	-0.103430	-0.615639
TERNA RETE ELETTRICA NAZ	5.694945	1.167069	1.362050	0.363804	-1.066610
ACEA	14.504954	2.798744	7.832966	0.442614	-0.794330
DEA CAPITAL DEAD - DELIST.08/03/23	1.340550	0.178294	0.031789	0.599871	0.779876
BANCA MEDIOLANUM	7.141700	1.011828	1.023796	0.021745	-0.486772
BANCA INTERMOBILIARE DEAD - DELIST.29/04/22	0.356606	0.466187	0.217330	1.532356	1.175172
TAMBURI INV.PARTNERS	6.120119	1.957132	3.830367	0.037034	-0.904828
MEDIOBANCA BC.FIN	8.699000	1.493904	2.231749	-0.203222	-0.025233
EQUITA GROUP	3.165784	0.419748	0.176188	-0.395800	0.269090
ANIMA HOLDING	4.638646	1.336647	1.786625	1.212170	0.995525
TELECOM ITALIA RSP	0.514002	0.209076	0.043713	0.562545	-0.272576
TELECOM ITALIA	0.582695	0.283506	0.080376	0.612907	-0.629683
ENERVIT	3.235321	0.346624	0.120148	-0.709165	0.889448
VALSOIA	14.325688	4.086977	16.703380	0.798329	0.104614
CENTRALE DEL LATTE D'ITALIA	2.938147	0.381608	0.145624	0.603316	-0.272911
HERA	2.883083	0.488721	0.238848	0.412479	-0.566291
IREN	2.015106	0.468129	0.219145	-0.154106	-0.895532
ITALGAS	5.183407	0.500225	0.250225	-1.059667	2.330691
EL EN	7.291531	3.972370	15.779721	0.648420	-0.591060
AMPLIFON	21.274807	11.345721	128.725380	0.353407	-0.980064
DE LONGHI	23.904679	5.300733	28.097773	0.693590	0.355578
BORGOSESIA RSP DEAD - DELIST.28/07/21	1.146606	0.575671	0.331397	-0.169711	-1.454260
CNH INDUSTRIAL N V DEAD - DELIST.02/01/24	9.048991	2.986969	8.921981	0.512810	-0.738923
FIDIA	4.171294	2.241621	5.024866	0.192644	-1.363740
INTERPUMP GROUP	30.653119	13.564946	184.007766	0.543526	-0.502746
INTEK GROUP	0.383114	0.217762	0.047420	1.918373	2.733719
ENAV	4.079811	0.495163	0.245186	0.847035	1.557386
POSTE ITALIANE	8.387081	1.741592	3.033144	0.353637	-0.880576
CATTOLICA ASSICURAZIONI DEAD - DELIST.12/08/22	6.718807	1.221212	1.491359	-0.017930	0.418094
RIZZOLI CRER.DLSM.GP.	0.877957	0.244242	0.059654	0.467783	-0.864755
CAIRO COMMUNICATION	2.911083	1.267704	1.607073	0.310094	-1.237333
MONRIF	0.155820	0.082099	0.006740	0.521342	-0.559867
GAMBERO ROSSO	0.621649	0.257713	0.066416	1.292406	1.142324
UNIPOL GRUPPO FINANZIARIO	4.198294	0.704169	0.495854	-0.629898	0.023796
ASSICURAZIONI GENERALI	15.904862	2.247727	5.052277	-0.287363	-0.620888
SNAM	4.322461	0.485369	0.235583	0.056990	-0.855391
ENI	13.249807	2.345625	5.501955	-1.089529	0.748824
TOD'S	49.698991	17.215786	296.383278	0.555026	-0.293746
RECORDATI INDUA.CHIMICA	35.930826	9.718776	94.454611	-0.259230	-0.486472
RISANAMENTO	0.076969	0.044713	0.001999	0.413456	-1.092234
BRIOSCHI SVILUPPO IMMBL	0.075202	0.014692	0.000216	0.103295	-0.461426
BEEWIZE	1.150991	0.499904	0.249904	1.219741	1.354699
EXPRIVIA	1.159339	0.439966	0.193570	0.668018	-0.556406
AUTOGRILL DEAD - DELIST.24/07/23	7.045413	1.492984	2.229001	-0.318971	0.638456
JUVENTUS FOOTBALL CLUB	0.496552	0.278045	0.077309	0.737102	-0.464444
SS LAZIO	1.005469	0.342836	0.117536	-0.113835	-0.841288
CLASS EDITORI	0.287739	0.271063	0.073475	1.779843	2.802106
BASTOGI	1.023376	0.408057	0.166510	1.517760	2.320110
CEMENTIR HOLDING	6.455991	1.379205	1.902206	0.343473	-0.214320
UNIPOLSAI	2.207174	0.270418	0.073126	-0.606393	0.482934
BUZZI	19.390917	3.484178	12.139499	0.011362	-0.140288
CREDITO EMILIANO	6.015541	1.152434	1.328103	0.036087	-0.842092
DANIELI	19.695780	3.934828	15.482875	-0.066755	-0.033475
ITALMOBILIARE	23.611101	4.768686	22.740362	-0.363569	0.090950
VINCENZO ZUCCHI	2.573556	0.981493	0.963329	1.353889	1.545892
WEBUILD	2.310560	0.883904	0.781286	0.659650	-0.610216
VIANINI INDR.	1.051156	0.104511	0.010922	-0.339664	-0.373306
EDISON RSP	1.033775	0.272386	0.074194	0.668177	-0.340265
RATTI	3.149972	0.851149	0.724455	0.717711	-0.119614
GABETTI PROPERTY SLTN.	0.735721	0.465784	0.216955	1.265553	1.221991
MFE B	7.290950	2.876552	8.274552	0.062410	-0.782322
ERG	19.260963	6.902228	47.640756	0.305632	-1.138100
CEMBRE	21.597248	6.432711	41.379765	0.271319	-0.763624
SABAF	15.755275	4.832453	23.352603	0.755862	-0.243830
BEGHELLI	0.338270	0.088412	0.007817	-0.355713	-0.937885
SOL	13.202110	5.758107	33.155797	1.099705	0.177887
DATALOGIC	16.457477	7.297539	53.254078	0.619324	-0.426909
BIESSE	19.613028	9.526640	90.756869	1.311636	0.891511
SAFILO GROUP	2.453856	2.018159	4.072967	1.179604	0.317440

