Filter of functors and model categories

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Contents

1 Introduction

1.1 Motivation

The filter quotient is a construction from topos theory. The idea is to quotient out the partial order of the subobjects of the terminal object by a filter, reducing the truth values in this topos [Joh02, Example 2.1.13]. One obtains a new topos, but because the quotiented category generally doesn't have infinite colimits anymore, it is not a Grothendieck topos. Nima Rasekh used the same construction, this time on an $(\infty, 1)$-topos, to obtain non-presentable examples [Ras21]. Recently, Michael Shulman proved that a particular kind of topos, called a type-theoretic model topos, can interpret Martin-Löf type theory with some additional structures [Shu19]. Using the filter quotient on type-theoretic model toposes, we hope that we could obtain interesting examples of non-presentable $(\infty, 1)$-toposes which could interpret Martin-Löf type theory and keep part of the additional structures. In this paper, we will do the first step of this work, and look at how the filter quotient interacts with homotopical structures, concentrating on model categories. We will generalize the filter quotient to obtain the notion of a filter of functors, which can be used in a greater variety of categories.

1.2 Main results

In the first section, we define the notion of a filter of functors, which is a generalization of the filter quotient. We obtain in the second section the following results of preservation of homotopical structures:

Theorem (Theorem 40)**.** *Let $\mathcal{C}$ be a category of fibrant objects and F a filter of functors such that the quotient functor $Q_F : \mathcal{C} \to \mathcal{C}_F$ preserves finite limits and such that fibrations are dense in F. Then, $\mathcal{C}_F$ is a category of fibrant objects.*

Corollary (Corollary 41)**.** *Let $\mathcal{C}$ be a category of fibrant objects and F a filter on the subobjects of the terminal object. Then, $\mathcal{C}_F$ is a category of fibrant objects.*

Theorem (Theorem 54)**.** *Let $\mathcal{C}$ be a model category, F an ascending well-levelled filter of functors such that for any $p \in P$ the trivial p-fibrations are genuine and the cofibrations are dense in F, and suppose that Q_F preserves finite colimits. Then, the weakly induced model structure exists on $\mathcal{C}_F$.*

Theorem (Theorem 61)**.** *Let $\mathcal{C}$ be a model category, F an ascending or descending filter of functors in which the model structure is dense and such that Q_F preserves finite limits and colimits. Then, the induced model structure on $\mathcal{C}_F$ exists.*

In the third section, we define the notion of a filter of functors compatible with simpliciality, and we prove the following result:

Theorem (Theorem 89, Theorem 94)**.** *Let $\mathcal{C}$ be a simplicial model category, and F an ascending or descending filter of functors compatible with simpliciality, such that Q_F preserves finite colimits and limits and such that the model structure is dense in F.*

Then, $\mathcal{C}_F$ is a simplicial model category. Moreover, if $\mathcal{C}$ has finite descent, $\mathcal{C}_F$ has finite descent.

In the fourth section, we explore a particular class of examples: truncations of diagrams on model categories. We define the ascending partial order $P(\mathcal{D})$ associated with a small category $\mathcal{D}$, and how filters on $P(\mathcal{D})$ induce filters on categories of the form $\mathcal{C}^{\mathcal{D}}$. We then obtain the following results:

Theorem (Theorem 103)**.** *Let $\mathcal{C}$ be a model category such that for any small category $\mathcal{D}$, the injective model structure exists on $\mathcal{C}^{\mathcal{D}}$. Then, for any small category $\mathcal{D}$, any filter F on $P(\mathcal{D})$, endowing $\mathcal{C}^{\mathcal{D}}$ with the injective model structure, the weakly induced model structure exists on $(\mathcal{C}^{\mathcal{D}})_F$.*

Theorem (Theorem 109)**.** *Let $\mathcal{C}$ be a complete and cocomplete model category with a strict initial object, $\mathcal{D}$ a Reedy category and F a left Reedy connected filter on $P(\mathcal{D})$. Then, endowing $\mathcal{C}^{\mathcal{D}}$ with the Reedy model structure, there is an induced model structure on $(\mathcal{C}^{\mathcal{D}})_F$.*

Theorem (Theorem 118)**.** *Let $\mathcal{C}$ be a cofibrantly generated model category, $\mathcal{D}$ a small category and F a left closed filter on $P(\mathcal{D})$. Then, endowing $\mathcal{C}^{\mathcal{D}}$ with the projective model structure, the induced model structure exists on $(\mathcal{C}^{\mathcal{D}})_F$*

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2 Reviewing the Filter Construction

2.1 Classical construction

Definition 1. For $\mathcal{C}$ a category, $A \in \operatorname{Obj} \mathcal{C}$ an object, two monomorphisms $i : B \hookrightarrow A$, $j : C \hookrightarrow A$, we have the relation $i \leq j$ if there exists $k : B \to C$ such that $j \circ k = i$. This relation forms a preorder, and a subobject of A is an equivalence class of monomorphisms into A with respect to this relation.

