Bayesian Statistics course

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BAYESIAN MODELS FOR FUNCTIONAL DATA WITH WEARABLE APPLICATIONS

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PRESENTATION OVERVIEW

INTRODUCTION

Project Objective

Examine different methods for ***Scalar-on-Function Regression*** techniques.

Contribute to Parkinson's Research, advancing collective knowledge and potentially inspiring innovative approaches.

DATASET

Dataset

Our analysis employs the ***Personalized Parkinson Project***, a study conducted on ***650 patients*** affected by Parkinson's Disease and carried out at the Radboud University Medical Center in Nijmegen, ***Netherlands***.

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Specifically

During a 2-year follow-up period, starting from October 1^{st} 2017 to December 31^{st} 2023, partecipants undergo ***three extensive annual assessments*** and continuously wear a ***multi-sensor investigational research device***.

DATASET

We work with a piece of data with **160 patients** associated to variables like:

- Impact of the ***tremor*** on daily activities;
- Level of ***activities of daily living***;
- ***Year*** of onset of the ***first symptoms***;
- ***Age*** and ***gender*** of the patients;
- ***Medication*** group;
- Z-scored retrained ***classifier*** for the ***tremor***;
- ***WearingOff***, i.e the ***decreasing effectiveness*** of treatments.

PREPROCESSING

We preprocess our dataset to handle its **complexity**:

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   - ***Kalman Filter***;
   - ***Savitzky–Golay (Savgol) Filter***.

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1. **Smoothing stage**:
   - ***Kalman Filter***;
   - ***Savitzky–Golay (Savgol) Filter***.

2. **Registration stage**:
   Extraction of the ***Karcher mean***, i.e. the "average curve" to which other curves are aligned at patient level.

THE MODEL

Scalar-on-Function Regression Model

$$Y_i = \mu + \int_T x_i(t)\,\beta(t)\,dx + \varepsilon_i\,, \quad i = 1, \ldots, n, \quad t \in T$$

where:

- Y_i is the scalar response;
- $\mu \in \mathbb{R}$;
- $T \subset \mathbb{R}^d$ is a product of compact intervals of $\mathbb{R}$;
- $x_i(t) \in L^1(T)$ is the functional predictor;
- β is a separable functional coefficient on T;
- $\varepsilon_1, \ldots, \varepsilon_n$ are independent random variables with a $\mathcal{N}(0, \sigma^2)$ distribution.

STARTING POINT

First step: develop and test the model on **synthetic data**, generated as follows:

- K Gaussian kernel;
- $\boldsymbol{\mu} = \mathbf{0}$;
- $\boldsymbol{x_1}, \dots, \boldsymbol{x_n} \sim GP(\boldsymbol{\mu}, K)$;
- σ^2 is known;
- $\beta(t) \approx \beta_{true}(t) = \sin\left(\frac{\pi}{4} t\right), \;\; t \in T$.

UNCONSTRAINED MODEL

Unconstrained Model

$$
\begin{aligned}
Y_i \mid \beta &\overset{ind}{\sim} \mathcal{N}\left(\mu + \int_T x_i(t)\,\beta(t)\,dx,\ \sigma^2\right) \\
\beta &\sim \mathrm{GP}\,(0, K)
\end{aligned}
$$

where:

- $\sigma^2 \in \mathbb{R}^+$;
- K is the continuous covariance function with a Gaussian kernel.

UNCONSTRAINED MODEL

Given the ***Likelihood*** and the ***Prior*** described above, the **Posterior** and the **Marginal distribution of Y** are:

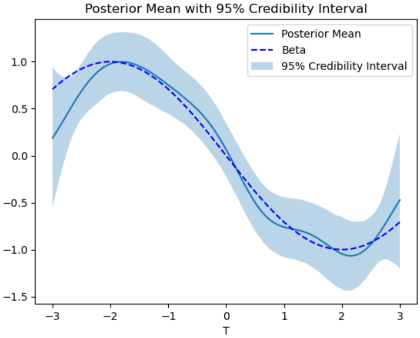
$$\beta \,|\, Y \;\sim\; \mathrm{GP}(m, K^*)$$
$$Y \;\sim\; \mathcal{N}_n(0, \Sigma + \sigma^2 \mathbf{I}_n)$$

where:

- $m(t) = Lx(t)' \left(\Sigma + \sigma^2 \mathbf{I}_n\right)^{-1} Y$;
- $K^*(s,t) = K(s,t) - Lx(s)' (\Sigma + \sigma^2 \mathbf{I}_n)^{-1} Lx(t)$;
- $\Sigma \in \mathbb{R}^{n \times n}$.