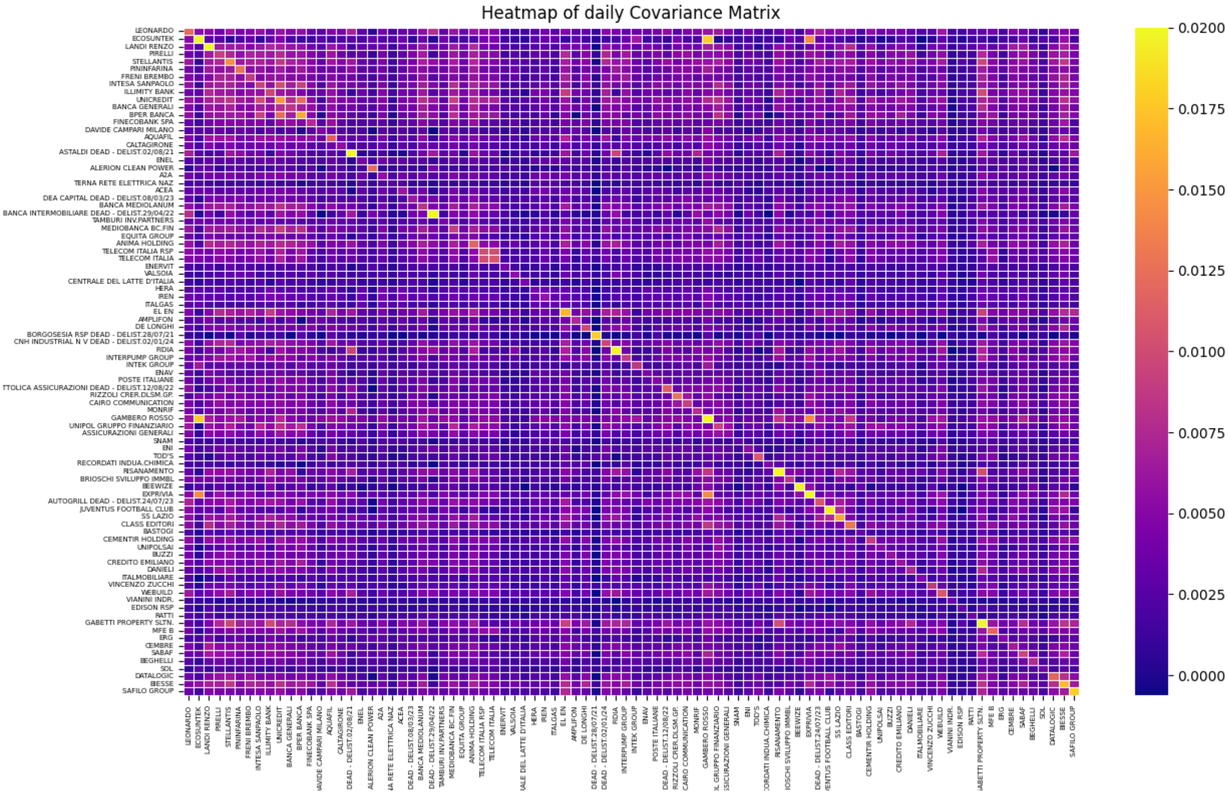

Figure 17: Heat map of daily Covariance Matrix. A.5a

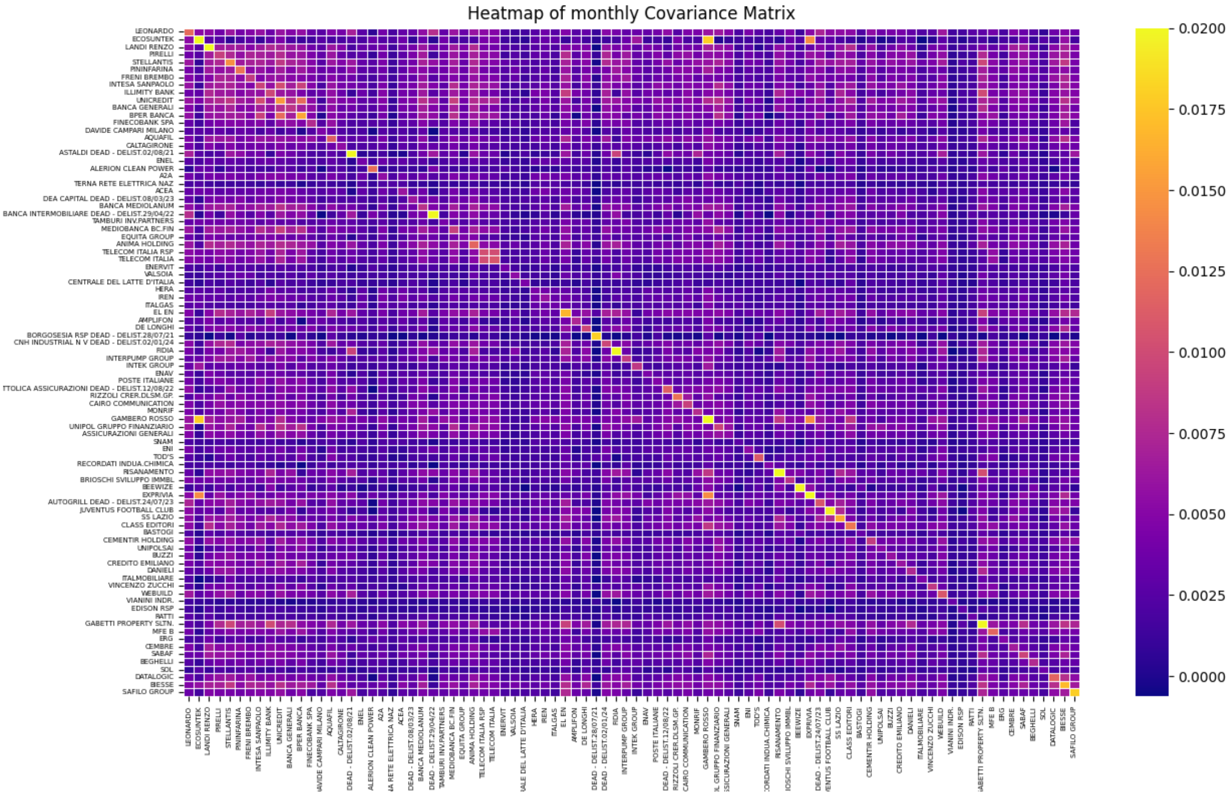

