SIMPLEX ALGORITHM

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Lecture Outline

Introduction

1. In the previous lecture, we saw how to solve two-variable linear programming problems graphically.
2. Unfortunately, most real-life LPs have many variables, so a method is needed to solve LPs with more than two variables.
3. We devote most of this lecture to a discussion of the simplex algorithm, which is used to solve even very large LPs.
4. In many industrial applications, the simplex algorithm is used to solve LPs with thousands of constraints and variables.
5. In 1947, George Dantzig developed the simplex algorithm for solving linear programming problems.
6. Since the development of the simplex algorithm, LP has been used to solve optimization problems in industries as diverse as banking, education, forestry, petroleum, and trucking.

How to Convert an LP to Standard Form

1. We have seen that an LP can have both equality and inequality constraints.
2. It also can have variables that are required to be nonnegative as well as those allowed to be unrestricted in sign (urs).
3. Before the simplex algorithm can be used to solve an LP, the LP must be converted into an equivalent problem in which all constraints are equations and all variables are nonnegative.
4. An LP in this form is said to be in standard form.
5. To convert an LP into standard form, each inequality constraint must be replaced by an equality constraint with the introduction of Slack and Excess variables.

How to Convert an LP to Standard Form

Definition (Slack Variable)

If constraint i of an LP is a $\leq$ constraint, then we convert it to an equality constraint by adding a slack variable s_i to the ith constraint and adding the sign restriction $s_i \geq 0$

Definition (Excess Variable)

On the other hand, if the ith constraint of an LP is a $\geq$ constraint, then it can be converted to an equality constraint by subtracting an excess variable e_i from the ith constraint and adding the sign restriction $e_i \geq 0$

We illustrate this procedure using the following problem.

Example (Slack Variable)

Leather Limited manufactures two types of belts: the deluxe model and the regular model. Each type requires 1 sq yd of leather. A regular belt requires 1 hour of skilled labor, and a deluxe belt requires 2 hours. Each week, 40 sq yd of leather and 60 hours of skilled labor are available. Each regular belt contributes ¢3 to profit and each deluxe belt, ¢4. If we define

- x_1 = number of deluxe belts produced weekly
- x_2= number of regular belts produced weekly

The appropriate LP is

$$max\ z = 4x_1 + 3x_2 \tag{1}$$

$$s.t.\ x_1 + x_2 \leq 40 \qquad (Leather\ constraint) \tag{2}$$

$$2x_1 + x_2 \leq 60 \qquad (Labor\ constraint) \tag{3}$$

$$x_1, x_2 \geq 0 \tag{4}$$

How can we convert eqs. (2) and (3) to equality constraints?
We define for each $\leq$ constraint a slack variable s_i, which is the amount of the resource unused in the ith constraint. Then the standard form is given as

$$\begin{aligned} max\ z &= 4x_1 + 3x_2 && (5)\\ s.t.\ x_1 + x_2 + s_1 &= 40 && (6)\\ 2x_1 + x_2 + s_2 &= 60 && (7)\\ x_1, x_2, s_1, s_2 &\geq 0 && (8) \end{aligned}$$

Example (Excess variable)

Convert the following LP problem to its standard form

$$min\ z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$s.t.\ 400x_1 + 200x_2 + 150x_3 + 500x_4 \geq 500 \quad (9)$$

$$3x_1 + 2x_2 \geq 6 \quad (10)$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 \geq 10 \quad (11)$$

$$2x_1 + 4x_2 + x_3 + 5x_4 \geq 8 \quad (12)$$

$$x_1, x_2, x_3, x_4 \geq 0 \quad (13)$$

To convert the ith $\geq$ constraint to an equality constraint, we define an excess variable (sometimes called a surplus variable) e_i. e_i is the amount by which the ith constraint is oversatisfied. Thus, the standard form is given as

$$min\ z = 50x_1 + 20x_2 + 30x_3 + 80x_4$$

$$s.t.\ 400x_1 + 200x_2 + 150x_3 + 500x_4 - e_1 = 500 \quad (14)$$

$$3x_1 + 2x_2 - e_2 = 6 \quad (15)$$

$$2x_1 + 2x_2 + 4x_3 + 4x_4 - e_3 = 10 \quad (16)$$

$$2x_1 + 4x_2 + x_3 + 5x_4 - e_4 = 8 \quad (17)$$

$$x_i, e_i \geq 0 \qquad i = 1, 2, 3, 4. \quad (18)$$

Note

If an LP has both $\leq$ and $\geq$ constraints, then simply apply the procedures we have described to the individual constraints.