We will designate the initial object by $\emptyset$, the terminal object by e, and the partial order of the subobjects of e by $\operatorname{Subobj}(e)$.
In this paper, all the non-small categories will have finite limits and colimits and $\operatorname{Subobj}(e)$ will be small, excepted when stated otherwise.

Some of the results in this part are not new, and can be found for example in [MLM94, 5.9].

Lemma 2. *For U an object of $\mathcal{C}$, U is a subobject of e if and only if for any other element A, there is at most one morphism from A to U. Notably, if U is a subobject of e, for a morphism $f : A \to B$, there is at most one map $\tilde{f} : A \to B \times U$ such that $\pi_B \circ \tilde{f} = f$, and it exists if and only if there is a map $A \to U$.*

Proof. The map $i : U \to e$ is mono if and only if for any pair of morphisms $f, g : A \to U$, $i \circ f = i \circ g$ if and only if $f = g$. But we always have $i \circ f = i \circ g$, so i is mono if and only if there is at most one morphism $A \to U$.

Now, if U is a subobject of e, then for a morphism $f : A \to B$, there is at most one lift $\tilde{f} : A \to B \times U$ such that $\pi_B \circ \tilde{f} = f$ by the universal property of the product. □

Lemma 3. *An object U of $\mathcal{C}$ is a subobject of e if and only if the diagonal map $D_U : U \to U \times U$ is an isomorphism.*

Proof. If U is a subobject of e, $U \times U$ is also a subobject of e, and we have a morphism $U \times U \to U$ and a morphism $U \to U \times U$, hence they are isomorphic.

If the diagonal map $D_U : U \to U \times U$ is an isomorphism, it is an epimorphism and the two projections $U \times U \to U$ are equal, hence for A an object of $\mathcal{C}$, two morphisms $f, g : A \to U$, we have (f, g) composed with the left projection which is equal to (f, g) composed with the right projection, and therefore $f = g$. □

From now on, we will suppose that we fixed a representative for any equivalence class of subobjects of e, and to simplify notations and proofs, for two subobjects $U \leq V$ of e, any object A, we will suppose that $A \times U \times V = A \times U$, and the projection $A \times U \times V \to A \times U$ is simply the identity. The letters U, V, W, X, Y, Z will always designate elements in the filter, excepted if stated otherwise.

Definition 4. For a filter F on this partial order and two objects A, B, we define the following relation: for $U, V \in F$, two maps

$$\begin{aligned} f &: A \times U \to B \times U \\ g &: A \times V \to B \times V, \end{aligned}$$

we have $f \sim g$ if and only if there exists $W \in F$, $W \leq U \times V$ such that $f \times W = g \times W$.

Lemma 5. *This relation is an equivalence relation, and if we have four maps*

$$\begin{aligned} f &: A \times U \to B \times U \\ f' &: A \times U' \to B \times U' \\ g &: B \times V \to C \times V \\ g' &: B \times V' \to C \times V', \end{aligned}$$

with $f \sim f'$, $g \sim g'$, and $W \leq U \times V$, $W' \leq U' \times V'$, then

$$(g \times W) \circ (f \times W) \sim (g' \times W') \circ (f' \times W').$$

We are now ready to define the category $\mathcal{C}_F$:

Definition 6 (Filter quotient)**.** We define the category $\mathcal{C}_F$ in the following way:

$$\operatorname{Obj} \mathcal{C}_F = \operatorname{Obj} \mathcal{C}$$
$$\mathcal{C}_F(A, B) = (\coprod_{U \in F} \mathcal{C}(A \times U, B \times U))_{\sim}$$

and the composition is as in the previous lemma. We then have a functor $Q_F : \mathcal{C} \to \mathcal{C}_F$, sending any object on itself and any morphism on its equivalence class.

Remark 7. The usual definition of the filter quotient is different from what we present here: the morphisms are not the equivalence classes of morphisms of the form $A \times U \to B \times U$ but of the form $A \times U \to B$. Because any map $A \times U \to B$ can be lifted as a map $A \times U \to B \times U$ and because the projections $B \times U \to B$ are monomorphisms, these two constructions are equivalent. The formulation used in this paper seems simpler and will be easier to generalize in the next section.

