# Generating Very Wide Small-World Bipartite Networks

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August 17, 2020

## 1 Overview

Small-world networks are mathematical graphs that are mostly locally connected with several long-range connections scattered in. They are optimal for balancing low wiring cost with high global efficiency (the inverse of the mean shortest path between two random nodes) [Achard and Bullmore, 2007]. The neuron-synapse connectivity in the human brain forms a small-world network [Humphries et al., 2006].

In this document, we present an efficient method of generating very wide small-world bipartite networks. Our method also allows the user to approximately set the sparsity of the network. We emphasize that this method produces just one class of small-world networks. Changing the specific metric used can enable alternate small-world topologies.

## 2 Algorithm

Consider a bipartite network with $n_1$ nodes in set $A$ and $n_2$ nodes in set $B$. Label the first class of nodes as $\{x_1, x_2, \cdots, x_{n_1}\}$ and the second class of nodes as $\{y_1, y_2, \cdots, y_{n_2}\}$.

Since distance-dependent networks induce small-worldness [She et al., 2016], we define the one-dimensional metric function $d : A \times B \to \mathbb{R}_{\geq 0}$ by

$$d(x_i, y_j) = ||i - j| - 0.5|n_2 - n_1|| \tag{1}$$

for all $i \in \{1, 2, \cdots, n_1\}$ and $j \in \{1, 2, \cdots, n_2\}$.

We define the probability of an edge existing between $x_i$ and $y_j$ by

$$\begin{aligned} P(x_i, y_j) &= \alpha \left( d(x_i, y_j) + \beta \right)^{-\lambda} && (2) \\ &= \alpha \left( ||i - j| - 0.5|n_2 - n_1|| + \beta \right)^{-\lambda}, && (3) \end{aligned}$$

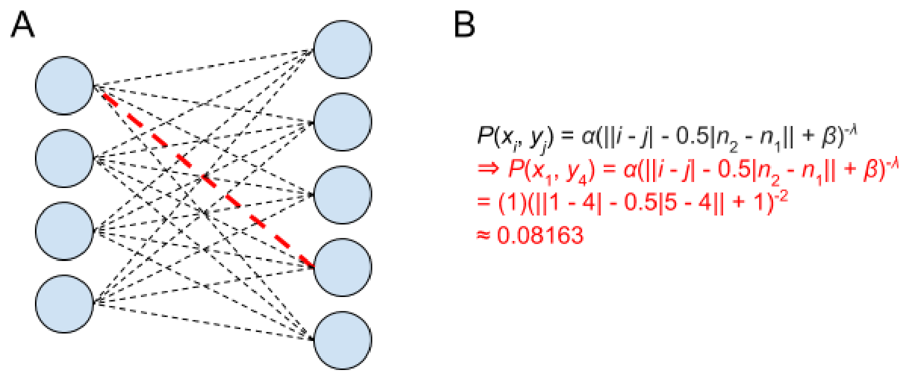


Figure 1: (A) An example bipartite network, where each dotted line represents a potential connection. (B) The probability that $x_1$ is connected to $y_4$ in the graph presented in (A) for $\alpha = 1$, $\beta = 1$, and $\lambda = 2$.

where $\alpha, \beta, \lambda \in \mathbb{R}_+$. See Figure 1 for an example.

To create a bipartite network of class sizes $n_1$ and $n_2$ with desired sparsity $s$, we first calculate the corresponding number of nonzero values by $r = n_1 n_2(1 - s)$. Let $N$ be a discrete random variable defining the total number of edges in a bipartite network configured using Eq. 3. See that

$$\mathbf{E}[N] = \sum_{i,j} P(x_i, y_j). \tag{4}$$

Fix $\alpha$ and $\beta$. Hence, to form a network with $r$ nonzero entries, we must solve for $\lambda$ in

$$\begin{aligned} r &= \sum_{i,j} P(x_i, y_j) &(5)\\ &= \sum_{i,j} \alpha\left(||i - j| - 0.5|n_2 - n_1|| + \beta\right)^{-\lambda}. &(6) \end{aligned}$$

We use $\alpha = 1$ and $\beta = 1$ in our experiments.

When varying $\lambda$ and making $r$ unknown, see that $r$ decreases monotonically. This fact enables us to efficiently approximate $\lambda$ (when $\lambda$ is unknown and $r$ is known). We use a binary search with initial lower bound $\lambda_{\min} = 10^{-5}$, initial upper bound $\lambda_{\max} = 5$, and error $\epsilon = |r - \sum_{i,j} P(x_i, y_j)|$. We cease the binary search when $\epsilon < 10$.

Using our newly calculated $\lambda$, we generate probability matrix $\mathbf{P} \in \mathbb{R}^{n_1 \times n_2}$, where $\mathbf{P}_{ij} = P(x_i, y_j)$. We then generate a reference matrix $\mathbf{A} \in \mathbb{R}^{n_1 \times n_2}$, where $\mathbf{A}_{ij} = \text{Uniform}[0, 1]$. We calculate Boolean mask matrix $\mathbf{M} \in \mathbb{R}^{n_1 \times n_2}$ by

$$\mathbf{M}_{ij} = \begin{cases} 0, & \mathbf{P}_{ij} < \mathbf{A}_{ij} \\ 1, & \mathbf{P}_{ij} \geq \mathbf{A}_{ij} \end{cases}. \tag{7}$$

This mask is the final connectivity matrix for our small-world bipartite network.

In practice, for very large networks (20,000+ nodes), we produce these masks and store their sparse COO representations sequentially. This allows us to generate small-world bipartite networks with 100,000+ nodes.

# References


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