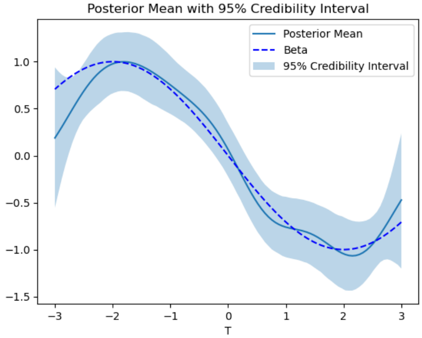
Additional details here

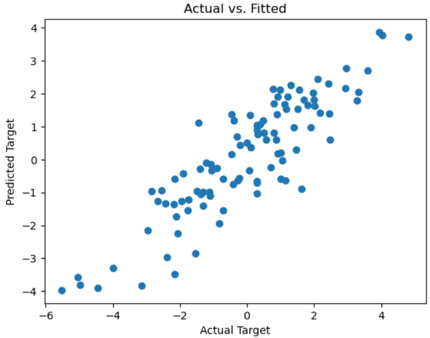
UNCONSTRAINED MODEL



Results obtained using ***synthetic*** data.

UNCONSTRAINED MODEL





Results obtained using ***synthetic*** data.

UNCONSTRAINED MODEL

Approaching the Constrained Model

The ***functional response*** assumes ***null value in certain time intervals***.

Let $T_0 \subset T$ be the union of such intervals, we constrain the ***functional coefficient*** β_t to be ***zero*** in T_0, namely:

$$\beta(t) = \beta_t \approx 0 \, \forall \, t \in T_0.$$

CONSTRAINED MODEL

RKHS in a nutshell

Let X be an arbitrary set and H a Hilbert space of real-valued functions on X. H is called a ***Reproducing Kernel Hilbert Space*** (RKHS) with kernel K if:

$$\forall f \in H \quad \forall x \in X \quad \langle f(\cdot)\,,\, K(x,\cdot)\rangle = f(x)$$

(reproducing property)

CONSTRAINED MODEL

- Denote H the RKHS with kernel K, define $H^0 := \{f \in H\colon f\,|_{T_0} = 0\}$ and let $\boldsymbol{P}$ be the ***orthogonal projection*** of H onto H^0.

CONSTRAINED MODEL

- Denote H the RKHS with kernel K, define $H^0 := \{f \in H\colon f\,|_{T_0} = 0\}$ and let $\boldsymbol{P}$ be the ***orthogonal projection*** of H onto H^0.
- H can be identified with an Hilbert space $\mathcal{H}$ of real-valued functions on T such that the ***map*** $\boldsymbol{\Theta}$ is a ***linear isometry***:

$$\begin{aligned} \Theta\colon \mathcal{H} &\to H \\ g(t) &\mapsto \Theta(g)(t) := \mathbb{E}[g\beta_t] \end{aligned}$$

CONSTRAINED MODEL

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$$\begin{aligned} \Theta \colon \mathcal{H} &\to H \\ g(t) &\mapsto \Theta(g)(t) := \mathbb{E}[g\beta_t] \end{aligned}$$

- Hence $\boldsymbol{\beta_t^0}$ is the ***projection*** in $\mathcal{H}^{\mathbf{0}} = \Theta^{-1}(H^0)$ of $\beta_t \in \mathcal{H}$.
Namely,

$$\beta_t = \beta_t^0 + \beta_t^1 \quad a.s. \quad \text{with} \quad \beta_t^0 \in \mathcal{H}^0, \beta_t^1 \in \mathcal{H}^1 := (\mathcal{H}^0)^\perp$$

CONSTRAINED MODEL

Constrained Model

The model is:

$$
\begin{aligned}
Y_i \,|\, \beta^0,\ \mu,\ \sigma^2 &\overset{ind}{\sim} \mathcal{N}\left(\int_T \mu + x_i(t)\beta^0(t)\,dt, \sigma^2\right) \\
\beta^0 | \, \sigma^2, T_0 &\sim GP(0, \sigma^2 K^0) \\
p(\mu, \sigma^2, T_0) &\propto \frac{1}{\sigma^2} p(T_0)
\end{aligned}
$$

