HW 4 — PL 2025: Termination

Gabriel Giancarlo

September 22, 2025

Problem 4.1

Consider the following algorithm:

```
while b != 0:
    temp = b
    b = a mod b
    a = temp
return a
```

Under certain conditions (which?) this algorithm always terminates.
Find a measure function and prove termination.

Problem 4.2

Consider the following fragment of an implementation of merge sort:

```
function merge_sort(arr, left, right):
    if left >= right:
        return
    mid = (left + right) / 2
    merge_sort(arr, left, mid)
    merge_sort(arr, mid+1, right)
    merge(arr, left, mid, right)
```

Prove that
$$\varphi(left, right) = right - left + 1$$
is a measure function for merge_sort.

Solution 4.1

The given algorithm is the **Euclidean Algorithm** for computing the greatest common divisor (gcd) of two integers.

Conditions for Termination

The algorithm requires that $a, b \in \mathbb{N}$ with $b \geq 0$. In particular:

- a and b must be non-negative integers.

- If $b = 0$, the loop is skipped and the function returns a immediately.

Measure Function

We define the measure function
$$\varphi(a, b) = b.$$

Proof of Termination

At each iteration:
$$b \longmapsto a \bmod b,$$

where $0 \leq a \bmod b < b$.

Thus $\varphi(a, b)$ strictly decreases whenever $b \neq 0$, and it always remains a non-negative integer. Since $\mathbb{N}$ is well-founded under $<$, infinite descent is impossible. Therefore the algorithm must terminate.

Solution 4.2

We want to show that
$$\varphi(left, right) = right - left + 1$$
is a measure function for `merge_sort`.

Non-negativity

For all valid indices with $left \leq right$, we have

$$\varphi(left, right) = right - left + 1 \geq 1.$$

Thus φ always takes positive integer values.

Decrease on Recursive Calls

At each recursive step:
$$mid = \frac{left + right}{2}.$$

The recursive calls are

$$\texttt{merge\_sort}(arr, left, mid), \qquad \texttt{merge\_sort}(arr, mid + 1, right).$$

Their measures are:
$$\varphi(left, mid) = mid - left + 1,$$
$$\varphi(mid + 1, right) = right - (mid + 1) + 1 = right - mid.$$

Since $left \leq mid < right$, both of these values are strictly smaller than

$$\varphi(left, right) = right - left + 1.$$

Termination

Each recursive call strictly reduces the measure φ, which is bounded below by 1. By well-foundedness of $\mathbb{N}$ under $<$, recursion cannot proceed indefinitely. Therefore the algorithm always terminates.

Conclusion

- For the Euclidean algorithm, $\varphi(a, b) = b$ is a valid measure function, and the algorithm terminates for non-negative integer inputs with $b \geq 0$.

- For merge sort, $\varphi(left, right) = right - left + 1$ is a valid measure function, and recursion always terminates because the subproblems are strictly smaller.