Mate 1: Curs #1

Profesor: Radu Gologan

1 Octombrie 2019

1 Multimi

Definim o **multime** A formata din elemente distincte ce au o proprietate prin notatia: $A = \{x \,|\, x \text{ } are \text{ } proprietatea \text{ } P\}$
Definim relatia de **incluziune**: $\subseteq$ pe doua multimi A si B astfel:
$A \subseteq B \iff (\forall a \in A \implies a \in B)$
Definim urmatoarele operatii:

- **reuniune**: $A \cup B = \{x \,|\, x \in A \ sau \ x \in B\}$
- **intersectie**: $A \cap B = \{x \,|\, x \in A \ si \ x \in B\}$
- **diferenta**: $A \backslash B = \{x \,|\, x \in A \ si \ x \notin B\}$
- **complementara**: $B \subseteq A \iff C_A^B = A \backslash B$

Definim **produs cartezian** intre doua multimi A si B astfel:
$A \times B = \{(a, b) \,|\, a \in A \ si \ b \in B\}$

Fie I o familie de indici(multime) si $A_i, \ i \in I$ o familie de multimi indexate de I. Definim urmatoarele operatii:

- **Reuniunea tuturor multimilor**: $\bigcup\limits_{i \in I} A_i = \{a \,|\, \exists i \in I, a \in A_i\}$
- **Intersectia tuturor multimilor**: $\bigcap\limits_{i \in I} A_i = \{a \,|\, \forall i \in I, a \in A_i\}$

Exemplu: $\bigcup\limits_{x \in \mathbb{R}} \{x\} = \mathbb{R}$

2 Relatii

Definim o **relatie** pe o multime X o submultime $R \subseteq X \times X$:
$R = \{(x, y) \,|\, x \in X \ si \ y \in X\}$
Vom nota $(x, y) \in R \iff xRy$
Definim o **relatie de echivalenta** pe X, o relatie R cu urmatoarele proprietati:

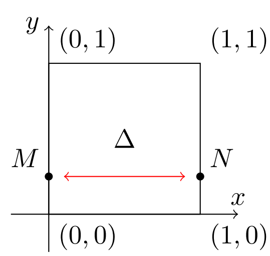
- **reflexivate**: $xRx, \forall x \in X$
- **simetrie**: $xRy \implies yRx,\ \forall (x,y) \in R$
- **tranzitivitate**: $xRy\ si\ yRz \implies xRz,\ \forall (x,y),(y,z) \in R$

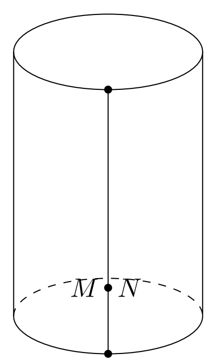
Exemple:

- Relatia de egalitate
- Relatia de echivalenta a figurilor geometrice
- Fie $n \in \mathbb{N}^*$, introducem pe $\mathbb{Z}$ relatia:
  $m, p \in \mathbb{Z},\ m \equiv p(mod\ n) \iff n|(m-p)$
  Pentru $i = \overline{0, n-1}$, definim **clasa de resturi a lui** i: $\hat{i} = \{i + n \cdot k \mid k \in \mathbb{Z}\}$
  si **multimea claselor de resturi modulo** n: $\mathbb{Z}_n = \{\hat{0}, \hat{1}, ..., \widehat{n-1}\}$

In general, pentru (X, R) o relatie de echivalenta, vom nota:
$\forall x \in X, \hat{x} = \{y \mid yRx\} \implies X = \bigcup_{x \in X} \hat{x}\ si\ x \not{R} y \implies \hat{x} \cap \hat{y} = \varnothing$
vom nota **multimea claselor de echivalenta** sau **spatiul cat** $\{\hat{x}\}_{x \in X} = X/_R$.
Exemple:

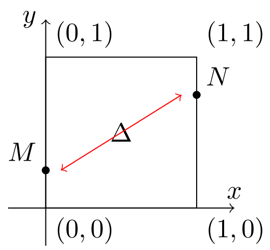
- Fie $\Delta = \{(x,y) \mid x, y \in [0,1]\}$ suprafata unui patrat si ρ o relatie de echivalenta astfel incat:
  Daca $M = (x_1, y_1)\ cu\ x_1 \in (0,1)\ si\ y_1 \in [0,1]\ si\ N = (x_2, y_2)\ cu\ x_2 \in (0,1)\ si\ y_2 \in [0,1] \implies M \rho N \iff M = N$
  Daca $M = (x_1, y_1)\ cu\ x_1 \in \{0,1\}\ si\ y_1 \in [0,1]\ si\ N = (x_2, y_2)\ cu\ x_2 \in \{0,1\}\ si\ y_2 \in [0,1] \implies M \rho N \iff y_1 = y_2$

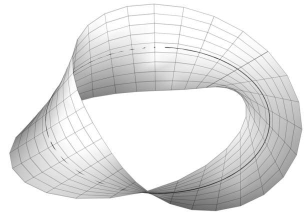
  

  Se observa din constructie ca $\Delta/_\rho =\ suprafata\ laterala\ a\ unui\ cilindru$

