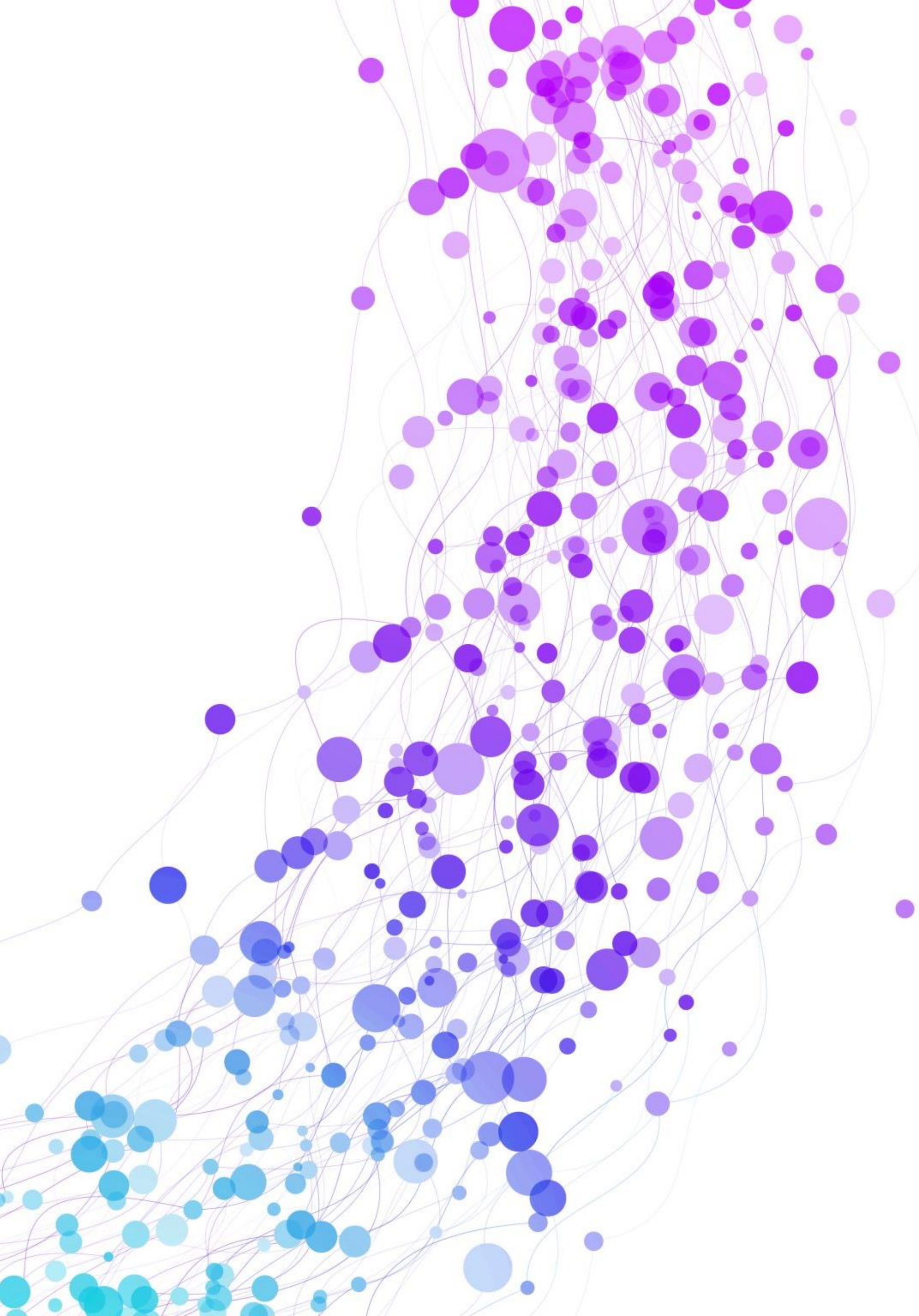


SOLVING BAYESIAN INVERSE PROBLEM VIA VARIATIONAL AUTOENCODER

Gabriele Bruni
Riccardo Rota
Nicola Zanetti

INVERSE PROBLEM

- **Classical inverse problem**:

$$\boldsymbol{y} = F(\boldsymbol{u}) + \boldsymbol{e}$$

$\boldsymbol{y}$ observed variable;

$\boldsymbol{u}$ parameter of interest (PoI);

$\boldsymbol{e}$ error;

F parameter to observable map (PtO).

Goal: to determine the PoI $\boldsymbol{u}$ given the observed data $\boldsymbol{y}$.

- **Bayesian inverse problem**:

$$p(\boldsymbol{U}|\boldsymbol{Y}) \propto p(\boldsymbol{Y}|\boldsymbol{U}) * p(\boldsymbol{U})$$

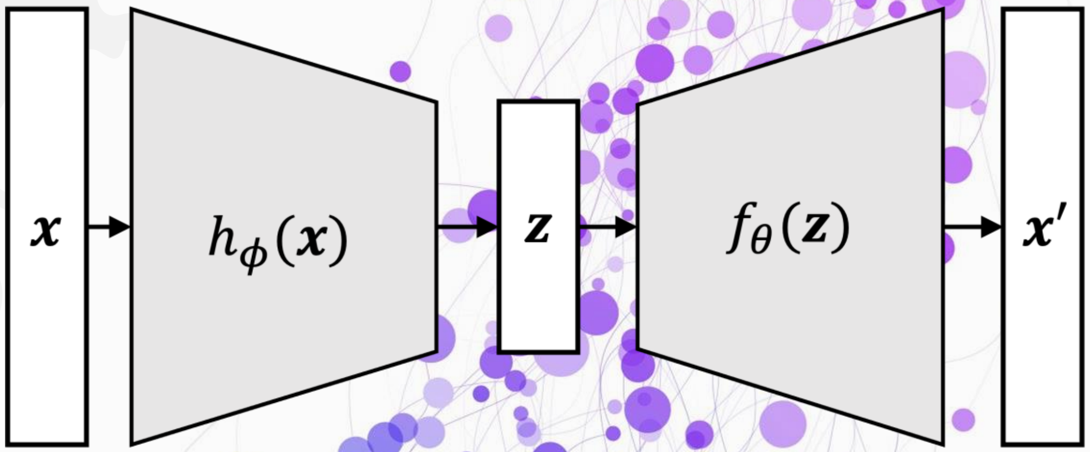
$\boldsymbol{Y}$ and $\boldsymbol{U}$ are random variables representing the observed variable and the parameter of interest.