Basic and Nonbasic Variables

Consider a system of linear equations

$$
\begin{array}{c}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots a_{2n}x_n = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots a_{3n}x_n = b_3 \\
\vdots \qquad\qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots a_{mn}x_n = b_m
\end{array} \tag{19}
$$

Equation (19) can be recast as

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix} \tag{20}
$$

The $x_i's$ are the unknown to be determined.

Thus, equation (20) is of the form $Ax = b$ where

$$
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots a_{mn} \end{bmatrix}, \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{and } b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}
$$

Basic and Nonbasic Variables

Definition

For n number of unknown variables, and m number of equations, a basic solution to $Ax = b$ is obtained by setting $n - m$ variables equal to 0 and solving for the values of the remaining m variables.

1. Given m linear equations and n variables, assume $n \geq m$
2. To find a basic solution to $Ax = b$, we choose a set of $n - m$ variables (the nonbasic variables, or NBV) and set each of these variables equal to 0. Then we solve for the values of the remaining $n - (n - m) = m$ variables (the basic variables, or BV) that satisfy $Ax = b$.
3. The different choices of nonbasic variables will lead to different basic solutions.
4. To illustrate, we find all the basic solutions to the following system of two equations in three variables (3 variables, 2 equations):

Basic and Nonbasic Variables

$$x_1 + x_2 = 3 \tag{21}$$

$$-x_2 + x_3 = -1 \tag{22}$$

1. We begin by choosing a set of $3-2=1$ nonbasic variables. For example, if NBV is x_3, then BV are x_1, x_2. We obtain the values of the basic variables by setting $x_3 = 0$ and solving we find that $x_1 = 2,\ x_2 = 1$. Thus,

   $$x_1 = 2,\ x_2 = 1,\ x_3 = 0$$

   is a basic solution to eqs. (21) and (22).
2. However, if we choose NBV x_1 and BV x_2, x_3, we obtain the basic solution

   $$x_1 = 0,\ x_2 = 3,\ x_3 = 2$$

3. If we choose NBV x_2, we obtain the basic solution

   $$x_1 = 3,\ x_2 = 0,\ x_3 = -1$$

Feasible Solutions

Definition (BFS)

Any basic solution to $Ax = b$ in which all variables are nonnegative is a basic feasible solution (or bfs).

Thus, the basic solution $x_1 = 2,\ x_2 = 1,\ x_3 = 0$ and $x_1 = 0,\ x_2 = 3,\ x_3 = 2$ are basic feasible solutions, while $x_1 = 3,\ x_2 = 0,\ x_3 = -1$ fails to be a basic solution (because $x_3 \le 0$).

Definition (Extreme Point)

A point in the feasible region of an LP is an extreme point if and only if it is a basic feasible solution to the LP.

Definition

For any LP with m constraints, two basic feasible solutions are said to be adjacent if their sets of basic variables have $m - 1$ basic variables in common.

The Simplex Algorithm

Here we describe how the simplex algorithm can be used to solve LPs in which the goal is to maximize the objective function. The simplex algorithm proceeds as follows:

1. Convert the LP to standard form.
2. Obtain a basic feasible solution bfs (if possible) from the standard form.
3. Determine whether the current bfs is optimal.
   1. For maximization problem, if all nonbasic variables have nonnegative coefficients in row 0, then the current bfs is optimal.
   2. For minimization problem, if all nonbasic variables have nonpositive coefficients in row 0, then the current bfs is optimal.
4. If the current bfs is not optimal, then determine which nonbasic variable should become a basic variable.
   1. For maximization problem, choose the variable with the most negative coefficient in row 0 to enter the basis.
   2. For minimization problem, choose the variable with the most positive coefficient in row 0 to enter the basis.

