Optimization for ML (contd)

CS771: Introduction to Machine Learning

Plan today

- Optimization methods
  - Stochastic gradient descent
  - Co-ordinate descent
  - Alternating optimization
  - Second order optimization
  - Some practical aspects
  - Constrained optimization
    - Projected gradient descent
    - Lagrangian based method
  - Optimization of non-differentiable functions
    - Subgradient descent

Faster GD: Stochastic Gradient Descent (SGD)

- Consider a loss function of the form $L(\boldsymbol{w}) = \frac{1}{N}\sum_{n=1}^{N} \ell_n(\boldsymbol{w})$

Writing as an average instead of sum. Won't affect minimization of $L(\boldsymbol{w})$

- The gradient in this case can be written as

$$\boldsymbol{g} = \nabla_{\boldsymbol{w}} L(w) = \nabla_{\boldsymbol{w}}[\frac{1}{N}\sum_{n=1}^{N} \ell_n(\boldsymbol{w})] = \frac{1}{N}\sum_{n=1}^{N} \boldsymbol{g}_n$$

Expensive to compute – requires doing it for all the training examples in each iteration ☹

Gradient of the loss on n^{th} training example

- Stochastic Gradient Descent (SGD) approximates $\boldsymbol{g}$ using a single training example

- At iter. $\boldsymbol{t}$, pick an index $\boldsymbol{i} \in \{1,2,\ldots,N\}$ uniformly randomly and approximate $\boldsymbol{g}$ as

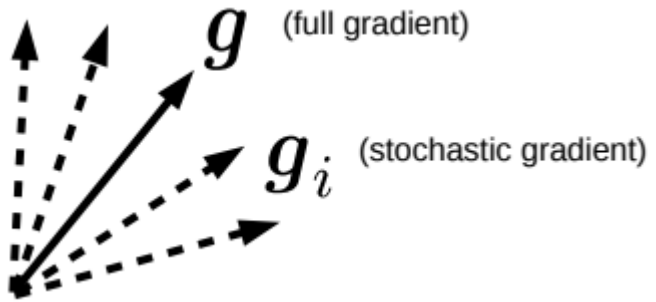
$$\boldsymbol{g} \approx \boldsymbol{g}_i = \nabla_{\boldsymbol{w}} \ell_i(\boldsymbol{w})$$

Can show that $\boldsymbol{g}_i$ is an unbiased estimate of $\boldsymbol{g}$, i.e., $\mathbb{E}[\boldsymbol{g}_i] = \boldsymbol{g}$

- May take more iterations than GD to converge but each iteration is much faster ☺
  - SGD per iter cost is $O(D)$ whereas GD per iter cost is $O(ND)$

Minibatch SGD

- Gradient approximation using a single training example may be noisy



The approximation may have a high variance – may slow down convergence, updates may be unstable, and may even give sub-optimal solutions (e.g., local minima where GD might have given global minima)

- We can use $B > 1$ unif. rand. chosen train. ex. with indices $\{i_1, i_2, \ldots, i_B\} \in \{1,2, \ldots, N\}$
- Using this "minibatch" of examples, we can compute a minibatch gradient

$$\boldsymbol{g} \approx \frac{1}{B}\sum_{b=1}^{B} \boldsymbol{g}_{i_b}$$

- Averaging helps in reducing the variance in the stochastic gradient
- Time complexity is $O(BD)$ per iteration in this case

Some Practical Aspects: Iterate Averaging for SGD

- SGD iterates $\boldsymbol{w}^{(1)}, \boldsymbol{w}^{(2)}, \boldsymbol{w}^{(3)}, \ldots$ can be noisy (recall SGD computes gradients using randomly picked single training example, or a small minibatch)
- Polyak-Ruppert Averaging: Average SGD iterates and use the average in the end

Stochastic gradient on a single/minibatch of examples

SGD/mini-batch SGD update at iteration $t+1$

$$\boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \eta_t \boldsymbol{g}^{(t)}$$

This way of computing the average is the same as doing $\bar{\boldsymbol{w}}^{(t+1)} = \sum_{i=1}^{t+1} \boldsymbol{w}^{(i)}$ but to avoid storing the previous weights, so we compute a running average

Sometimes, we don't start averaging from iteration 1 but after some warm-up iterations

Averaged weight vector at previous iteration t

Running average weight vector at iteration $t+1$

$$\bar{\boldsymbol{w}}^{(t+1)} = \frac{t}{t+1}\bar{\boldsymbol{w}}^{(t)} + \frac{1}{t+1}\boldsymbol{w}^{(t+1)}$$

- Averaging is quite popular for SGD. Stochastic Weighted Averaging (SWA) is another such recently proposed scheme (similar to Polyak-Ruppert Averaging) used for deep neural networks

