Summary

Cond.	Types	Applications
Weak	$X_n \xrightarrow{d} X$ $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$	Central Limited Theorem
Inbetween	$X_n \xrightarrow{p} X$ $\lim_{n\to\infty} P(\lvert X_n - X\rvert \geq \epsilon) = 0$	Weak law of large number (WLLN) Consistency of estimator
Strong	$X_n \xrightarrow{r^{th}} X$ $\lim_{n\to\infty} \mathrm{E}[\lvert X_n - X\rvert^r] = 0$	MSE of estimator (Efficiency)
Strong	$X_n \xrightarrow{a.s} X$ $P\left(\lim_{n\to\infty} X_n = X\right) = 1$	Strong law of large number (SLLN)

6 Homework

- Homework 2.1 (scores=5x3=15)
- Homework 2.2 (scores=5)
- Bonus: Proof of Lindeberg–Lévy CLT (10 scores)

Homework 2.1 (scores=5x3=15)

HW 1: Converge to **a constant** in distribution/probability

If random variables $X_1, X_2, X_3, \ldots$ be a sequence of i.i.d. Bernoulli$(1/n)$.
Questions:

1. Verify if $X_n \xrightarrow{d} 0$, *i.e.*, $\lim_{n\to\infty} F_{X_n}(x) = F_X(x = 0)$? (5 scores)
2. Verify if $X_n \xrightarrow{p} 0$, *i.e.*, $\lim_{n\to\infty} P(|X_n - 0| \geq \epsilon) = 0$? (5 scores)
3. Verify if $X_n \xrightarrow{a.s.} 0$, *i.e.*, $P(\lim_{n\to\infty} X_n = 0) = 1$? To get a full score, you have to provide: 1.theoretical proof & 2. simulation....(5 scores)

```python
# Provide histrogram density of $|X_n| > \epsilon $
N_list = [1,10,1000] # < You have to change this ... Just look at the formular .. what is the range of n..
P_count_n = []
P_count_temp = 0
N_size=1000

for N_select in N_list:
    XN = np.random.binomial(size=N_size, p=1/N_select, n=1)
    counts = 1*(XN > 0)

    P_count = counts.sum()/N_size
    P_count_temp += P_count
    P_count_n.append(P_count_temp)


P_count_n
plt.figure(figsize=(12,6))
plt.plot(N_list, P_count_n, color="blue", linewidth=2, label="Uniform")
plt.ylabel("$P_n$")
plt.xlabel("Iterations")
plt.grid()
plt.legend()
plt.show()
```

Homework 2.2 (scores=5)

HW 2: Convergence in mean $\xRightarrow{Imply}$ convergence in probability

If $X_n \xrightarrow{L^r} X$ for some $r \geq 1$, then $X_n \xrightarrow{p} X$.

Show that $\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$. Hint! Since $|X_n - X|$ is an RV $\in \mathbb{R}^+$, this can be done easily using Markov Inequality.

Bonus: Proof of Lindeberg–Lévy CLT (10 scores)

Bonus 1: Lindeberg–Lévy Central Limit Theorem (CLT) [2]

Let $X_1, X_2, \ldots$ be a sequence of iid random variables with mean $\mathrm{E}[X_n] = \mu \leq \infty$ and variance $\mathrm{Var}[X_n] = \sigma^2 \leq \infty$. Then, the random variable Z_n, defined as,

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sigma\sqrt{n}} \tag{37}$$

converges **in distribution** to the standard normal random variable X, *i.e.*, $Z_n \xrightarrow{d} X$ where $X \sim \mathcal{N}(0, 1^2)$. That is, $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$.

Note that $X_1, X_2, \ldots$ can have any distribution.
Unlike the estimator settings, $X_1, X_2, \ldots$ is unrelated to X.

Show that this theorem is true (using Continuity Theorem). Hint in Slide 150

Soln. Bonus 1: Sketch of how you may do this & given info ...

- If $X \sim \mathcal{N}(0, 1^2)$, $M_X(t) = \mathrm{E}[e^{tX}] = e^{\frac{t^2}{2}}$
- Use the fact that if $\lim_{n\to\infty} M_{Z_n}(t) = M_X(t) = e^{\frac{t^2}{2}}$, then $Z_n \xrightarrow{d} X$.
- Taylor series expansion centering around a: $f(q) = \sum_i \frac{f^i(q=a)(q-a)^i}{i!}$
- $\lim_{n\to\infty} \left(1 + \frac{t^2}{2n}\right)^n = e^{\frac{t^2}{2}}$ (L'Hôpital's rule)

Goal: show that $\lim_{n\to\infty} M_{Z_n}(t) = e^{\frac{t^2}{2}}$.

Sketch the steps ...

1. $\Rightarrow$ Derive $M_{Z_n}(t) = \mathrm{E}[e^{tZ_n}]$ (4 scores)
2. $\Rightarrow$ Use Taylor expansion to calculate $\mathrm{E}[e^{tZ_n}]$ (4 scores)
3. $\Rightarrow$ Take the limit of $n \to \infty$ (2 scores)