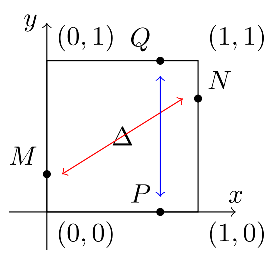
- Fie $\Delta = \{(x,y) \mid x, y \in [0,1]\}$ suprafata unui patrat si ρ o relatie de echivalenta astfel incat:
  Daca $M = (x_1, y_1)\ cu\ x_1 \in (0,1)\ si\ y_1 \in [0,1]\ si\ N = (x_2, y_2)\ cu\ x_2 \in (0,1)\ si\ y_2 \in [0,1] \implies M \rho N \iff M = N$
  Daca $M = (x_1, y_1)\ cu\ x_1 \in \{0,1\}\ si\ y_1 \in [0,1]\ si\ N = (x_2, y_2)\ cu\ x_2 \in \{0,1\}\ si\ y_2 \in [0,1] \implies M \rho N \iff y_1 = 1 - y_2$

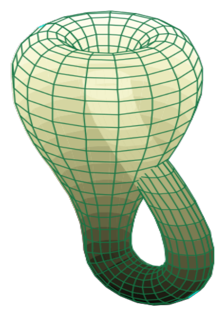




Se observa din constructie ca $\Delta/_{\rho} = \ suprafata\ definita\ de\ o\ banda\ Mobius$

- Fie $\Delta = \{(x, y) \,|\, x, y \in [0, 1]\}$ suprafata unui patrat si ρ o relatie de echivalenta astfel incat:
  Daca $M = (x_1, y_1)\ cu\ x_1 \in (0, 1)\ si\ y_1 \in (0, 1)\ si\ N = (x_2, y_2)\ cu\ x_2 \in (0, 1)\ si\ y_2 \in (0, 1) \implies M\rho N \iff M = N$
  Daca $M = (x_1, y_1)\ cu\ x_1 \in \{0, 1\}\ si\ y_1 \in [0, 1]\ si\ N = (x_2, y_2)\ cu\ x_2 \in \{0, 1\}\ si\ y_2 \in [0, 1] \implies M\rho N \iff y_1 = 1 - y_2$
  Daca $M = (x_1, y_1)\ cu\ x_1 \in [0, 1]\ si\ y_1 \in \{0, 1\}\ si\ N = (x_2, y_2)\ cu\ x_2 \in [0, 1]\ si\ y_2 \in \{0, 1\} \implies M\rho N \iff x_1 = x_2$





Se observa din constructie ca $\Delta/_{\rho} = \ suprafata\ definita\ de\ o\ sticla\ Klein$

3 Relatii de ordine

(X, R) o relatie pe X se numeste **relatie de ordine** daca si numai daca are urmatoarele proprietati:

- Reflexivitate
- **Antisimetrie:** $xRy\ si\ yRx \implies x = y$
- Tranzitivitate

O relatie de ordine se numeste **relatie de ordine totala** daca:
$\forall x, y \in X, xRy\ sau\ yRx$
O relatie de ordine se numeste **relatie de ordine partiala** daca nu este o relatie de ordine totala

Exemple:

- Notam $\mathscr{P}(X) = \{A \,|\, A \subseteq X\}, |X| = n \implies |\mathscr{P}(X)| = 2^n$ **multimea tuturor submultimilor lui X**
  $ARB \iff A \subseteq B$ este o relatie de ordine partiala.

De obicei, relatiile de ordine se noteaza $xRy \iff x \leqslant y$

Daca $(X, \leqslant)$ este o relatie de ordine si $A \subseteq X$, numim $M \in X$ **majorant pentru A** atunci cand $x \leqslant M,\ \forall x \in A$ si $m \in X$ **minorant pentru A** atunci cand $m \leqslant x,\ \forall x \in A$
Definim sup_A ca fiind cel mai mic majorant al multimii A, daca acesta exista, si inf_A ca fiind cel mai mare minorant al multimii A, daca acesta exista.
Explicitam:
$sup_A = S \in X \iff x \leqslant S\ \forall x \in A\ si\ daca\ \forall S' \in X\ a.i.\ x \leqslant S' \implies S \leqslant S'$.
Analog se defineste si inf_A.
In $\mathbb{R}$, $A \subseteq \mathbb{R}$, $sup_A = S \in \mathbb{R} \iff a \leqslant S, \forall a \in A\ si\ \forall \epsilon > 0, \exists a \in A\ a.i.\ S - \epsilon \leqslant a$.

Exercitiu:
Fie $(a_n)_n,\ (b_n)_n$ doua siruri marginite in $\mathbb{R}$. Vom demonstra ca $sup(a_n + b_n) \leqslant sup(a_n) + sub(b_n)$.

Demonstratie:
$a_i \leqslant sup(a_n), \forall i\ si\ b_i \leqslant sup(b_n), \forall i \implies a_i + b_i \leqslant sup(a_n) + sup(b_n), \forall i \implies sup(a_n + b_n) \leqslant sup(a_n) + sup(b_n)$