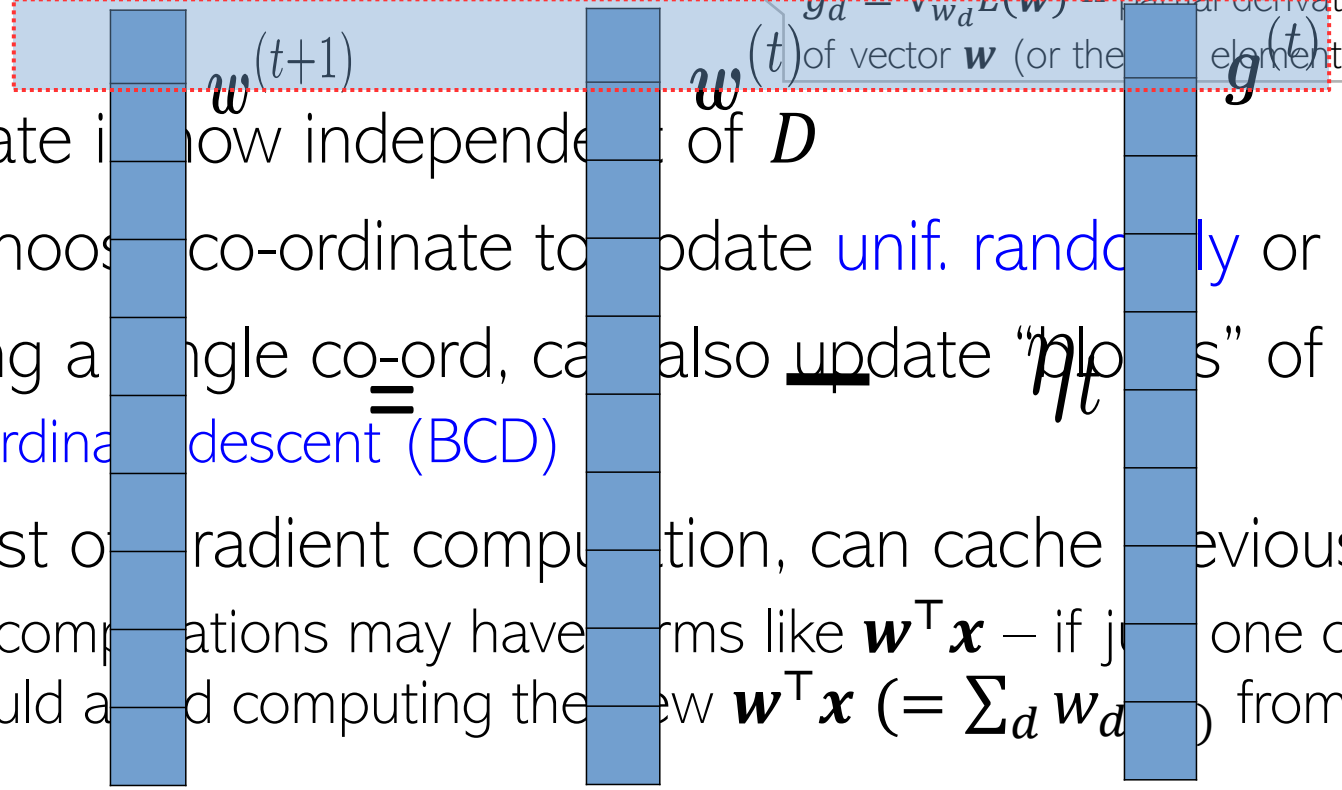
Averaging Weights Leads to Wider Optima and Better Generalization (Izmailov et al, UAI 2018)

Co-ordinate Descent (CD)

- Standard gradient descent update for : $\boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \eta_t \boldsymbol{g}^{(t)}$
- CD: In each iter, update only one entry (co-ordinate) of $\boldsymbol{w}$. Keep all others fixed

$$w_d^{(t+1)} = w_d^{(t)} - \eta_t g_d^{(t)} \qquad d \in \{1,2,\ldots,D\}$$

$g_d = \nabla_{w_d} L(\boldsymbol{w})$ -- partial derivative w.r.t. the d^{th} element of vector $\boldsymbol{w}$ (or the d^{th} element of the gradient vector g)



- Cost of each update is now independent of D
- In each iter, can choose co-ordinate to update unif. randomly or in cyclic order
- Instead of updating a single co-ord, can also update "blocks" of co-ordinates
  - Called block co-ordinate descent (BCD)
- To avoid $O(D)$ cost of gradient computation, can cache previous computations
  - Recall that grad. computations may have terms like $\boldsymbol{w}^\top \boldsymbol{x}$ – if just one co-ordinate of $\boldsymbol{w}$ changes, we should avoid computing the new $\boldsymbol{w}^\top \boldsymbol{x}$ $(= \sum_d w_d x_d)$ from scratch

Alternating Optimization (ALT-OPT)

- Consider opt. problems with several variables, say two variables $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$

Each variable can be scalar or vector or matrix or tensor

$$\{\hat{\boldsymbol{w}}_1, \hat{\boldsymbol{w}}_2\} = \arg\min_{\boldsymbol{w}_1, \boldsymbol{w}_2} \mathcal{L}(\boldsymbol{w}_1, \boldsymbol{w}_2)$$

E.g. first order optimality doesn't give a closed form solution when solving for both variables jointly

- Often, this "joint" optimization is hard/impossible to solve
- We can take an alternating optimization approach to solve such problems

ALT-OPT

1. Initialize one of the variables, e.g., $\boldsymbol{w}_2 = \boldsymbol{w}_2^{(0)}, t = 0$
2. Solve $\boldsymbol{w}_1^{(t+1)} = \arg\min_{\boldsymbol{w}_1} \mathcal{L}(\boldsymbol{w}_1, \boldsymbol{w}_2^{(t)})$ // $\boldsymbol{w}_2$ "fixed" at its most recent value $\boldsymbol{w}_2^{(t)}$
3. Solve $\boldsymbol{w}_2^{(t+1)} = \arg\min_{\boldsymbol{w}_2} \mathcal{L}(\boldsymbol{w}_1^{(t+1)}, \boldsymbol{w}_2)$ // $\boldsymbol{w}_1$ "fixed" at its most recent value $\boldsymbol{w}_1^{(t+1)}$
4. $t = t + 1$. Go to step 2 if not converged yet.

