ELE639 - Lab 1 - Report

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Abstract

This experiment will be used to study the operating and critical characteristics of a variety of feedback-loop modes. We used SIMULINK to create three different models, a P, PI, and PD controller, the characteristics that we were interested in studying included; K_{op}, K_{crit}, ω_{crit}, and G_m of the various modes and the implications this has on our control system.

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1 Background & Lab Specifications

1.1 Transfer Functions

The starting point for any control system is the *Transfer Function* for the actuator we desire to control. For this lab we were presented with two transfer functions and were advised to use them by feeding the out of $G_1(s)$ into the input of $G_2(s)$ where the output would be measured. This is depicted in Figure 1. We can simplify the system by multiplying the transfer functions together and treat them as one, and we will for our calculations, in MATLAB it is easier to leave them as two separate equations. Also, G_{open} represents the open loops transfer function for our system.

$$G_1(s) = \frac{Y(s)}{U(s)} = \frac{1.3}{s^2 + 9s + 2} \tag{1}$$

$$G_2(s) = \frac{Y(s)}{U(s)} = \frac{2}{s^2 + 2s + 2.1} \tag{2}$$

$$G_{open}(s) = G_1(s) \bullet G_2(s) = \frac{2.6}{s^4 + 11s^3 + 22.1s^2 + 22.9s + 4.2} \tag{3}$$

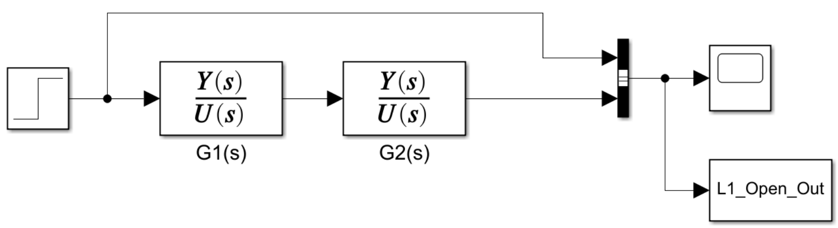

Figure 1: Block Diagram of Open Loop System

1.2 Routh Array

The *Routh Array* is a process of calculating coefficients and building the Routh array based on the denominator of the systems transfer function. We will use our system for example purposes. Table 1 will contain the equations for calculating a fourth order Routh array, for the purposes of this lab we were required to calculate a fifth order Routh array for one of the controllers. An example of a calculated Routh is show in Table 2.

If the transfer function of our system, $G(s)$ is:

$$G_1(s) = \frac{N(s)}{Q(s)} = \frac{A}{a_n s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + a_{n-3}s^{n-3} + a_{n-4}s^{n-4}} \tag{4}$$

s^n	a_n	a_{n-2}	a_{n-4}
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}
s^{n-2}	b_1	b_2	b_3
s^{n-3}	c_1	c_2	c_3

Table 1: Routh Array

Where the equations for b_1, b_2, c_1, etc, are as follows:

$$b_1 = \frac{-|(a_n \bullet a_{n-3}) - (a_{n-1} \bullet a_{n-2})|}{a_{n-1}} \tag{5}$$

$$b_2 = \frac{-|(a_n \bullet a_{n-5}) - (a_{n-1} \bullet a_{n-4})|}{a_{n-1}} \tag{6}$$

$$c_1 = \frac{-|(a_{n-1} \bullet b_2) - (b_1 \bullet a_{n-3})|}{b_1} \tag{7}$$

$$c_2 = \frac{-|(a_{n-1} \bullet b_3) - (b_1 \bullet a_{n-5})|}{b_1} \tag{8}$$

1.2.1 Example

If the transfer function of our system, $G(s)$ is:

$$G_1(s) = \frac{N(s)}{Q(s)} = \frac{2.6}{s^4 + 11s^3 + 22.1s^2 + 22.9s + 4.2} \tag{9}$$

s^4	1	22.1	4.2
s^3	11	22.9	0
s^2	20.02	4.2	0
s^1	20.59	0	0
s^0	4.2	0	0

Table 2: Calculated Routh Array

1.3 Routh Hurwitz Stability Criterion

The *Routh Hurwitz Stability Criterion* states that if all of the coefficients in the first column of the *Routh Array* are positive then it is said that the system is stable. This is a beneficial property when we want to calculate the bounds of our control system

1.4 Controllers

In this lab we experiment primarily with three different control modes for feedback-loop/closed-loop control systems: P, PI, and PD, below the transfer functions and system diagrams will be provided and explained.

