HYPRE PCG TEST PROBLEM

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Let $A = -\Delta + aI$ on a discrete 3-D periodic domain. Specifically, if u is a discrete function define

$$\begin{aligned}(Au)(j_1, j_2, j_3) = au(j_1, j_2, j_3) + 6u(j_1, j_2, j_3) - \\ u(j_1 - 1, j_2, j_3) - u(j_1 + 1, j_2, j_3) - u(j_1, j_2 - 1, j_3) - \\ u(j_1, j_2 + 1, j_3) - u(j_1, j_2, j_3 - 1) - u(j_1, j_2, j_3 + 1),\end{aligned} \quad (1)$$

where $0 \leq j_i < N_i$, for $i = 1, 2, 3$, and index arithmetic is done modulo N_1, N_2 and N_3, respectively.

Let $u_{n_1 n_2 n_3}$, with $0 \leq n_i < N_i$, $i = 1, 2, 3$, denote the eigenfunctions of A:

$$u_{n_1 n_2 n_3}(j_1, j_2, j_3) = \exp\bigl(2\pi i(j_1 n_1/N_1 + j_2 n_2/N_2 + j_3 n_3/N_3)\bigr). \quad (2)$$

Note the the functions exhibit required periodicity when allowing the indices j_i to range over all integers. Then

$$Au_{n_1 n_2 n_3} = \lambda_{n_1 n_2 n_3} u_{n_1 n_2 n_3} \quad (3)$$

with

$$\lambda_{n_1 n_2 n_3} = a + 4\bigl(\sin^2(\pi n_1/N_1) + \sin^2(\pi n_2/N_2) + \sin^2(\pi n_3/N_3)\bigr) \quad (4)$$

The condition number of A is

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \frac{12 + a}{a}, \quad (5)$$

which can be made arbitrarily large taking $a > 0$ small. Note that A is singular when $a = 0$ with nullspace constant functions.

For the test of the Hypre PCG solver consider the problem

$$Ax = \lambda_{n_1 n_2 n_3} u_{n_1 n_2 n_3}, \quad (6)$$

which has the solution $u_{n_1 n_2 n_3}$.