Figure 18: Heat map of monthly Covariance Matrix. A.5b

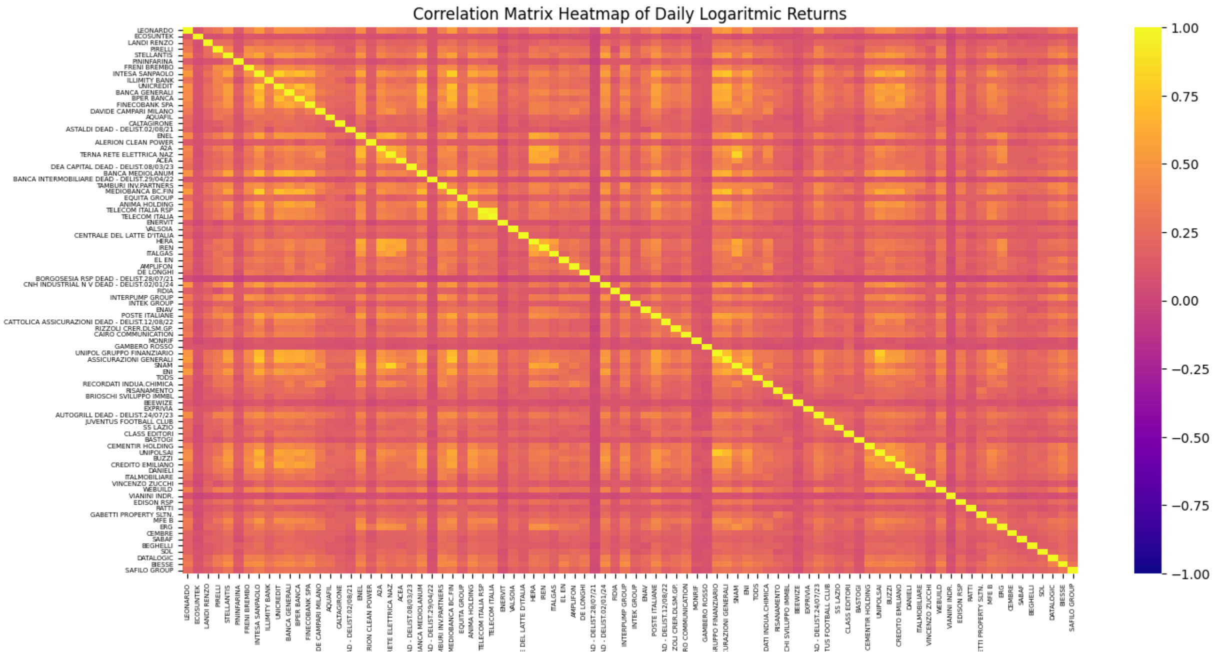

Figure 19: Heatmap of Correlation