CONSTRAINED MODEL

Constrained Model

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$$\beta^0 |\, \sigma^2, T_0 \sim GP(0, \sigma^2 K^0)$$
$$p(\mu, \sigma^2, T_0) \propto \frac{1}{\sigma^2} p(T_0)$$

The **Prior for T_0** is:

$$p(T_0) := e^{-\alpha r(T_0)}, \quad \alpha \in \mathbb{R}^+$$

CONSTRAINED MODEL

Posterior of β and conditional of Y

$$\beta^0 \,|\, Y, \mu, \sigma^2, T_0 \sim \mathrm{GP}(m, K^*)$$
$$Y \,|\, \mu, \sigma^2, T_0 \sim \mathcal{N}_n\left(\mu \mathbf{1}_n,\, \sigma^2 M\right)$$

where:

- $m(t) = L^0 x(t)' \, M^{-1} \, (Y - \mu \mathbf{1}_n)$;
- $K^*(s,t) = \sigma^2 \left[K^0(s,t) - L^0 x(s)' \, M^{-1} L^0 x(t) \right]$;
- $M = \Sigma^0 + \mathbf{I}_n$;
- $Y = (Y_1, ..., Y_n)$.

Additional details here.

CONSTRAINED MODEL

Metropolis Hastings

- T is a ***discrete grid*** of $\boldsymbol{g}$ time instants.

CONSTRAINED MODEL

Metropolis Hastings

- T is a ***discrete grid*** of $\boldsymbol{g}$ time instants.
- T_0 is a ***vector of entries*** $\{0,1\}^g$ such that $T_0[i] = 1$ if $T[i] \in T_0$.

CONSTRAINED MODEL

Metropolis Hastings

- T is a ***discrete grid*** of $\boldsymbol{g}$ time instants.
- T_0 is a ***vector of entries*** $\{0,1\}^g$ such that $T_0[i] = 1$ if $T[i] \in T_0$.

➥ **Algorithm**:
- Set $\boldsymbol{g}$ and α and compute $\mathbf{k}$;
- Initialize the chain with $\mathbf{T_0^{(0)}}$ a random combination of $\boldsymbol{g}$ zeros and ones with exactly $\mathbf{k}$ runs;
- While $i < 5.000$: generate a candidate $\mathbf{T_0^{new}}$ for $T_0^{(i+1)}$, by randomly choosing if:
  - → ***removing/adding*** a 1 in a run's end;
  - → ***splitting*** a run of ones by setting to 0 a random internal point.

  Accept it with rate:

$$\text{rate} = \frac{p(Y|\, T_0^{new})p(T_0^{new})}{p(Y|\, T_0^{(i)})p(T_0^{(i)})}$$

CONSTRAINED MODEL

Metropolis Hastings - customized

- T is a ***discrete grid*** of **g** time instants;
- T_0 is a ***vector of entries*** $\{0,1\}^g$ such that $T_0[i] = 1$ if $T[i] \in T_0$.

➥ **Algorithm**:

- Set $\boldsymbol{g}$ and α and compute $\mathbf{k}$;
- Initialize the chain with $\mathbf{T_0^{(0)}} = (0, 1, 0, 1, ..., 0, 1)$;
- While $i < 5.000$: generate a candidate $\mathbf{T_0^{new}}$ by randomly choosing if:
  - → ***removing/adding*** a 1 in a run's end;
  - → ***splitting*** a run of ones by setting to 0 a random internal point.

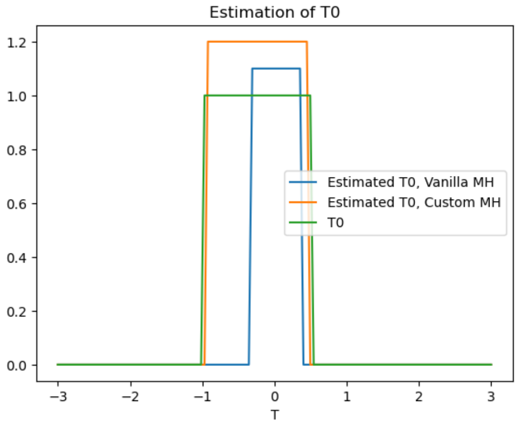
  Accept it if:

$$\text{rate} = \frac{p(Y|T_0^{new})p(T_0^{new})}{p(Y|T_0^{(i)})p(T_0^{(i)})} > \gamma^{(i)}, \quad \gamma^{(i)} = 1.0 - 0.02 \cdot \frac{(s-A)}{s}$$

where A is the number of acceptances in the last s steps.