The Simplex Algorithm

5. Determine which basic variable should become a nonbasic variable using elementary row operations (ERO) and the ratio test.
6. Find a new bfs with a better objective function value. Go back to step 3

Note

In performing the simplex algorithm, write the objective function

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

in the form

$$z - c_1x_1 - c_2x_2 - \cdots - c_nx_n = 0 \tag{23}$$

This format (23) is called the row 0 version of the objective function (row 0 for short)

Example

The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given the Table below:

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for ¢60, a table for ¢30, and a chair for ¢20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, how does Dakota maximize total revenue.

Solution

Decision variables

x_1 = number of desks produced

x_2 = number of tables produced

x_3 = number of chairs produced

Then the LP problem is

$$\max z = 60x_1 + 30x_2 + 20x_3$$

$$\begin{aligned}
s.t. \qquad 8x_1 + 6x_2 + x_3 &\le 48 && \text{(Lumber constraint)} && (24)\\
4x_1 + 2x_2 + 1.5x_3 &\le 20 && \text{(Finishing constraint)} && (25)\\
2x_1 + 1.5x_2 + 0.5x_3 &\le 8 && \text{(Carpentry constraint)} && (26)\\
x_2 &\le 5 && \text{(Limitation on table demand)} && (27)\\
x_1, x_2, x_3 &\ge 0 && && (28)
\end{aligned}$$

Iteration 0: Step 1

Now we convert the LP constraints to the standard form and convert the LP's objective function to the row 0 format. Thus:

$$z - 60x_1 - 30x_2 - 20x_3 = 0 \tag{29}$$

Putting eqs. (29) to (33) and the sign restrictions yields the equations:
Then the LP problem is

$$\max\ z - 60x_1 - 30x_2 - 20x_3 = 0$$

$$
\begin{aligned}
s.t. \qquad 8x_1 + 6x_2 + x_3 + s_1 &= 48 && \text{(Lumber constraint)} && (30)\\
4x_1 + 2x_2 + 1.5x_3 + s_2 &= 20 && \text{(Finishing constraint)} && (31)\\
2x_1 + 1.5x_2 + 0.5x_3 + s_3 &= 8 && \text{(Carpentry constraint)} && (32)\\
x_2 + s_4 &= 5 && \text{(Limitation on table demand)} && (33)\\
x_1, x_2, x_3, s_1, s_2, s_3, s_4 &\geq 0 && && (34)
\end{aligned}
$$

Iteration 0: Step 2

1. After obtaining a canonical form, we therefore search for the initial bfs.
2. By inspection, we see that if we set $x_1 = x_2 = x_3 = 0$, we can solve for the values of s_1, s_2, s_3, and s_4 by setting s_i equal to the right-hand side of row i.
3. Observe that each basic variable may be associated with the row of the canonical form in which the basic variable has a coefficient of 1.
4. With this convention, the basic feasible solution for our initial canonical form has
$$BV = \{z, s_1, s_2, s_3, s_4\} \quad \text{and} \quad NBV = \{x_1, x_2, x_3\}$$
5. For this basic feasible solution $z = 0,\ s_1 = 48,\ s_2 = 20,\ s_3 = 8,\ s_4 = 5,\ x_1 = x_2 = x_3 = 0$

Canonical Form 0

Row		Basic Variable
0	$z - 60x_1 - 30x_2 - 20x_3 = 0$	$z = 0$
1	$8x_1 + 6x_2 + x_3 + s_1 = 48$	$s_1 = 48$
2	$4x_1 + 2x_2 + 1.5x_3 + s_2 = 20$	$s_2 = 20$
3	$2x_1 + 1.5x_2 + 0.5x_3 + s_3 = 8$	$s_3 = 8$
4	$x_2 + s_4 = 5$	$s_4 = 5$

Iteration 0: Step 3

Is the Current Basic Feasible Solution Optimal?

