Homework 5: Lambda Calculus Workout

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1 Workout Problem

Evaluate the following lambda calculus expression step by step:

$$(\lambda f.\lambda x.f(f(x)))(\lambda f.\lambda x.(f(f(fx))))$$

2 Solution

Let $M = \lambda f.\lambda x.f(f(x))$ and $N = \lambda f.\lambda x.(f(f(fx)))$.
We need to evaluate MN.

$$\begin{aligned} MN &= (\lambda f.\lambda x.f(f(x)))(\lambda f.\lambda x.(f(f(fx)))) && (1)\\ &\rightsquigarrow \lambda x.(\lambda f.\lambda x.(f(f(fx))))((\lambda f.\lambda x.(f(f(fx))))x) && (2)\\ &= \lambda x.(\lambda f.\lambda x.(f(f(fx))))(f(f(fx))) && (3)\\ &\rightsquigarrow \lambda x.f(f(f(f(f(fx))))) && (4) \end{aligned}$$

3 Step-by-Step Explanation

1. **Initial expression:** $(\lambda f.\lambda x.f(f(x)))(\lambda f.\lambda x.(f(f(fx))))$
2. **First -reduction:** Apply the function $M = \lambda f.\lambda x.f(f(x))$ to the argument $N = \lambda f.\lambda x.(f(f(fx)))$.

   This substitutes N for f in M:

   $$\lambda x.N(N(x))$$
3. **Expand N:** Replace N with its definition:

   $$\lambda x.(\lambda f.\lambda x.(f(f(fx))))((\lambda f.\lambda x.(f(f(fx))))x)$$
4. **Second -reduction:** Apply the inner function to x:

   $$(\lambda f.\lambda x.(f(f(fx))))x \rightsquigarrow \lambda x.(f(f(fx)))$$

   But wait, this creates a variable capture issue. Let me be more careful.

5. **Correct approach:** Let's rename variables to avoid capture:

$$(\lambda f.\lambda x.(f(f(fx))))x = (\lambda f.\lambda y.(f(f(fy))))x \rightsquigarrow \lambda y.(f(f(fy)))$$

6. **Final result:** The expression reduces to:

$$\lambda x.f(f(f(f(f(fx)))))$$

4 Mathematical Interpretation

This expression represents the composition of two functions:

- The first function $M = \lambda f.\lambda x.f(f(x))$ applies its argument twice
- The second function $N = \lambda f.\lambda x.(f(f(fx)))$ applies its argument three times
- When composed, the result applies the argument six times total

This demonstrates the power of lambda calculus to represent complex computations through simple function composition and application, which was Church's original vision for a foundation of mathematics based purely on functions.