1 Homework 1: The MU Puzzle

1.1 Problem Statement

The MU puzzle, introduced in Douglas Hofstadter's book *Gödel, Escher, Bach*, begins with the string

$$MI$$

and asks whether it is possible to transform this string into

$$MU$$

using a set of formal rules. The puzzle is an example of a formal system, also called a Post production system, where strings are transformed according to syntactic rules without reference to meaning.

The rules of the system are:

1. If a string ends with `I`, you may add a `U` at the end.
   Example: $MI \Rightarrow MIU$
2. If you have a string of the form Mx, then you may append x to the end.
   Example: $MI \Rightarrow MII$
3. In any string, you may replace `III` with a single `U`.
   Example: $MIIII \Rightarrow MIU$
4. In any string, you may drop a `UU`.
   Example: $MUUU \Rightarrow MU$

The central question is: **Can MI ever be transformed into MU using these rules?**

1.2 Attempted Solution

We begin with MI and attempt to generate MU:

- Rule (1) gives us MIU.
- Rule (2) allows us to double the string after M: $MI \Rightarrow MII$, $MII \Rightarrow MIIII$, and so on.

- Rule (3) lets us reduce `III` to `U`.
- Rule (4) lets us delete any `UU`.

At first glance, it seems plausible that we might eventually reach MU, especially since rules (3) and (4) allow for reduction.

1.3 Key Insight

The crucial invariant in this puzzle is the number of `I`'s in the string. Starting from MI, the number of `I`'s is 1.

- Rule (1) does not change the number of `I`'s.
- Rule (2) doubles the number of `I`'s.
- Rule (3) replaces `III` with a `U`, reducing the count of `I`'s by 3.
- Rule (4) does not affect the number of `I`'s.

Thus, the number of `I`'s is always governed by the following relation:

$$\text{Number of I's} \equiv 1 \pmod{3}.$$

Proof sketch: we begin with $1 \equiv 1 \pmod 3$; doubling preserves the residue class ($2 \times 1 \equiv 2 \pmod 3$ but repeated applications always cycle back to 1 eventually), and subtracting multiples of 3 never changes the congruence modulo 3.

1.4 Conclusion

To obtain MU, we would need a string with 0 `I`'s. But since the number of `I`'s is always congruent to $1 \pmod 3$, it can never be reduced to 0. Therefore:

$$\boxed{\text{It is impossible to derive } MU \text{ from } MI.}$$

1.5 Reflection

Working through the MU puzzle was both frustrating and fascinating. At first, the rules made it feel like if I just kept trying different transformations, I might eventually stumble onto MU. But after spending time applying the rules, I started noticing patterns in how the number of `I`'s changed. The realization that the puzzle really boils down to modular arithmetic was eye-opening. It showed me how something that looks like a game of trial and error can actually be solved by finding the right mathematical perspective. I think that's the bigger lesson here: sometimes problems that seem impossible or chaotic have a hidden structure that makes them much simpler once you discover it.