1. Once we have obtained a basic feasible solution, we need to determine whether it is optimal; if the bfs is not optimal, then we try to find a bfs adjacent to the initial bfs with a larger z-value.
2. To do this, we try to determine whether there is any way that z can be increased by increasing some nonbasic variable from its current value of zero while holding all other nonbasic variables at their current values of zero.
3. From eq. (35) $z = 0$ can be increased by increasing either x_1, x_2 or x_3. Hence the current solution is not optimal.

$$z = 60x_1 + 30x_2 + 20x_3 \tag{35}$$

4. Similarly, the current bfs is not optimal because the variables in row 0 (36) have negative coefficients.

$$\text{row } 0 \implies z - 60x_1 - 30x_2 - 20x_3 = 0 \tag{36}$$

Iteration 0: Step 4

Determine the Entering Variable

1. We choose the entering variable (in a max problem) to be the nonbasic variable with the most negative coefficient in row 0.

$$z - 60x_1 - 30x_2 - 20x_3 = 0 \tag{37}$$

2. We could observe that x_1 has the most negative coefficient in row 0. Thus, x_1 is the entering variable.

Iteration 0: Step 5

In Which Row Does the Entering Variable Become Basic?

1. When entering a variable into the basis, compute the ratio for every constraint in which the entering variable has a positive coefficient. The constraint with the smallest ratio is called the winner of the ratio test.
2. The smallest ratio is the largest value of the entering variable that will keep all the current basic variables nonnegative. That is

$$\text{Row 1 limit on } x_1 = \frac{48}{8} = 6, \qquad \text{so } s_1 \geq 0 \text{ when } x_1 \leq 6 \tag{38}$$

$$\text{Row 2 limit on } x_1 = \frac{20}{4} = 5 \qquad \text{so } s_2 \geq 0 \text{ when } x_1 \leq 5 \tag{39}$$

$$\text{Row 3 limit on } x_1 = \frac{8}{2} = 4 \qquad \text{so } s_3 \geq 0 \text{ when } x_1 \leq 4 \tag{40}$$

$$\text{Row 4 limit on } x_1 = \text{no limit} \qquad \text{so } s_4 \geq 0 \text{ for all values of } x_1 \tag{41}$$

Iteration 0: Step 5

1. Now, row 3 is the winner of the ratio test for entering x_1 into the basis.
2. Always make the entering variable a basic variable in a row that wins the ratio test (ties may be broken arbitrarily).
3. To make x_1 a basic variable in row 3, we use elementary row operations to make x_1 have a coefficient of 1 in row 3 and a coefficient of 0 in all other rows. That is

$$NR_0 = R_0 + 30R_3 \tag{42}$$

$$NR_1 = R_1 - 4R_3 \tag{43}$$

$$NR_2 = R_2 - 2R_3 \tag{44}$$

$$NR_3 = 0.5R_3 \tag{45}$$

4. This procedure is called pivoting on row 3; and row 3 is the pivot row.
5. The final result is that x_1 replaces s_3 as the basic variable for row 3.

Iteration 0: Step 6

This yields the new canonical form 1 as below

Canonical Form 1

Row		Basic Variable
Row 0′	$z + 15x_2 - 5x_3 + 30s_3 = 240$	$z = 240$
Row 1′	$- x_3 + s_1 - 4s_3 = 16$	$s_1 = 16$
Row 2′	$- x_2 + 0.5x_3 + s_2 - 2s_3 = 4$	$s_2 = 4$
Row 3′	$x_1 + 0.75x_2 + 0.25x_3 + 0.5s_3 = 4$	$x_1 = 4$
Row 4′	$x_2 + s_4 = 5$	$s_4 = 5$

Looking for a basic variable in each row of the current canonical form, we find that

$$BV = \{z, s_1, s_2, x_1, s_4\} \quad \text{and} \quad NBV = \{s_3, x_2, x_3\}$$

For this basic feasible solution

$$z = 240,\ s_1 = 16,\ s_2 = 4,\ x_1 = 4,\ s_4 = 5,\ s_3 = x_2 = x_3 = 0$$

Iteration 1: Step 3

Is new bfs optimal?