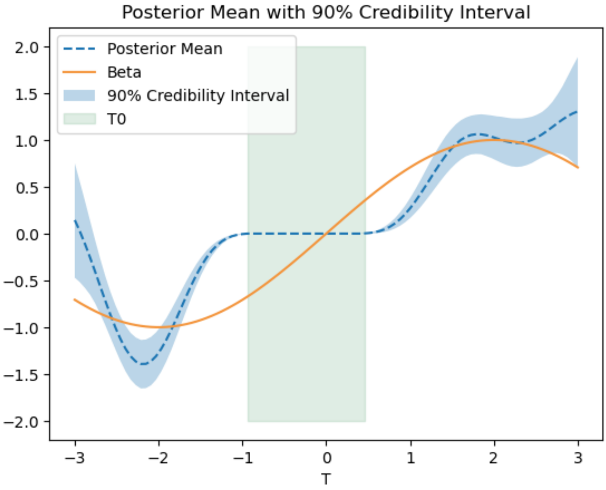
CONSTRAINED MODEL

Efficiency of the ***Customized MH*** (500 iterations; burn-in 100) versus the ***RWMH*** (2000 iterations; burn-in 1000).



Results obtained using ***synthetic*** data.

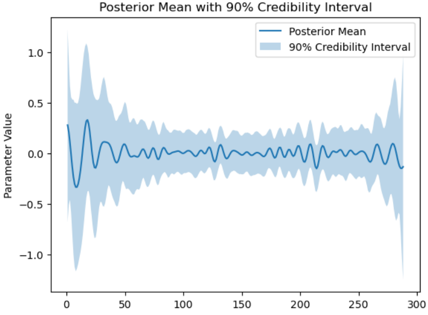
CONSTRAINED MODEL



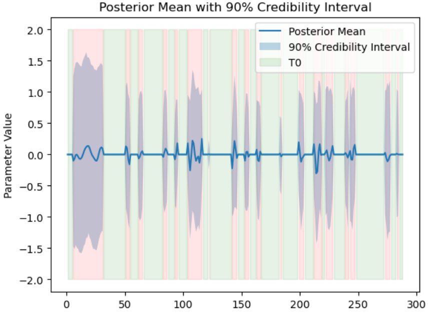
Results obtained using ***synthetic*** data.

APPLICATION ON REAL-WORLD DATA

Unconstrained Model



Constrained Model

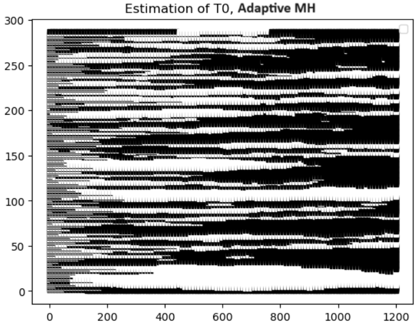


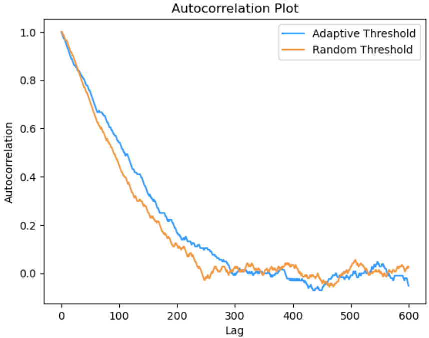
Results obtained using ***preprocessed real-world*** data.

- Y_i : Parkinson's severity for patient i;
- To classify patients, we use a **logit transformation**;
- $x_i(t)$: tremor probability (Z-score) for patient i;
- $i \in \{1, ..., 159\}$, $t \in T$.

APPLICATION ON REAL-WORLD DATA

Convergence diagnostics





Results obtained using ***preprocessed real-world*** data.