- Usually converges to a local optima. But very very useful. Will see examples later
  - Also related to the Expectation-Maximization (EM) algorithm which we will see later

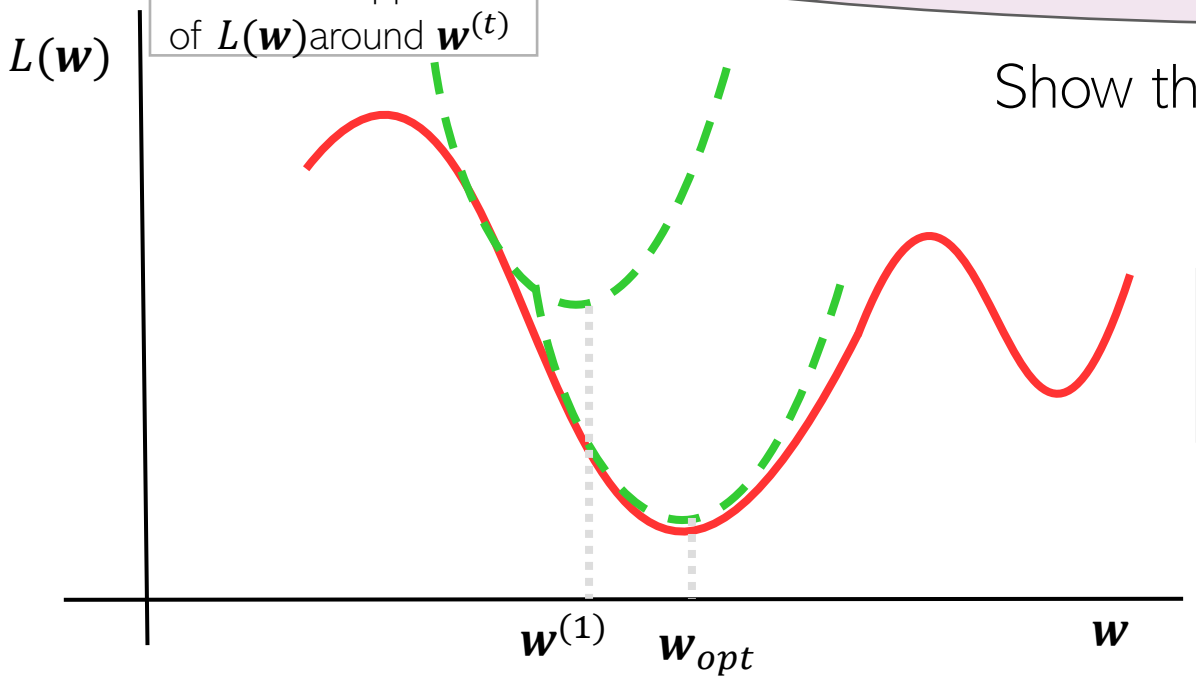
Second Order Methods: Newton's Method

- Unlike GD and its variants, Newton's method uses second-order information (second derivative, a.k.a. the Hessian). Iterative method, just like GD
- Given current $\boldsymbol{w}^{(t)}$, minimize the quadratic (second-order) approx. of $\boldsymbol{L(w)}$

Quadratic approx

$$\boldsymbol{w}^{(t+1)} = \arg\min_{\boldsymbol{w}} \left[L\left(\boldsymbol{w}^{(t)}\right) + \nabla L\left(\boldsymbol{w}^{(t)}\right)^{\top}\left(\boldsymbol{w} - \boldsymbol{w}^{(t)}\right) + \frac{1}{2}\left(\boldsymbol{w} - \boldsymbol{w}^{(t)}\right)^{\top}\nabla^2 L\left(\boldsymbol{w}^{(t)}\right)\left(\boldsymbol{w} - \boldsymbol{w}^{(t)}\right) \right]$$

First order approx. of $L(\boldsymbol{w})$ around $\boldsymbol{w}^{(t)}$



Show that $\boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \left(\nabla^2 L\left(\boldsymbol{w}^{(t)}\right)\right)^{-1} \nabla L\left(\boldsymbol{w}^{(t)}\right)$

$= \boldsymbol{w}^{(t)} - \left(\boldsymbol{H}^{(t)}\right)^{-1}\boldsymbol{g}^{(t)}$

Converges much faster than GD (very fast for convex functions). Also no "learning rate". But per iteration cost is slower due to Hessian computation and inversion

Faster versions of Newton's method also exist, e.g., those based on approximating Hessian using previous gradients (see L-BFGS which is a popular method)

Some Practical Aspects: Assessing Convergence

- Various ways to assess convergence, e.g. consider converged if
  - The objective's value (on train set) ceases to change much across iterations

$$L(\boldsymbol{w}^{(t+1)}) - L(\boldsymbol{w}^{(t)}) < \epsilon$$ (for some small pre-defined ϵ)

  - The parameter values cease to change much across iterations

$$\| \boldsymbol{w}^{(t+1)} - \boldsymbol{w}^{(t)} \| < \tau$$ (for some small pre-defined τ)

  - Above condition is also equivalent to saying that the gradients are close to zero

$$\| \boldsymbol{g}^{(t)} \| \rightarrow 0$$

Caution: May not yet be at the optima. Use at your own risk!