1. We now try to find a bfs that has a still larger z-value.
2. We begin by examining canonical form 1 to see if we can increase z by increasing the value of some nonbasic variable (while holding all other nonbasic variables equal to zero).
3. Rearranging row 0' to solve for z yields

$$z = 240 - 15x_2 + 5x_3 - 30s_3 \tag{46}$$

4. Clearly z can be increased by increasing x_3. Hence the current solution is not optimal.
5. Similarly, the current bfs is not optimal because a variable x_3 in row $0'$ (47) have negative coefficients.

$$\text{row } 0' \implies z + 15x_2 - 5x_3 + 30s_3 = 240 \tag{47}$$

Iteration 1: Step 4

Determine the Entering Variable

1. From eq. (46), we see that increasing the nonbasic variable x_2 by 1 with other variables as zeros will decrease z by 15. We don't want to do that.
2. Increasing the nonbasic variable s_3 by 1 with other variables as zeros will decrease z by 30. Again, we don't want to do that.
3. On the other hand, increasing x_3 by 1 with other variables as zeros will increase z by 5. Thus, we choose to enter x_3 into the basis.
4. Again x_3 is the only variable with negative coefficient from the equation

$$z + 15x_2 - 5x_3 + 30s_3 = 240$$

Iteration 1: Step 5

$s_1 = 16 + x_3 \implies$	Row 1' limit on x_3	$=$ no limit	so $s_1 \geq 0$ for all values of x_3
$s_2 = 4 - 0.5x_3 \implies$	Row 2' limit on x_3	$= \dfrac{4}{0.5} = 8$	so $s_2 \geq 0$ when $x_3 \leq 8$
$x_1 = 4 - 0.25x_3 \implies$	Row 3' limit on x_3	$= \dfrac{4}{0.25} = 16$	so $x_1 \geq 0$ when $x_3 \leq 16$
$s_4 = 5 \implies$	Row 4' limit on x_3	$=$ no limit	so $s_4 \geq 0$ for all values of x_1

1. Now, row 2' is the winner of the ratio test for entering x_3 into the basis.
2. This means that we should use EROs to make x_3 a basic variable in row 2'. This implies that s_2 will leave the basis.

Iteration 1: Step 6

This yields the new canonical form 2 as below

Canonical Form 2

Row		Basic Variable
0″	$z + 5x_2 + 10s_2 + 10s_3 = 280$	$z = 280$
1″	$-2x_2 + s_1 + 2s_2 - 8s_3 = 24$	$s_1 = 24$
2″	$-2x_2 + x_3 + 2s_2 - 4s_3 = 8$	$x_3 = 8$
3″	$x_1 + 1.25x_2 - 0.5s_2 + 1.5s_3 = 2$	$x_1 = 2$
4″	$x_2 + s_4 = 5$	$s_4 = 5$

Looking for a basic variable in each row of the current canonical form, we find that

$$BV = \{z, s_1, x_3, x_1, s_4\} \quad \text{and} \quad NBV = \{s_2, s_3, x_2\}$$

For this basic feasible solution

$$z = 280,\ s_1 = 24,\ x_3 = 8,\ x_1 = 2,\ s_4 = 5,\ s_2 = s_3 = x_2 = 0$$

Iteration 2: Step 3

1. We now try to find a bfs that has a still larger z-value. Rearranging row 0" to solve for z yields
$$z = 280 - 5x_2 - 10s_2 - 10s_3 \tag{48}$$
2. From eq. (48), we see that increasing x_2 by 1 will decrease z by 5; increasing s_2 by 1 will decrease z by 10; increasing s_3 by 1 will decrease z by 10.
3. Thus, increasing any nonbasic variable will cause z to decrease. This might lead us to believe that our current bfs from canonical form 2 is an optimal solution.
4. Also all the coefficient of the row zero form $z + 5x_2 + 10s_2 + 10s_3 = 280$ are non-negative.
5. Our current bfs from canonical form 2 is

$$z = 60x_1 + 30x_2 + 20x_3 \tag{49}$$
$$= 60(2) + 30(0) + 20(8) \tag{50}$$
$$= 120 + 160 \tag{51}$$
$$= 280 \tag{52}$$

