Lab Nr. 2, Numerical Calculus

Taylor Polynomials and Series

- **Short theory review**

Theorem 1. **[Taylor's Theorem]** *Let $f : [a,b] \to \mathbb{R}$ be a function with $n+1$ continuous derivatives on $[a,b]$, for some $n \ge 0$ and let $x, x_0 \in [a,b]$. Then*

$$\begin{aligned} f(x) &= T_n f(x) + R_{n+1}(x), \quad \text{with} \\ T_n f(x) &= f(x_0) + \frac{(x-x_0)}{1!} f'(x_0) + \ldots + \frac{(x-x_0)^n}{n!} f^{(n)}(x_0) \quad \text{and} \\ R_{n+1}(x) &= \frac{1}{n!} \int_{x_0}^{x} (x-t)^n f^{(n+1)}(t)dt \;=\; \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi),\ \xi \text{ between } x \text{ and } x_0. \end{aligned} \tag{1}$$

Using Taylor's theorem with $x_0 = 0$, we obtain MacLaurin's formula:

$$f(x) \;=\; f(0) + \frac{x}{1!} f'(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\xi),\ \xi \text{ between } x \text{ and } x_0.$$

MacLaurin's formula for some common functions:

$$e^x \;=\; 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} e^{\xi_x}, \tag{2}$$

$$\cos x \;=\; 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + (-1)^n \frac{x^{2n}}{(2n)!} + (-1)^{n+1} \frac{x^{2n+2}}{(2n+2)!} \cos \xi_x, \tag{3}$$

$$\sin x \;=\; x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \sin \xi_x, \tag{4}$$

$$\ln(1+x) \;=\; x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1} \frac{x^n}{n} + (-1)^n \frac{x^{n+1}}{n+1} \frac{1}{(1+\xi_x)^{n+1}}, \tag{5}$$

$$(1+x)^a \;=\; 1 + \binom{a}{1} x + \binom{a}{2} x^2 + \ldots + \binom{a}{n} x^n + \binom{a}{n+1} \frac{x^{n+1}}{(1+\xi_x)^{n+1-a}}, \tag{6}$$

$$\binom{a}{k} \;=\; \frac{a(a-1)\cdots(a-k+1)}{k!},\ k = 1, 2, 3, \ldots,\ a \in \mathbb{R}.$$

As $n \to \infty$, the Taylor series on the right-hand-sides of $(2) - (4)$ converge for $x \in \mathbb{R}$, and those in $(5) - (6)$, for $|x| < 1$.

- **Matlab function** *taylor*

Applications

1. Use formula (2) to do the following:

a) Graph on the same set of axes the function e^x and its Taylor polynomials of degrees $1, 2, 3$ and 4, for $x \in [-3, 3]$;

b) Approximate e with 6 correct decimals.

2. Using formula (4), do the following:

a) Graph on the same set of axes the function $\sin x$ and its Taylor polynomials of degree 3 and 5, for $x \in [-\pi, \pi]$;

b) Approximate $\sin\left(\frac{\pi}{5}\right)$ with 5 correct decimals;

c) What happens for $x = \frac{10\pi}{3}$? Explain the phenomenon and find a solution.

3. **a)** Using formula (5), graph on the same set of axes the function $\ln(1+x)$ and its Taylor polynomials of degrees 2 and 5, for $x \in [-0.9, 1]$;

b) How many terms would be necessary in (5) to approximate $\ln 2$ with 5 correct decimals?

c) Use formula (5) to find a Taylor series expansion for the function $\ln(1-x)$.

d) Use formula (5) and the one found in part **c)** to derive a Taylor series expansion for the function $\ln \frac{1+x}{1-x}$.

e) Use the formula found in part **d)** to approximate $\ln 2$ with 5 correct decimals. How many terms are necessary?

<u>**Optional**</u>

4. Use formula (6) to approximate $\sqrt[3]{999}$ with 10 correct decimals (**Caution!!** The series in (6) converges *only* for $|x| < 1$).