MIXED EFFECT MODEL

Mixed model with random effect on Gender

$$Y_{ij} = \int_T x_{ij}(t)\beta(t)\,dt + z_{ij}\theta_j + \theta_0 + \varepsilon_i, \quad i = 1, \ldots, 159,\ t \in T$$

$$\beta \mid \mu, \sigma^2, T_0 \sim GP(0, \sigma^2 K^0)$$

$$\theta_j \mid \mu_s, \sigma_s^2 \sim \mathcal{N}(\mu_s, \sigma_s^2)$$

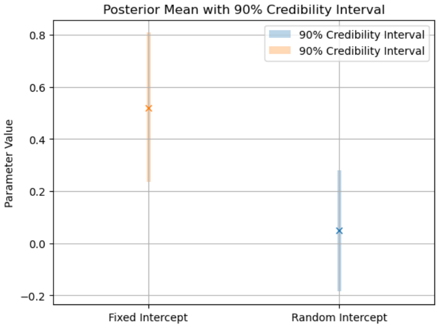
$$\theta_0 \mid \mu_0, \sigma_0^2 \sim \mathcal{N}(\mu_0, \sigma_0^2)$$

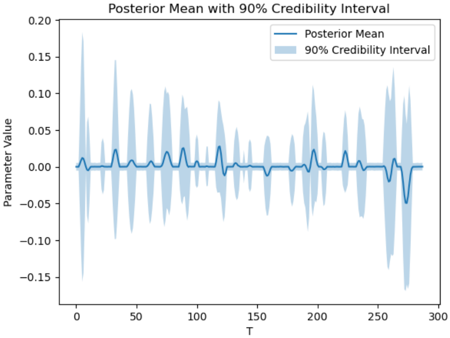
$$\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

where:

- $x_{ij}(t)$: tremor probability for the i^{th} patient, belonging to group j;
- Y_{ij} : severity of the i^{th} patient, belonging to group j;
- z_{ij}: grouping variable for groups $\{$men, women$\}$.

MIXED EFFECT MODEL





Results obtained using ***preprocessed real-world*** data.

MIXED EFFECT MODEL

Mixed model with random effect on Levodopa treatments

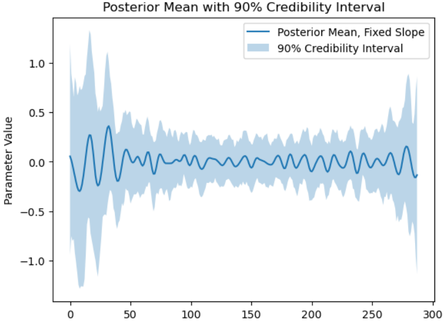
$$Y_{ij} = \int_T x_{ij}(t)\theta(t)\,dt + \int_T x_{ij}(t)\gamma_j(t)\,dt + \varepsilon_i, \qquad i = 1, \ldots, 136, t \in T$$

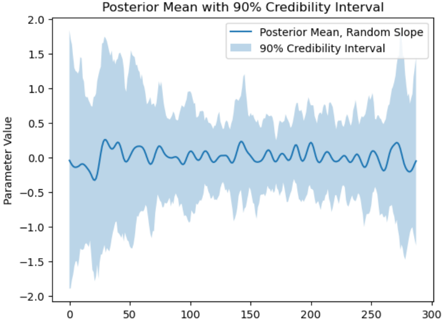
$$\begin{aligned} \theta \,|\, \sigma^2, T_0 &\sim GP(0, \sigma^2 K^0) \\ \gamma_j | \sigma_0^2 &\sim GP(0, \sigma_0^2 K^0), \\ \varepsilon_i &\sim \mathcal{N}(0, \sigma^2) \end{aligned}$$

where:

- $x_{ij}(t)$: tremor probability for the i^{th} patient, belonging to group j;
- Y_{ij}: severity of the i^{th} patient, belonging to group j;
- j represents the grouping induced by different numbers of doses received.

MIXED EFFECT MODEL





Results obtained using ***preprocessed real-world*** data.