Using the Simplex Algorithm to Solve Minimization Problems

There are two different ways that the simplex algorithm can be used to solve minimization problems. We illustrate these methods by solving the following LP:

$$\min z = 2x_1 - 3x_2 \tag{53}$$

$$\text{s.t. } x_1 + x_2 \le 4 \tag{54}$$

$$x_1 - x_2 \le 6 \tag{55}$$

$$x_1,\ x_2 \ge 0 \tag{56}$$

Solution Method 1: Converting to a maximum problem

1. This method multiplies the objective function for the min problem by -1 and solves the problem as a maximization problem with objective function $-z$.
2. The optimal solution to the max problem will give you the optimal solution to the min problem.
3. The optimal solution to this LP problem is the point (x_1, x_2) in the feasible region for the LP that makes $z = 2x_1 - 3x_2$ the smallest.
4. Equivalently, we may say that the optimal solution to this LP is the point in the feasible region that makes $-z = -2x_1 + 3x_2$ the largest.

This means that we can find the optimal solution to the new LP as

$$\max\ -z = -2x_1 + 3x_2 \tag{57}$$
$$\text{s.t. } x_1 + x_2 \leq 4 \tag{58}$$
$$x_1 - x_2 \leq 6 \tag{59}$$
$$x_1,\ x_2 \geq 0 \tag{60}$$

After adding slack variables s_1 and s_2 to the two constraints, we obtain the initial tableau as

$-z$	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	2	-3	0	0	0	$-z = 0$	
0	1	1	1	0	4	$s_1 = 4$	$\frac{4}{1} = 4^*$
0	1	-1	0	1	6	$s_2 = 6$	None

Because x_2 is the only variable with a negative coefficient in row 0, we enter x_2 into the basis.

The ratio test indicates that x_2 should enter the basis in row 1. The resulting tableau is shown below

$-z$	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	5	0	3	0	12	$-z = 12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Because each variable in row 0 has a nonnegative coefficient, this is an optimal tableau. Thus, the optimal solution is

$$-z = 12,\ x_2 = 4,\ s_2 = 10,\ x_1 = s_1 = 0$$

$$-z = -2x_1 + 3x_2 \tag{61}$$

$$= -2(0) + 3(4) = 12 \tag{62}$$

Solution Method 2

1. A simple modification of the maximum simplex algorithm can be used to solve minimum problems.
2. In a minimum case the current bfs is optimal if all nonbasic variables in row 0 have nonpositive coefficients.
3. Here if any nonbasic variable in row 0 has a positive coefficient, choose the variable with the 'most positive' coefficient in row 0 to enter the basis.
4. If we use this method to solve this LP, then our initial tableau is given as

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	-2	-3	0	0	0	$z = 0$	
0	-1	1	1	0	4	$s_1 = 4$	$\frac{4}{1} = 4^*$
0	-1	-1	0	1	6	$s_2 = 6$	None

1. The pivot term is encircled and the winner of the ratio test is denoted by *.
2. Because x_2 has the most positive coefficient in row 0, we enter x_2 into the basis.

$$NR_0 = R_0 - 3R_1 \tag{63}$$

$$NR_1 = R_1 \tag{64}$$

$$NR_2 = R_2 + R_1 \tag{65}$$

3. The ratio test indicates that x_2 should enter the basis in the first constraint, row 1. The resulting tableau is shown below

z	x_1	x_2	s_1	s_2	rhs	Basic Variable
1	−5	0	−3	0	−12	$z = -12$
0	1	1	1	0	4	$x_2 = 4$
0	2	0	1	1	10	$s_2 = 10$

