18 November, 2021

1 Homework #1, AA2021-2022: MIMO System Identification and Deconvolution

Multiple Input Multiple Output (MIMO) are multichannel systems that model various applications. The Homework's objective is to get practice and evaluate the MSE in MIMO estimation and deconvolution by using Matlab. The general $M \times N$ MIMO system is described as a set of MN filters that combine the input signals $x_1[k], \ldots, x_N[k]$ onto the ith signal as output

$$y_i[k] = \sum_{\ell=0}^{N} h_{i,\ell}[k] * x_\ell[k] + w_i[k],$$

this can be represented as

$$\begin{bmatrix} y_1[k] \\ \vdots \\ y_M[k] \end{bmatrix} = \begin{bmatrix} h_{11}[k] & \ldots & h_{1N}[k] \\ \vdots & \ddots & \vdots \\ h_{M1}[k] & \ldots & h_{MN}[k] \end{bmatrix} * \begin{bmatrix} x_1[k] \\ \vdots \\ x_N[k] \end{bmatrix} + \begin{bmatrix} w_1[k] \\ \vdots \\ w_M[k] \end{bmatrix}$$

or more compactly for the kth sample

$$\mathbf{y}[k] = \mathbf{H}[k] * \mathbf{x}[k] + \mathbf{w}[k]$$

where $\mathbf{H}[0], \ldots, \mathbf{H}[K-1]$ is the length-K MIMO filter response.

The MIMO system considered in this homework exemplifies a digital communication system. The input samples $\mathbf{x}[0], ..., \mathbf{x}[P-1]$ are of length P, where the first Q samples are called pilot or training samples, and they are known to both the transmitter and the receiver sides. The remaining $P-Q$ samples represent the unknown transmitted information. In this context, the student is requested to

(a) estimate the MIMO channel $\hat{\mathbf{H}}[k]$ using the known Q pilots;

(b) estimate the unknown signal $\mathbf{x}[Q+1], ..., \mathbf{x}[P]$ through deconvolution using the previously estimated MIMO channel $\hat{\mathbf{H}}[k]$

The MIMO filter response is modelled as

$$h_{i,j}[k] = \alpha^{|i-j|} \times \beta^k \quad \text{for} \quad k = 0, 1, 2, ...$$

where $0 \leq \alpha \leq 1$ is the coupling effect among MIMO lines ($\alpha \approx 0$ denotes weak coupling, and $\alpha \approx 1$ denotes strong coupling), and the second term account for the time decaying. The noise is Gaussian, temporally white, i.e., $\mathbb{E}[\mathbf{w}[k]\mathbf{w}[\ell]] = \mathbf{C}\delta[\ell - k]$, and mutually correlated among paired lines, i.e.,

$$\mathbf{w}[k] \sim \mathcal{N}(\mathbf{0}, \mathbf{C}) \text{ where } [\mathbf{C}]_{i,j} = \begin{cases} \rho & i \neq j \\ 1 & i = j \end{cases}$$

where $|\rho| \leq 1$ is the correlation coefficient. The input signal $\mathbf{x}[k]$ contains both pilot and information samples, and can be generated from uncorrelated and white random processes:

$$\mathbf{x}[k] \sim \mathcal{N}(\mathbf{0}, \sigma_x^2 \mathbf{I}).$$

1.1 Noise generation

Generate using Matlab code a correlated noise for any arbitrary choice of ρ and compute the sample covariance $\hat{\mathbf{C}}$ from a set of L samples $\{\mathbf{w}[k]\}_{k=1}^{L}$ and compare with the true covariance $\mathbf{C}$ for $\texttt{L} = \texttt{linespace}(10, 200, 10)$ and $\rho = \texttt{linespace}(0, 0.99, 10)$ to evaluate how close the sample covariance $\hat{\mathbf{C}}$ is to $\mathbf{C}$ after defining a proper metric that is represented graphically vs ρ for the values of L.

1.2 MIMO estimation

Let us consider an $M = N = 4$ MIMO filter response. Estimate the 4×4 MIMO filter response under the following settings:

(i) consider a memoryless filter, i.e., $K = 1$. After defining the MLE of the MIMO filter and the corresponding filter estimate $\hat{\mathbf{H}}$, evaluate numerically and analytically the MSE of the MIMO filter vs the signal to noise ration (SNR) defined as $SNR = \sigma_x^2/\sigma_w^2 = \sigma_x^2$, and plot (in dB-dB[1] scale) the MSE vs $\mathtt{SNR} = -\mathtt{10} : \mathtt{2} : \mathtt{30}$ dB for $\mathtt{Q} = \mathtt{linespace}(\mathtt{1}, \mathtt{50}, \mathtt{5})$ and $\alpha = \mathtt{linespace}(\mathtt{0}, .\mathtt{99}, \mathtt{5})$. Compare all results with the CRB.

(ii) consider a memory filter, i.e., $K = 4$. Repeat the same exercise above in point (i) for memory causal filters with different coefficients $\beta^{\mathbf{k}} = \{\mathtt{0.9}^{\mathbf{k}}, \mathtt{0.5}^{\mathbf{k}}, \mathtt{0.1}^{\mathbf{k}}\}$ for $\mathbf{k} = [0, 1, 2, 3]^T$. After defining the problem and its Matlab solution, the student must include in the final Hw-Report a clear definition of the theoretical structure used to solve the problem.
Hint: it is possible to reduce the above problem to a simple linear algebraic filtering by a careful design of convolution-matrices.

1.3 MIMO deconvolution

Let us consider a MIMO filter with no-memory (i.e., $h_{i,j} = \alpha^{|i-j|}$) as in the exercise in Sec.1.2(i), with an input signal $\mathbf{x}[k]$ of length P. In a communication system, the first Q samples are known to both the transmitter and receiver and are used to estimate the MIMO filter response, while the remaining are unknown to the receiver and represent the transmitted information. For a fixed P, the choice of the number Q is important and must be selected carefully as tradeoff between accurate MIMO filter response estimation (higher values of Q) and low overhead (lower values of Q). Therefore, the goal is to estimate the MIMO filter $\hat{\mathbf{H}}$ using the first Q pilot samples $\mathbf{x}[k]$ and use it to estimate the unknown information samples $\hat{\mathbf{x}}[k]$ from $\mathbf{y}[k]$, with $k = \{Q+1, Q+2, ..., P\}$ using both MLE and MMSE criteria. Evaluate numerically and analytically the MSE vs $SNR = \sigma_x^2$ and plot (in dB-dB scale) the MSE vs $\mathtt{SNR} = -\mathtt{10} : \mathtt{2} : \mathtt{30}$ for $\mathtt{P} = \mathtt{200}$, $\mathtt{Q} = \mathtt{linespace}(\mathtt{1}, \mathtt{P}, \mathtt{20})$, $\rho = .1$, and $\alpha = .5$. Define a proper metric that capture the tradeoff in the selection of the parameter Q and represent it graphically vs SNR, and discuss the results.

[1] The decibel scale is a logarithmic scale that can better capture exponentially growing/decreasing values: $P_{dB} = 10\log(P_{linear})$. To obtain a dB-dB scale plot on Matlab, you can use the function `plot(10log10(SNR), 10log10(MSE))`