CONCLUSIONS

Tables of **results**:

Models with Analytical Posterior	Accuracy	LPML
Unconstrained	62.3%	-206.61
Constrained	70.4%	-223.37
Constrained, Customized MH	70.0%	-218.18

Models with No Analytical Posterior	Accuracy	LPML
Unconstrained, Random Slope, 1 Group	62.5%	-188.82
Unconstrained, Random Slope, 2 Groups	65.4%	-204.68
Constrained, Random Intercept	67.3%	-167.08

CONCLUSIONS

Possible **extensions** of the models:

- ***Multiple*** Scalar-on-Function regression;
- Incorporation of ***more complex*** hierarchical structures;
- Use of ***Metropolis-within-Gibbs*** to update the estimate of parameters in Mixed Effect model;
- ***Further enhancement*** of estimation methods for T_0.

CONCLUSIONS

Overall, the **results** presented are ***satisfactory*** in terms of:

- Accuracy;
- Information obtained.

Drawbacks:
High computational costs impose many ***limits*** to our research.

ACKNOWLEDGMENTS and QUESTIONS

Thank you all for having attended this presentation.

We appreciate your attention and are available
for any ***questions*** you may have.

REFERENCES

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J.Q. Shi, B. Wang, ***Curve prediction and clustering with mixtures of Gaussian process functional regression models***, Springer Science, 2008.

DETAILS

- $m(t) = Lx(t)' \, (\Sigma + \sigma^2 \mathbf{I}_n)^{-1} \, Y;$
- $K^*(s,t) = K(s,t) - Lx(s)' \, (\Sigma + \sigma^2 \mathbf{I}_n)^{-1} \, Lx(t).$

where:

- $Lx_i(t) = \int_T K(t,s) \, x_i(s) \, ds \qquad i = 1, \ldots, n, \qquad t \in T \, ;$
- $Y = (Y_1, \ldots, Y_n)';$
- $\Sigma_{ij} = R(x_i, x_j) \qquad i, j = 1, \ldots, n \, ;$
- $R(x_i, x_j) = \int_T \int_T K(s,t) \, x_i(s) \, x_j(t) \, ds \, dt \qquad i, j = 1, \ldots, n \qquad t \in T \, .$

Main slide here

DETAILS

- $m(t) = L^0 x(t)' \, M^{-1} \, (Y - \mu \mathbf{1}_n)$;
- $K^*(s,t) = \sigma^2 \left[K^0(s,t) - L^0 x(s)' \, M^{-1} L^0 x(t) \right]$.

where:

- $L^0 x_i(t) = \int_T K^0(t,s) \, x_i(s) \, ds, \quad i = 1, \ldots, n, \quad t \in T$;
- $Y = (Y_1, \ldots, Y_n)'$;
- $M = \Sigma^0 + \mathbf{I}_n$;
- $\Sigma^0_{ij} = \int_T \int_T K^0(s,t) x_i(s) x_j(t) ds dt, \;\; i,j = 1, \ldots, n, \quad t \in T$.

Main slide here

DETAILS

The idea is to find β_t^0 as the projection in $\mathcal{H}^0$ of $\beta_t \in \mathcal{H}$. Thanks to the isometry Θ, the following commutative diagram is well-defined:

$$\begin{array}{ccc} \mathcal{H} & \xrightarrow{\Theta} & H \\ \downarrow{\scriptstyle \mathcal{P}} & & \downarrow{\scriptstyle P} \\ \mathcal{H}^0 & \xrightarrow{\Theta} & H^0 \end{array}$$

Then we can decompose the random process β_t:

$$\beta_t = \mathcal{P}\beta_t + \mathcal{Q}\,\beta_t = \beta_t^0 + \beta_t^1 \quad a.s.$$

Finally, for how we define H^0 and as a consequence of the reproducity property:

$$\forall f \in H^0, \forall \tau \in T^0 \quad f(\tau) = 0 \iff \langle f(\tau), K_\tau(s)\rangle = 0$$

allowing to deduce:

$$H^1 = \overline{span\{K_\tau : \tau \in T_0\}}^H.$$

DETAILS

Let $\mathcal{S}_k = \{T_0 \in T, r(T_0) = k\}$ and c_k its cardinality, we have:

$$\mathbb{P}(r(T_0) = k) = \sum_{T_0 \in \mathcal{S}_k} p(T_0) = \frac{c_k e^{-\alpha k}}{\sum_{i=1}^{g} c_i e^{-\alpha i}} \tag{1}$$

As initial value of $r(T_0)$ we use the expected number of runs, namely we fix α and g and take the k that maximize (1).