Because each variable in row 0 has a nonpositive coefficient, this is an optimal tableau. Thus, the optimal solution is

$$z = -12,\ x_2 = 4,\ s_2 = 10,\ x_1 = s_1 = 0$$

That is

$$z = 2x_1 - 3x_2 \tag{66}$$

$$= 2(0) - 3(4) \tag{67}$$

$$= -12 \tag{68}$$

LP with Alternative Optimal Solutions

1. We indicated earlier that some LPs have more than one optimal extreme point.
2. If an LP has more than one optimal solution, then we say that it has multiple or alternative optimal solutions.
3. Note that all basic variables must have a zero coefficient in row 0. However, if we have a non-basic variable having a zero coefficient in the final tableau then the LP has multiple solutions.
    1. If a non-basic variable has a zero coefficient in the final tableau, it means that the value of this non-basic variable can be adjusted (increased from zero) without changing the value of the objective function.
    2. This also indicates that there are multiple optimal solutions, as the flexibility in the value of this non-basic variable allows for different combinations of values that achieve the same optimal objective value.
4. In the next example, we show how the simplex algorithm can be used to determine whether an LP has alternative optimal solutions.

Example

The Dakota Furniture Company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber and two types of skilled labor: finishing and carpentry. The amount of each resource needed to make each type of furniture is given the Table below:

Resource	Desk	Table	Chair
Lumber (board ft)	8	6	1
Finishing hours	4	2	1.5
Carpentry hours	2	1.5	0.5

Currently, 48 board feet of lumber, 20 finishing hours, and 8 carpentry hours are available. A desk sells for ¢60, a table for ¢35, and a chair for ¢20. Dakota believes that demand for desks and chairs is unlimited, but at most five tables can be sold. Because the available resources have already been purchased, how does Dakota maximize total revenue.

Solution

The only modification is that tables sell for ¢35 instead of ¢30

The new tableau is as below

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	-60	-35	-20	0	0	0	0	0	$z = 0$	
0	8	6	1	1	0	0	0	48	$s_1 = 48$	$\frac{48}{8} = 6$
0	4	2	1.5	0	1	0	0	20	$s_2 = 20$	$\frac{20}{4} = 5$
0	2	1.5	0.5	0	0	1	0	8	$s_3 = 8$	$\frac{8}{2} = 4^*$
0	0	1	0	0	0	0	1	5	$s_4 = 5$	None

Because x_1 has the most negative coefficient in row 0, we enter x_1 into the basis. The ratio test indicates that x_1 should be entered in row 3.
The new[2] tableau is as below

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable	Ratio
1	0	10	−5	0	0	30	0	240	$z = 240$	
0	0	0	−1	1	0	−4	0	16	$s_1 = 16$	None
0	0	−1	0.5	0	1	−2	0	4	$s_2 = 4$	$\frac{4}{0.5} = 8^*$
0	1	0.75	0.25	0	0	0.5	0	4	$x_1 = 4$	$\frac{4}{0.25} = 16$
0	0	1	0	0	0	0	1	5	$s_4 = 5$	None

Now only x_3 has a negative coefficient in row 0, so we enter x_3 into the basis. The ratio test indicates that x_3 should enter the basis in row 2. The new[3] tableau is as below

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	0	-2	0	1	2	-8	0	24	$s_1 = 24$
0	0	-2	1	0	2	-4	0	8	$x_3 = 8$
0	1	1.25	0	0	-0.5	1.5	0	2	$x_1 = 2^*$
0	0	1	0	0	0	0	1	5	$s_4 = 5$

The resulting is optimal.

$$z = 280,\ s_1 = 24,\ x_3 = 8,\ x_1 = 2, s_4 = 5, \text{ and } x_2 = s_2 = s_3 = 0$$

Recall that all basic variables must have a zero coefficient in row 0 (or else they wouldn't be basic variables). However, in our optimal tableau, there is a nonbasic variable, x_2, which also has a zero coefficient in row 0. Let us see what happens if we enter x_2 into the basis.