  - The objective's value has become small enough that we are happy with ☺
  - Use a validation set to assess if the model's performance is acceptable (early stopping)

Some Practical Aspects: Learning Rate (Step Size)

- Some guidelines to select good learning rate (a.k.a. step size) η_t

C is a hyperparameter

- For convex functions, setting η_t something like C/t or $C/\sqrt{t}$ often works well
  - These step-sizes are actually theoretically optimal in some settings. Moreover we want
    - $\eta_t \to 0$ as t becomes very very large
    - $\sum \eta_t = \infty$ (needed to ensure that we can potentially reach anywhere in the parameter space)
    - $\sum {\eta_t}^2 < \infty$ (to prevent indefinite oscillatory behavior around the optima)
  - Sometimes carefully chosen constant learning rates (usually small, or initially large and later small) also work well in practice
- Can also search for the "best" step-size by solving an opt. problem in each step

Also called "line search"

$$\eta_t = \arg\min_{\eta \geq 0} f\left(\mathbf{w}^{(t)} - \eta \cdot \mathbf{g}^{(t)}\right)$$

A one-dim optimization problem (note that $\mathbf{w}^{(t)}$ and $\mathbf{g}^{(t)}$ are fixed)

- A faster alternative to line search is the Armijo-Goldstein rule
  - Starting with current (or some large) learning rate (from prev. iter), and try a few values in decreasing order until the objective's value has a sufficient reduction

Some Practical Aspects: Adaptive Gradient Methods

- Can also use different learning rate in different dimensions

$$\boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \boldsymbol{e}^{(t)} \odot \boldsymbol{g}^{(t)} \qquad e_d^{(t)} = \frac{1}{\sqrt{\epsilon + \sum_{\tau=1}^{t} \left(g_d^{(\tau)}\right)^2}}$$

Vector of learning rates along each dimension

Element-wise product of two vectors

If some dimension had big updates recently (marked by large gradient values), slow down along those directions by using smaller learning rates - AdaGrad (Duchi et al, 2011)

- Can use a momentum term to stabilize gradients by reusing info from past grads
  - Move faster along directions that were previously good
  - Slow down along directions where gradient has changed abruptly

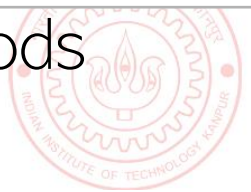
$$\boldsymbol{m}^{(t)} = \beta \boldsymbol{m}^{(t-1)} + (1-\beta)\boldsymbol{g}^{(t)}$$
$$\boldsymbol{w}^{(t+1)} \leftarrow \boldsymbol{w}^{(t)} - \eta \boldsymbol{m}^{(t)}$$

β usually set as 0.9

The "momentum" term. Set to 0 at initialization

In an even faster version of this, $\boldsymbol{g}^{(t)} = \nabla L(\boldsymbol{w}^{(t)})$ is replaced by the gradient computed at the next step if previous direction were used, i.e., $\nabla L(\boldsymbol{w}^{(t)} - \beta \boldsymbol{m}^{(t-1)})$. Called Nesterov's Accelerated Gradient (NAG) method

- Also exist several more advanced methods that combine the above methods
  - RMS-Prop: AdaGrad + Momentum, Adam: NAG + RMS-Prop
  - These methods are part of packages such as PyTorch, Tensorflow, etc



Constrained Optimization

Projected Gradient Descent

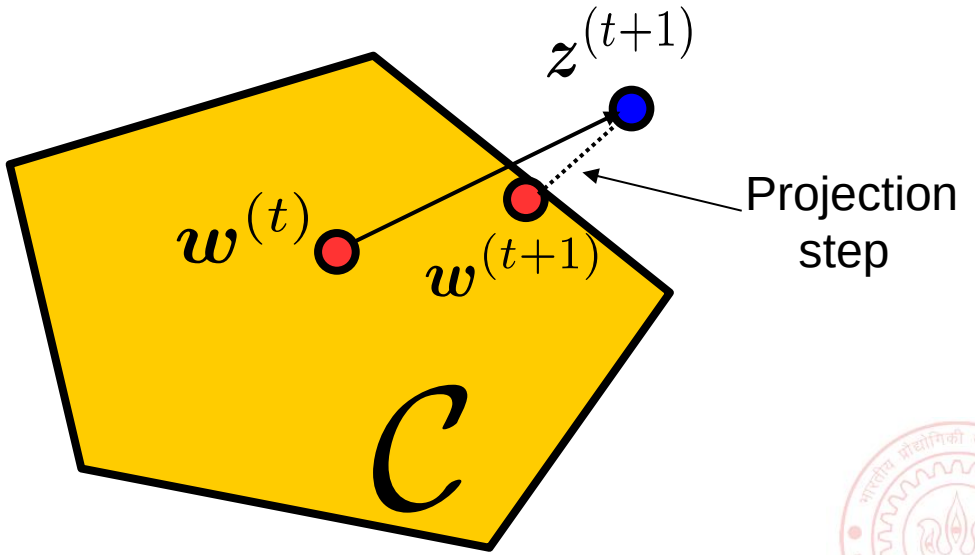
- Consider an optimization problem of the form

$$w_{opt} = \arg\min_{w \in \mathcal{C}} L(\boldsymbol{w})$$

- Projected GD is very similar to GD with an extra projection step

- Each iteration $\boldsymbol{t}$ will be of the form
  - Perform update: $\boldsymbol{z}^{(t+1)} = \boldsymbol{w}^{(t)} - \eta_t \boldsymbol{g}^{(t)}$
  - Check if $\boldsymbol{z}^{(t+1)}$ satisfies constraints
    - If $\boldsymbol{z}^{(t+1)} \in \mathcal{C}$, set $\boldsymbol{w}^{(t+1)} = \boldsymbol{z}^{(t+1)}$
    - If $\boldsymbol{z}^{(t+1)} \notin \mathcal{C}$, project as $\boldsymbol{w}^{(t+1)} = \Pi_{\mathcal{C}}[\boldsymbol{z}^{(t+1)}]$

Projection operator



Projected GD: How to Project?