1.4.1 Proportional

A proportional control system is one whose output is directly proportional to its error (the difference between the current state of the system and the reference input). A proportional controller has the following open and closed loop transfer function, as well as an associated system diagram, Figure 2:

$$G(s) = G_1(s) \bullet G_2(s) \tag{10}$$

$$G_{open}(s) = K_P G(s) \tag{11}$$

$$G_{cl}(s) = \frac{K_P G(s)}{1 + K_P G(s)} = \frac{G_{open}(s)}{1 + G_{open}(s)} \tag{12}$$

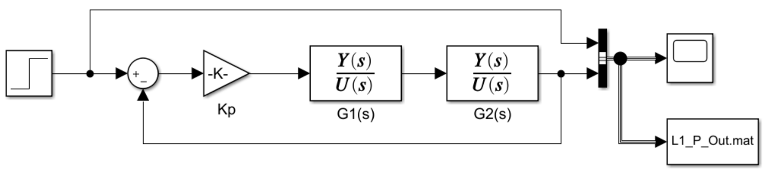

Figure 2: Block Diagram of Proportional Control System

1.4.2 Proportional + Integral

A PI control system is one whose input is not just directly proportional to its error, it is also dependent on the previous terms now, as we are performing an integral. A PI controller has the following open and closed loop transfer function, as well as an associated system diagram, Figure 3:

$$G_{open}(s) = K_P(1 + \frac{1}{K_I s})G(s) \tag{13}$$

$$G_{cl}(s) = \frac{K_P(1 + \frac{1}{K_I s})G(s)}{1 + K_P(1 + \frac{1}{K_I s})G(s)} = \frac{G_{open}(s)}{1 + G_{open}(s)} \tag{14}$$

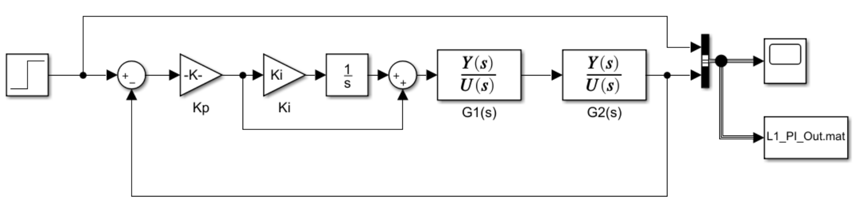

Figure 3: Block Diagram of Proportional + Integral Control System

1.4.3 Proportional + Derivative

A PI control system is one whose input is not just directly proportional to its error, it is now trying to predict the next term via the derivative. A PD controller has the following open and closed loop transfer function, as well as an associated system diagram, Figure 4:

$$G_{open}(s) = K_P(1 + K_D s)G(s) \tag{15}$$

$$G_{cl}(s) = \frac{K_P(1 + K_D s)G(s)}{1 + K_P(1 + K_D s)G(s))} = \frac{G_{open}(s)}{1 + G_{open}(s)} \tag{16}$$

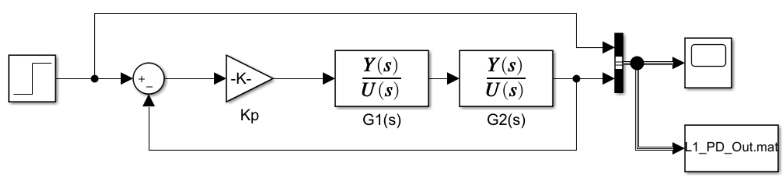

Figure 4: Block Diagram of Proportional + Derivative Control System

2 Results

2.1 Proportional

Required Equations:

$$G_{open}(s) = K_P G(s) = \frac{K_P 2.6}{s^4 + 11s^3 + 22.1s^2 + 22.9s + 4.2} \tag{17}$$

$$G_{cl}(s) = \frac{K_P G(s)}{1 + K_P G(s)} = \frac{K_P 2.6}{s^4 + 11s^3 + 22.1s^2 + 22.9s + (4.2 + K_P 2.6)} \tag{18}$$

Routh Array:

s^4	1	22.1	4.2 + 2.6K
s^3	11	22.9	0
s^2	20.02	4.2 + 2.6k	0
s^1	-1.43k + 20.59	0	0
s^0	4.2 + 2.6k	0	0

Table 3: Routh Array for P Control

Observed Values:

	$K_{P,crit}$	ω_{crit}
Experimental	14.413	1.457
Theoretical	14.3986	1.443

Table 4: Observed Values for P Control

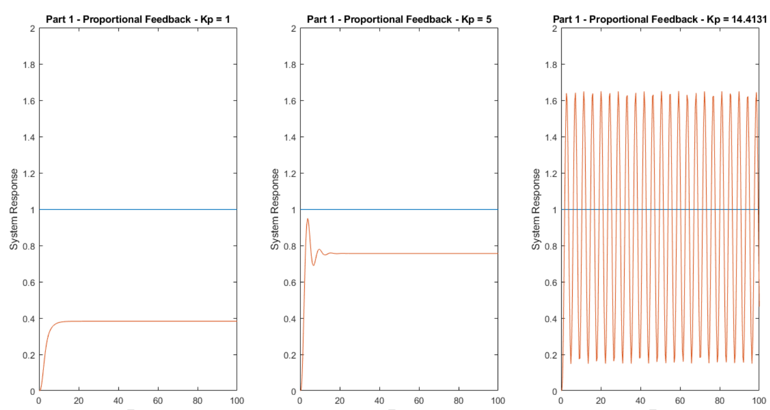

Figure 5: Plots of P Control

2.2 Proportional + Integral

Required Equations:

$$G_{open}(s) = K_P(1 + \frac{1}{K_I s})G(s) = \frac{K_P(1 + \frac{1}{K_I s})2.6}{s^4 + 11s^3 + 22.1s^2 + 22.9s + 4.2} \tag{19}$$

$$G_{cl}(s) = \frac{G_{open}(s)}{1 + G_{open}(s)} = \frac{K_P(1 + \frac{1}{K_I s})2.6}{5s^5 + 55s^4 + 110.5s^3 + 114.5s^2 + s(21 + K_P 13) + 2.6K_P} \tag{20}$$

Given Values:

$$K_I = 5s \tag{21}$$

Routh Array:

s^5	5	110.5	13 + 21K
s^4	55	114.5	2.6K
s^3	100.09	$\frac{702K+1155}{55}$	0
s^2	$\frac{2061061-140400k}{55}$	0	0
s^1	$\frac{702K+1155}{55}$	0	0
s^0	0	0	0

Table 5: Routh Array for PI Control

Observed Values:

	$K_{PD,crit}$	ω_{crit}
Experimental	12.205	1.313
Theoretical	14.680	2.051

Table 6: Observed Values for PI Control

The $K_{PI,crit}$ was calculated using the auxiliary equations for s^1 and ω_{crit} auxiliary equations for s^2.

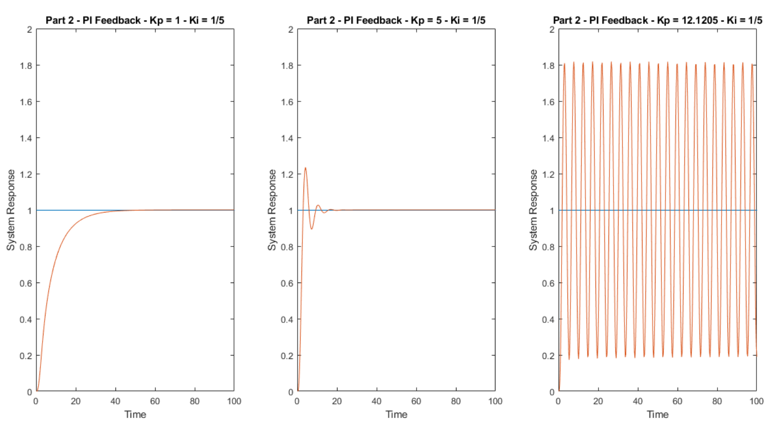

Figure 6: Plots of PI Control

2.3 Proportional + Derivative

Required Equations:

$$G_{open}(s) = K_P(1 + K_D s)G(s) = \frac{K_P(1 + K_D s)2.6}{s^4 + 11s^3 + 22.1s^2 + 22.9s + 4.2} \tag{22}$$

$$G_{cl}(s) = \frac{G_{open}(s)}{1 + G_{open}(s)} = \frac{K_P(1 + K_D s)2.6}{s^4 + 11s^3 + 22.1s^2 + s(22.9 + 5.2K_P) + (4.2 + 2.6K_P)} \tag{23}$$

Given Values:

$$K_D = 2s \tag{24}$$

Routh Array:

s^4	1	22.1	4.2 + 2.6K
s^3	11	22.9 + 5.2K	0
s^2	$\frac{1101-26K}{55}$	4.2 + 2.6k	0
s^1	$\frac{35568K+226719-1352K^2}{11010-260K}$	0	0
s^0	4.2 + 2.6k	0	0

Table 7: Routh Array for PD Control

Observed Values:

	$K_{PI,crit}$	ω_{crit}
Experimental	31.450	4.120
Theoretical	31.612	2.689

Table 8: Observed Values for PD Control

The $K_{PI,crit}$ was again calculated using the auxiliary equations for s^1 and ω_{crit} auxiliary equations for s^2