Matrix of Daily Returns. A.6a.

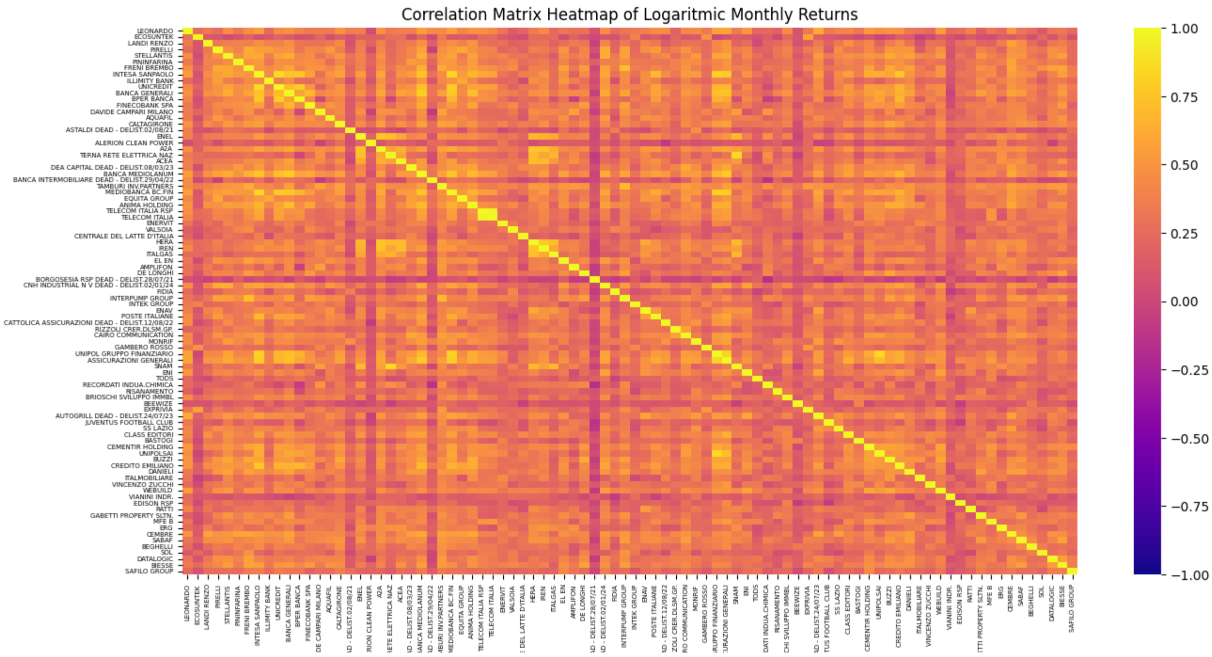

Figure 20: Heatmap of Correlation Matrix of Monthly Returns. A.6b.