- Here projecting a point means finding the "closest" point from the constraint set

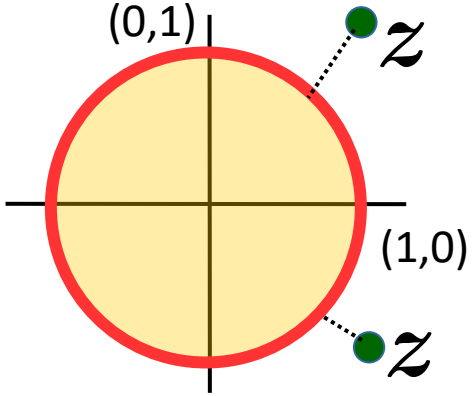
$$\Pi_{\mathcal{C}}[\boldsymbol{z}] = \arg\min_{\boldsymbol{w}\in\mathcal{C}} \|\boldsymbol{z}-\boldsymbol{w}\|^2$$

Another constrainted optimization problem! But simpler to solve! ☺

Projected GD commonly used only when the projection step is simple and efficient to compute

- For some sets $\mathcal{C}$, the projection step is easy

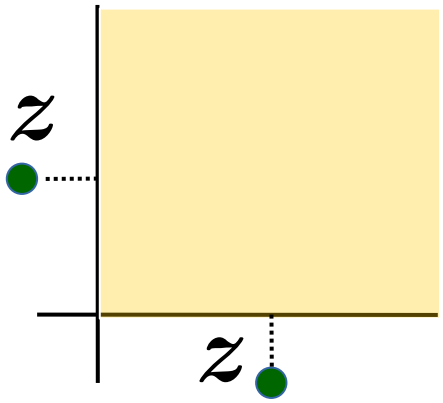
$\mathcal{C}$: Unit radius ℓ_2 ball



Projection = Normalize to unit Euclidean length vector

$$\hat{\mathbf{x}} = \begin{cases} \mathbf{x} & \text{if } \|\mathbf{x}\|_2 \le 1 \\ \frac{\mathbf{x}}{\|\mathbf{x}\|_2} & \text{if } \|\mathbf{x}\|_2 > 1 \end{cases}$$

$\mathcal{C}$: Set of non-negative reals



Projection = Set each negative entry in $\boldsymbol{z}$ to be zero

$$\hat{\mathbf{x}}_i = \begin{cases} \mathbf{x}_i & \text{if } \mathbf{x}_i \ge 0 \\ 0 & \text{if } \mathbf{x}_i < 0 \end{cases}$$

Constrained Opt. via Lagrangian

- Consider the following constrained minimization problem (using f instead of L)

$$\hat{\boldsymbol{w}} = \arg\min_{\boldsymbol{w}} f(\boldsymbol{w}), \quad \text{s.t.} \quad g(\boldsymbol{w}) \leq 0$$

- Note: If constraints of the form $g(\boldsymbol{w}) \geq 0$, use $-g(\boldsymbol{w}) \leq 0$
- Can handle multiple inequality and equality constraints too (will see later)
- Can transform the above into the following equivalent unconstrained problem

$$\hat{\boldsymbol{w}} = \arg\min_{\boldsymbol{w}} f(\boldsymbol{w}) + c(\boldsymbol{w})$$

$$c(\boldsymbol{w}) = \max_{\alpha \geq 0} \alpha g(\boldsymbol{w}) = \begin{cases} \infty, & \text{if } g(\boldsymbol{w}) > 0 \quad \text{(constraint violated)} \\ 0 & \text{if } g(\boldsymbol{w}) \leq 0 \quad \text{(constraint satisfied)} \end{cases}$$

- Our problem can now be written as

$$\hat{\boldsymbol{w}} = \arg\min_{\boldsymbol{w}} \left\{ f(\boldsymbol{w}) + \max_{\alpha \geq 0} \alpha g(\boldsymbol{w}) \right\}$$

Constrained Opt. via Lagrangian

The Lagrangian: $\mathcal{L}(\boldsymbol{w}, \boldsymbol{\alpha})$

- Therefore, we can write our original problem as

$$\hat{\boldsymbol{w}} = \arg\min_{\boldsymbol{w}} \left\{ f(\boldsymbol{w}) + \max_{\alpha \geq 0} \alpha g(\boldsymbol{w}) \right\} = \arg\min_{\boldsymbol{w}} \left\{ \max_{\alpha \geq 0} \{ f(\boldsymbol{w}) + \alpha g(\boldsymbol{w}) \} \right\}$$