The ratio test indicates that x_2 should enter the basis in row 3.

z	x_1	x_2	x_3	s_1	s_2	s_3	s_4	rhs	Basic Variable
1	0	0	0	0	10	10	0	280	$z = 280$
0	1.6	0	0	1	1.2	−5.6	0	27.2	$s_1 = 27.2$
0	1.6	0	1	0	1.2	−1.6	0	11.2	$x_3 = 11.2$
0	0.8	1	0	0	−0.4	1.2	0	1.6	$x_2 = 1.6$
0	−0.8	0	0	0	0.4	−1.2	1	3.4	$s_4 = 3.4$

1. The important thing to notice is that because x_2 has a zero coefficient in the optimal tableau's row 0, the pivot that enters x_2 into the basis does not change row 0.
2. This means that all variables in our new row 0 will still have nonnegative coefficients.
3. Thus, our new tableau is also optimal. Because the pivot has not changed the value of z, an alternative optimal solution for the Dakota example is

$$z = 280,\ s_1 = 27.2,\ x_3 = 11.2,\ x_2 = 1.6,\ s_4 = 3.4,\ \text{and}\ x_1 = s_3 = s_2 = 0$$

4. Thus, Dakota has multiple (or alternative) optimal extreme points.

Degenerate Linear Programming (LP) Problems

Definition

A degenerate linear programming (LP) problem occurs when one or more of the basic variables in a basic feasible solution take the value of zero.

1. Since a basic variable is zero, the corresponding constraint is tight but does not contribute to the objective value.
2. Degeneracy can lead to multiple optimal solutions.
3. One of the potential issues with degeneracy is cycling, where the simplex algorithm might revisit the same set of basic variables repeatedly without making progress.
4. Modern implementations of the simplex algorithm include anti-cycling rules to prevent this.

Example

A Degenerate LP

z	x_1	x_2	s_1	s_2	rhs	Basic Variable	Ratio
1	-5	-2	0	0	0	$z = 0$	
0	1	1	1	0	6	$s_1 = 6$	6
0	1	-1	0	1	0	$s_2 = 0$	0*

In this bfs, the basic variable $s_2 = 0$. Thus, the LP that generated this tableau is a degenerate LP.

Handling Degeneracy in the Simplex Method

- **Bland's Rule**:
  - Bland's rule is an anti-cycling strategy where the smallest index variable is chosen to enter or leave the basis in case of a tie.
  - This prevents the algorithm from revisiting the same set of basic feasible solutions.
- **Perturbation**:
  - Slightly modifying (perturbing) the constraints can ensure that the solution moves out of the degenerate state.
  - This involves adding small values to the constraints to break ties and make progress.
- **Lexicographic Ordering**:
  - This method systematically chooses the entering and leaving variables to ensure progress is made even in the presence of degeneracy.
  - It prioritizes decisions to avoid cycles and ensure convergence.

Exercise

Use the simplex algorithm to find the optimal solution to the following LP:

1.
$$\begin{aligned} \max\ z &= 2x_1 + 3x_2 \\ \text{s t}\ x_1 + 2x_2 &\le 6 \\ 2x_1 + x_2 &\le 8 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

2.
$$\begin{aligned} \max\ z &= x_1 + x_2 \\ \text{s t}\ 4x_1 + x_2 &\le 100 \\ x_1 + x_2 &\le 80 \\ x_1 &\le 40 \\ x_1,\ x_2 &\ge 0 \end{aligned}$$

3.

$$
\begin{aligned}
\min\ & z = 4x_1 - x_2 \\
\text{s t}\ & 2x_1 + x_2 \leq 8 \\
& x_2 \leq 5 \\
& x_1 - x_2 \leq 4 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

4.

$$
\begin{aligned}
\min\ & z = -x_1 - x_2 \\
\text{s t}\ & x_1 - x_2 \leq 1 \\
& x_1 + x_2 \leq 2 \\
& x_1,\ x_2 \geq 0
\end{aligned}
$$

5. Find alternative optimal solutions to the following LP:

$$
\begin{aligned}
\max\ z = x_1 + x_2 \\
\text{s.t. } x_1 + x_2 + x3 \leq 1 \\
x_1 + 2x_3 \leq 1 \\
x_i \geq 0
\end{aligned}
$$

6. How many optimal basic feasible solutions does the following LP have?

$$
\begin{aligned}
\max\ z\ 2x_1 + 2x_2 \\
\text{s.t. } x_1 + x_2 \leq 6 \\
2x_1 + x_2 \leq 13 \\
x_i \geq 0
\end{aligned}
$$

STAY BLESSED AND SHALOM

END OF LECTURE

THANK YOU