Goal: to determine the distribution of $\boldsymbol{U}|\boldsymbol{Y}$ given the distribution of $\boldsymbol{Y}$.

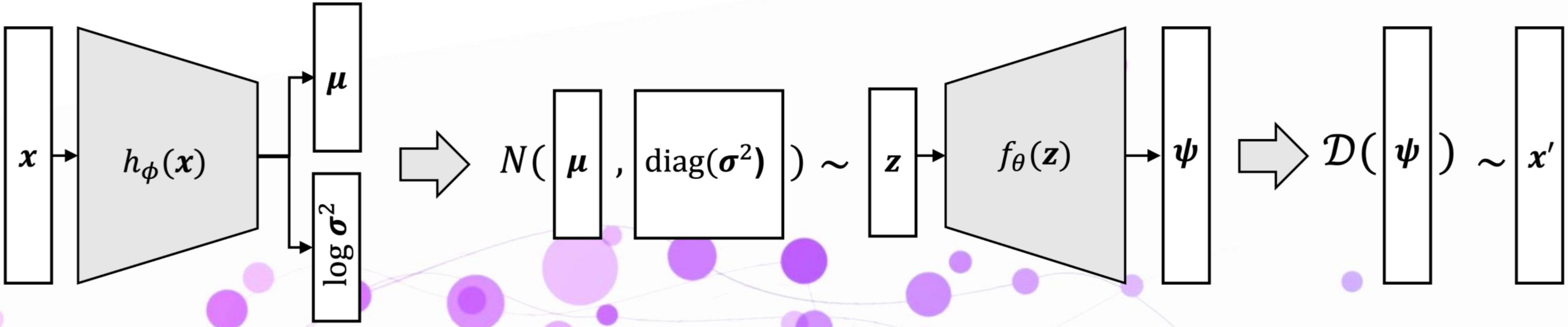
AUTOENCODER

Key Idea: to obtain a lower-dimensional representation of the input space.

- **Encoder**: a neural network h with parameter $\boldsymbol{\Phi}$ that takes a vector, $\boldsymbol{x}$, compress it into a smaller vector, $\boldsymbol{z}$.
- **Decoder**: f with parameters $\boldsymbol{\theta}$ that takes $\boldsymbol{z}$ and decompress it back into a vector $\boldsymbol{x}'$.
- $\boldsymbol{z}$ is called **latent variable** and it is a lower-dimensional representation of the data.



VARIATIONAL AUTOENCODER



The encoder takes $\boldsymbol{x}$ and generates the parameters of the latent distribution $\boldsymbol{z}$.

$\boldsymbol{z}$ is sampled from the distribution with the parameters given by the encoder.

The decoder maps $\boldsymbol{z}$ into the parameters of the distrbution used to reconstruct the original input.

$\boldsymbol{x}'$ is sampled from the distribution with the parameters given by the decoder.

VAE IN BAYESIAN INVERSE PROBLEMS

Key Idea: one-to-one correspondence between the frameworks of **inverse problem** and **VAE**.

- The observation $\boldsymbol{y}$ corresponds to the input $\boldsymbol{x}$ of the encoder.
- The latent variable $\boldsymbol{z}$ corresponds to the PoI $\boldsymbol{u}$.

In the **Bayesian framework**:

- $p(\boldsymbol{U}|\boldsymbol{Y})$ is given by the encoder: from $\boldsymbol{Y}$ we estimate the parameters of the distribution from which we sample $\boldsymbol{U}$.
- $p(\boldsymbol{Y}|\boldsymbol{U})$ is given by the decoder: from $\boldsymbol{U}$ we compute the parameters to reconstruct the distribution of $\boldsymbol{Y}$.

OUR APPLICATION

$$\begin{cases} -\nabla \cdot \left(q(\boldsymbol{x})\nabla y(\boldsymbol{x})\right) = f(\boldsymbol{x}) & in\ \Omega = [0,1]^2 \\ y(\boldsymbol{x}) = g(\boldsymbol{x}) & on\ \partial\Omega \end{cases}$$

- Steady state heat equation with non-constant diffusion coefficient $q(\boldsymbol{x})$.
- Observed variable: solution $y(\boldsymbol{x})$.
- We generate $y(\boldsymbol{x})$ using a Finite Element Method starting from a given prior distribution of $q(\boldsymbol{x})$.
- **Goal**: to estimate the posterior distribution of $q(\boldsymbol{x})$ given $y(\boldsymbol{x})$.
- **Result**: we will be able to make prediction on $q(\boldsymbol{x})$ from the observation of $y(\boldsymbol{x})$, and to estimate the accuracy of such prediction.

THANK YOU!

REFERENCES:

- Solving Bayesian Inverse Problems via Variational Autoencoders, H. Goh, S. Sheriffdeen, J. Wittmer, T. Bui-Thanh

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