- The Lagrangian is now optimized w.r.t. $\boldsymbol{w}$ and $\boldsymbol{\alpha}$ (Lagrange multiplier)
- We can define Primal and Dual problem as

$$\hat{\boldsymbol{w}}_P = \arg\min_{\boldsymbol{w}} \left\{ \max_{\alpha \geq 0} \{ f(\boldsymbol{w}) + \alpha g(\boldsymbol{w}) \} \right\} \quad \text{(Primal Problem)}$$

$$\hat{\boldsymbol{w}}_D = \arg\max_{\alpha \geq 0} \left\{ \min_{\boldsymbol{w}} \{ f(\boldsymbol{w}) + \alpha g(\boldsymbol{w}) \} \right\} \quad \text{(Dual Problem)}$$

Both equal if $f(\boldsymbol{w})$ and the set $g(\boldsymbol{w}) \leq 0$ are convex

$$\alpha_D g(\hat{\boldsymbol{w}}_D) = 0$$

complimentary slackness/Karush-Kuhn-Tucker (KKT) condition

Constrained Opt. with Multiple Constraints

- We can also have multiple inequality and equality constraints

$$\begin{aligned} \hat{\boldsymbol{w}} &= \arg\min_{\boldsymbol{w}} f(\boldsymbol{w}) \\ \text{s.t.} \quad & g_i(\boldsymbol{w}) \leq 0, \quad i = 1, \ldots, K \\ & h_j(\boldsymbol{w}) = 0, \quad j =, 1, \ldots, L \end{aligned}$$

- Introduce Lagrange multipliers $\boldsymbol{\alpha} = [\alpha_1, \alpha_2, \ldots, \alpha_K]$ and $\boldsymbol{\beta} = [\beta_1, \beta_2, \ldots, \beta_L]$
- The Lagrangian based primal and dual problems will be

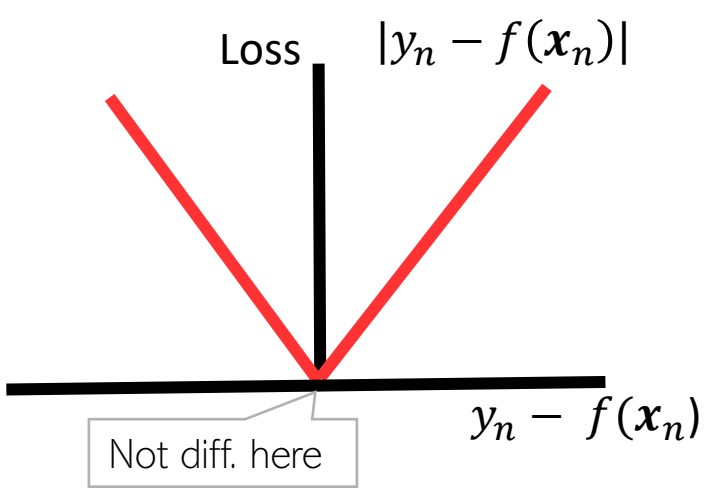
$$\hat{\boldsymbol{w}}_P = \arg\min_{\boldsymbol{w}} \left\{ \max_{\alpha \geq 0, \beta} \left\{ f(\boldsymbol{w}) + \sum_{i=1}^{K} \alpha_i g_i(\boldsymbol{w}) + \sum_{j=1}^{L} \beta_j h_j(\boldsymbol{w}) \right\} \right\}$$

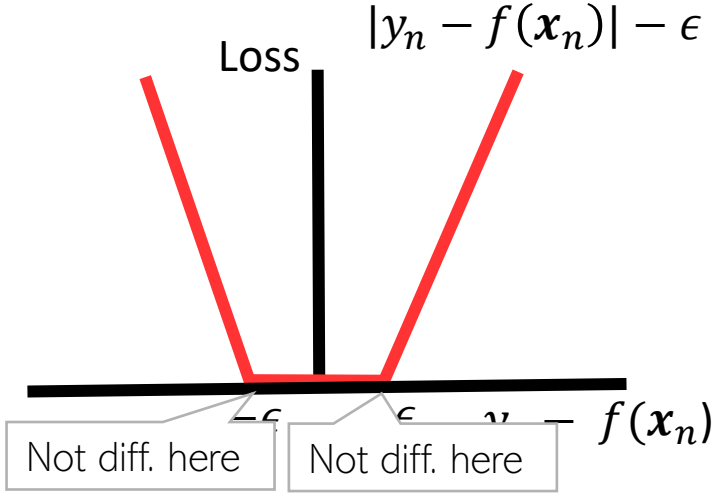
$$\hat{\boldsymbol{w}}_D = \arg\max_{\alpha \geq 0, \beta} \left\{ \min_{\boldsymbol{w}} \left\{ f(\boldsymbol{w}) + \sum_{i=1}^{K} \alpha_i g_i(\boldsymbol{w}) + \sum_{j=1}^{L} \beta_j h_j(\boldsymbol{w}) \right\} \right\}$$

Optimization of Non-differentiable Functions

Dealing with Non-differentiable Functions

- In many ML problems, the objective function will be non-differentiable
- Some examples that we have already seen: Linear regression with absolute loss, or ϵ-insensitive loss; even ℓ_1 norm regularizer is non-diff