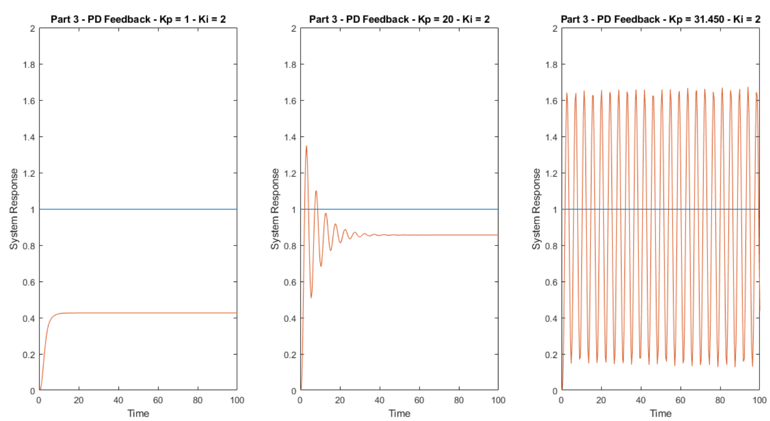

Figure 7: Plots of PD Control

3 Discussion

3.1 Q1

Are the theoretical results consistent with your recorded experimental values?
Our experimental and theoretical values are consistent. Our values are listed in the following table:

	Theoretical		Experimental	
	$K_{P,crit}$	ω_{crit}	$K_{P,crit}$	ω_{crit}
P	14.399	1.433	14.413	1.457
PI	14.680	2.051	12.1205	1.313
PD	31.612	2.689	31.450	4.120

Table 9: Gains and Oscillations observed in Lab

As observed, the experimental values are relatively consistent with the theory, we are attributing any error to rounding throughout the calculations for the theoretical values.

3.2 Q2

How did K_{crit} change under the different modes of control?
We see that K_{crit} varies from P to PI and from P to PD as well. As we change the mode of operation of our system to PI and include the integral term, it is observed that $K_{PI,crit}$ is lower than that of $K_{P,crit}$, the opposite is true between $K_{PD,crit}$ and $K_{P,crit}$, it is observed that $K_{PD,crit} >> K_{P,crit}$.

3.3 Q3

How did ω_{crit} change?
The same pattern that was observed for K_{crit} is also true for ω_{crit} as K_{crit} is directly related to the value of ω_{crit}. ω_{crit} is the frequency of oscillation when the system is marginally stable, implying a pair of system poles are located on the imaginary axis at a value of $\omega_{crit}\frac{rads}{s}$, K_{crit} corresponds to the proportional gain that results in a marginally stable system.

3.4 Q4

How did the Gain Margin change under the differnt modes of control?

For the P controller it was required that the operational gain was calculated via the measured K_{crit} and the provided G_m.

$$G_m = \frac{K_{crit}}{K_{op}} \Rightarrow K_{op} = \frac{K_{crit}}{G_m} \tag{25}$$

And then to use the calculated operational gain for the rest of the systems.

	Theoretical			Experimental		
	P	PI	PD	P	PI	PD
K_{crit}	14.399	14.680	31.612	14.413	12.1205	31.450
K_{OP}	3.59975	3.59975	3.59975	3.603	3.603	3.603
G_m	4	4.0781	8.7817	4	3.363	8.729

Table 10: Gain Margins observed in Lab

As observed in Table 10 the gain margin should decrease as the integral term is introduced because $K_{PI,crit} < K_{P,crit}$, and K_{OP} is fixed. Inversely, as we introduce the derivative term we see that the gain margin increases which makes sense because, $K_{PD,crit} > K_{P,crit}$, and K_{OP} is still fixed. The variance between the experimental and theoretical values is attributed to rounding error.

3.5 Q5

What are the implications of these changes on the relative stability and operation of a control system?

The implications that the different modes of operation imposes on a control system are the following:

- The *default*, P controller is incapable of producing the desired response before this particular system became unstable, this is because the P controller is not of a great enough order, the response can be observed in Figure 5.
- As an integral term is introduced it is observed that the system is capable producing the desired response before the system became marginally stable, also this system was capable of producing no oscillations in the response as seen in Figure 6. Since the PI controller introduces a new s term into the system in the form of the integrator, it is now capable of producing the desired response. But, since $K_{PI,crit} < K_{P,crit}$ we will also observe that $G_{PI,m} < G_{P,m}$ implying that the PI system has a smaller operating gain.
- As a derivative term is introduced it is also observed that the system is incapable of producing the desired response before this particular system became marginally stable, this can be observed in Figure 7. But, since $K_{PD,crit} > K_{P,crit}$ we will also observe that $G_{PD,m} > G_{P,m}$ implying that the PD system has a larger operating gain.