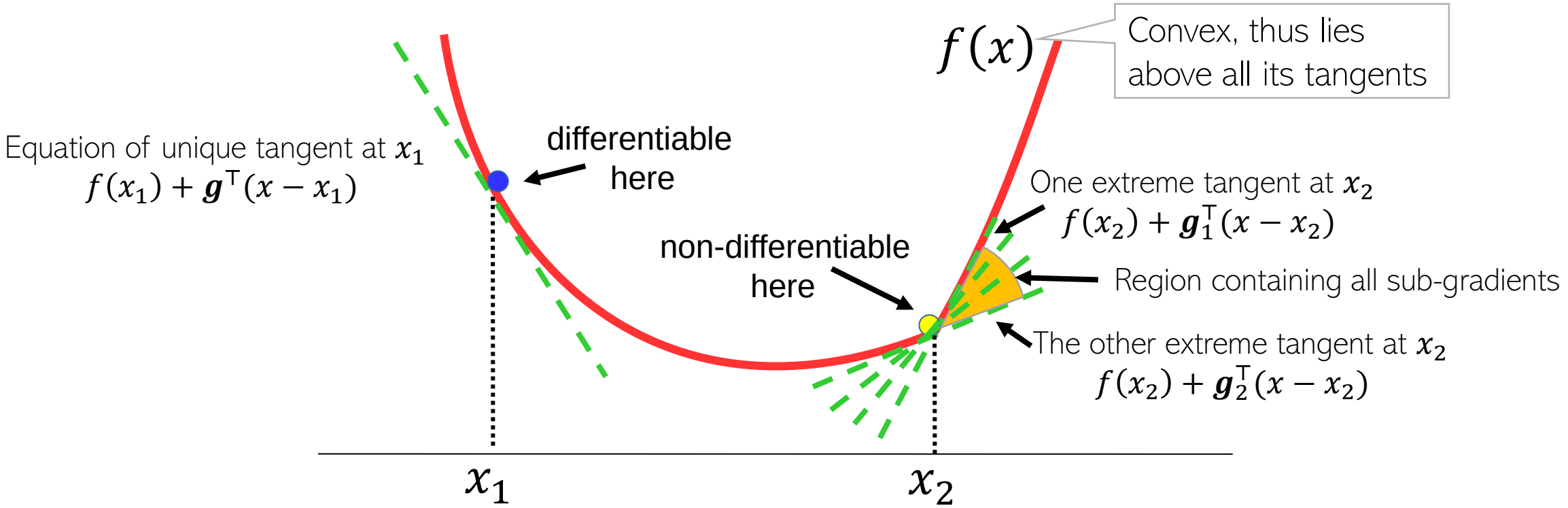




- Basically, any function in which there are points with kink is non-diff
  - At such points, the function is non-differentiable and thus gradients not defined
  - Reason: Can't define a unique tangent at such points

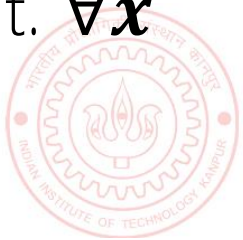
Sub-gradients

- For convex non-diff fn, can define sub-gradients at point(s) of non-differentiabilty



- For a convex, non-diff function $f(\boldsymbol{x})$, sub-gradient at $\boldsymbol{x}_*$ is <u>any</u> vector $\boldsymbol{g}$ s.t. $\forall \boldsymbol{x}$

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}_*) + \boldsymbol{g}^\top(\boldsymbol{x} - \boldsymbol{x}_*)$$



Sub-gradients, Sub-differential, and Some Rules

- Set of all sub-gradient at a non-diff point $\boldsymbol{x}_*$ is called the sub-differential

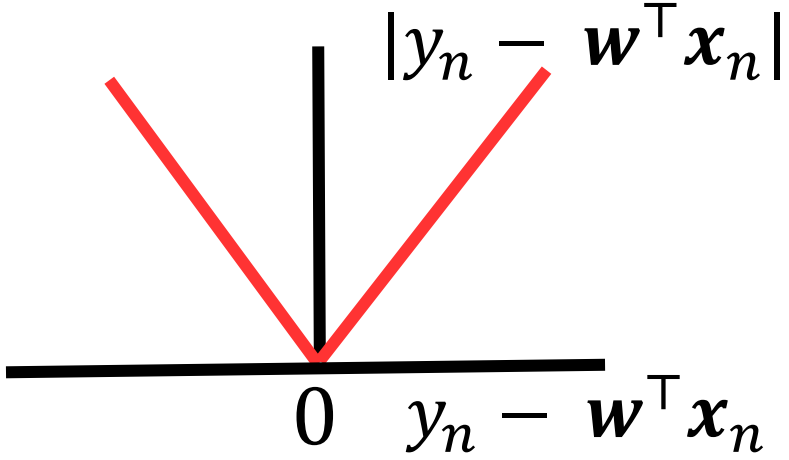
$$\partial f(\boldsymbol{x}_*) \triangleq \{\boldsymbol{g} : f(\mathbf{x}) \geq f(\boldsymbol{x}_*) + \boldsymbol{g}^\top(\boldsymbol{x} - \boldsymbol{x}_*) \quad \forall \mathbf{x}\}$$

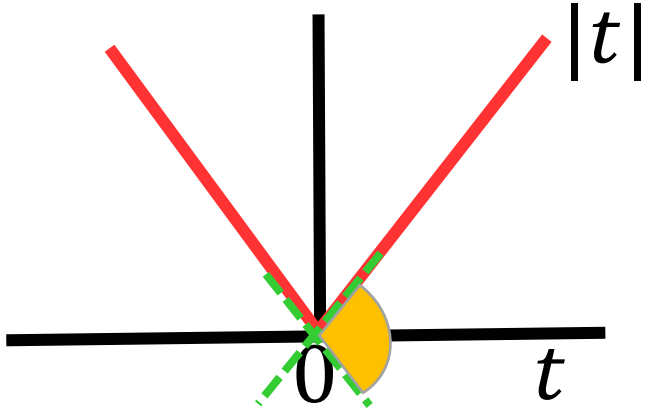
- Some basic rules of sub-diff calculus to keep in mind
  - Scaling rule: $\partial\big(c \cdot f(\mathbf{x})\big) = c \cdot \partial f(\mathbf{x}) = \{c \cdot \mathbf{v} : \mathbf{v} \in \partial f(\mathbf{x})\}$
  - Sum rule: $\partial\big(f(\mathbf{x}) + g(\mathbf{x})\big) = \partial f(\mathbf{x}) + \partial g(\mathbf{x}) = \{\mathbf{u} + \mathbf{v} : \mathbf{u} \in \partial f(\mathbf{x}), \mathbf{v} \in \partial g(\mathbf{x})\}$
  - Affine trans: $\partial f(\mathbf{a}^\top\mathbf{x} + b) = \mathbf{a} \cdot \partial f(t) = \{\mathbf{a} \cdot c : c \in \partial f(t)\}$, where $t = \mathbf{a}^\top\mathbf{x} + b$
  - Max rule: If $h(\boldsymbol{x}) = \max\{f(\boldsymbol{x}), g(\boldsymbol{x})\}$ then we calculate $\partial h(\boldsymbol{x})$ at $\boldsymbol{x}_*$ as
    - If $f(\boldsymbol{x}_*) > g(\boldsymbol{x}_*), \partial h(\boldsymbol{x}_*) = \partial f(\boldsymbol{x}_*)$, If $g(\boldsymbol{x}_*) > f(\boldsymbol{x}_*), \partial h(\boldsymbol{x}_*) = \partial g(\boldsymbol{x}_*)$
    - If $f(\boldsymbol{x}_*) = g(\boldsymbol{x}_*), \partial h(\boldsymbol{x}_*) = \{\alpha\mathbf{a} + (1-\alpha)\mathbf{b} : \mathbf{a} \in \partial f(\boldsymbol{x}_*), \mathbf{b} \in \partial g(\boldsymbol{x}_*), \alpha \in [0,1]\}$

The affine transform rule is a special case of the more general chain rule

- $\boldsymbol{x}_*$ is a stationary point for a non-diff function $f(\boldsymbol{x})$ if the zero vector belongs to the sub-differential at $\boldsymbol{x}_*$, i.e., $\mathbf{0} \in \partial f(\boldsymbol{x}_*)$

Sub-Gradient For Absolute Loss Regression





Using max rule of sub-differentials and using $|t| = \max\{t, -t\}$

$$\partial|t| = \begin{cases} 1 & \text{if } t > 0 \\ -1 & \text{if } t < 0 \\ [-1, +1] & \text{if } t = 0 \end{cases}$$

- The loss function for linear reg. with absolute loss: $L(\boldsymbol{w}) = |y_n - \boldsymbol{w}^\top \boldsymbol{x}_n|$
- Non-differentiable at $y_n - \boldsymbol{w}^\top \boldsymbol{x}_n = 0$
- Can use the affine transform and max rule of sub-diff calculus
- Assume $t = y_n - \boldsymbol{w}^\top \boldsymbol{x}_n$. Then $\partial L(\boldsymbol{w}) = -\boldsymbol{x}_n \partial|t|$
  - $\partial L(\boldsymbol{w}) = -\boldsymbol{x}_n \times 1 = -\boldsymbol{x}_n$ if $t > 0$
  - $\partial L(\boldsymbol{w}) = -\boldsymbol{x}_n \times -1 = \boldsymbol{x}_n$ if $t < 0$
  - $\partial L(\boldsymbol{w}) = -\boldsymbol{x}_n \times c = -c\boldsymbol{x}_n$ where $c \in [-1, +1]$ if $t = 0$

Sub-Gradient Descent

- Suppose we have a non-differentiable function $L(\boldsymbol{w})$
- Sub-gradient descent is almost identical to GD except we use subgradients

Sub-Gradient Descent

- Initialize $\boldsymbol{w}$ as $\boldsymbol{w}^{(0)}$
- For iteration $t = 0,1,2, \ldots$ (or until convergence)
  - Calculate the sub-gradient $\boldsymbol{g}^{(t)} \in \partial L(\boldsymbol{w}^{(t)})$
  - Set the learning rate η_t
  - Move in the opposite direction of subgradient

$$\boldsymbol{w}^{(t+1)} = \boldsymbol{w}^{(t)} - \eta_t \boldsymbol{g}^{(t)}$$

Optimization for ML: Some Final Comments

- Gradient methods are simple to understand and implement
- More sophisticated optimization methods also often use gradient methods
- Backpropagation algo used in deep neural nets is GD + chain rule of differentiation
- Use subgradient methods if function not differentiable
- Constrained optimization can use Lagrangian or projected GD
- Second order methods such as Newton's method faster but computationally expensive
- But computing all this gradient related stuff by hand looks scary to me. Any help?
  - Don't worry. Automatic Differentiation (AD) methods available now (will see them later)
  - AD only requires specifying the loss function (especially useful for deep neural nets)
  - Many packages such as Tensorflow, PyTorch, etc. provide AD support
  - But having a good understanding of optimization is still helpful