Proposition 8. *Suppose that $\mathcal{C}$ has all infinite limits. Then Q_F preserves limits if and only if F is principal.*

Proof. The initial object is always preserved by Q_F, and for any subobject U of e, the functor $- \times U : \mathcal{C} \to \mathcal{C}$ preserves limits of non-empty diagrams. If F is principal, it follows that Q_F preserves limits. Now, if F is not principal, we will show that $Q_F(\prod_{U \in F} U)$ is not isomorphic to the initial object. The initial object e doesn't have a map to $Q_F(\prod_{U \in F} U)$, because for any $V \in F$, there exists a subobject W strictly smaller, so V has no map to W in $\mathcal{C}$, hence no map to $\prod_{U \in F} U$. $\square$

Proposition 9. *If $\mathcal{C}$ is complete, the inclusion of the full subcategory* $\operatorname{Subobj}(e)$ *has a left adjoint L, defined as*

$$L(A) = \prod_{\substack{U \in \operatorname{Subobj}(e) \\ A \to U}} U.$$

In other words, the subobjects of e form a reflective subcategory.

Proof. L is a functor, because if we have a morphism $f : A \to B$, then for any subobject U of e, if there is a morphism $B \to U$, then we have a morphism $A \to U$ given by composition with f, so $L(A) \leq L(B)$. Moreover, for any subobject V of e, if there is a morphism $A \to V$, then $L(A) \leq V$, and if $L(A) \leq V$, because there is a morphism $A \to L(A)$, there is a morphism $A \to V$. Therefore, we have an adjunction. $\square$

Proposition 10. *Suppose that $\mathcal{C}$ is complete and cocomplete, and that $\emptyset$ is a subobject of e. If* $\operatorname{Subobj}(e)$ *is a Boolean algebra and Q_F preserves the infinite colimits, then F is principal.*

Proof. If $\mathcal{C}$ is complete, the subcategory of subobjects of e is reflective, so if $\mathcal{C}$ is cocomplete, this subcategory is also cocomplete, and we define X as the coproduct in the partial order of all the subobjects Y such that $\neg Y \in F$. Because for any subobject Y such that $\neg Y \in F$, $Q_F(Y)$ is isomorphic to $\emptyset$, $Q_F(X)$ should be isomorphic to $\emptyset$, hence $Q_F(\neg X) \cong \neg Q_F(X) \cong e$, and $\neg X \in F$. Therefore, the subset of objects Y such that $\neg Y \in F$ has a greatest object, thus F has a least object, and F is principal. □

Proposition 11. *If $\mathcal{C}$ is Cartesian closed, $\mathcal{C}_F$ is cartesian closed and Q_F preserves the exponentials.*

Lemma 12. *Let $L : \mathcal{C} \to \mathcal{D}$ be a functor, U a subobject of $e_{\mathcal{C}}$ such that for any object X in $\mathcal{C}$, the map*

$$(L(\pi_X), L(\pi_U)) : L(X \times U) \to L(X) \times L(U)$$

is an isomorphism. Then $L(U)$ is a subobject of $e_{\mathcal{D}}$.

Proof. The composition $L(U) \to L(U \times U) \to L(U) \times L(U)$ is simply the diagonal map, and it is a composition of two isomorphisms, therefore it is an isomorphism and $L(U)$ is a subobject of $e_{\mathcal{D}}$. □

Definition 13. If a functor $L : \mathcal{C} \to \mathcal{D}$ respects the condition of the previous lemma for all objects U in a filter F, we say that the functor L commutes with F.

Lemma 14. *Let $L : \mathcal{C} \to \mathcal{D}$ be a functor commuting with a filter F, and F' a filter in* $\mathrm{Subobj}(e_{\mathcal{D}})$ *containing $L(F)$. There is an induced functor $\hat{L} : \mathcal{C}_F \to \mathcal{D}_{F'}$.*

Proof. It is a direct consequence of the fact that the functors Q_F and $Q_{F'}$ are localizations, as we will prove it in Proposition 21. The functor is defined the same way as L on objects, and $\hat{L}([f]_A) = [L(f)]_{L(A)}$ for any morphism $f : A \times U \to B$. □

Proposition 15. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be two functors forming an adjunction, F a filter in* $\mathrm{Subobj}(e_{\mathcal{C}})$ *commuting with L, and F' the filter generated by $L(F)$ in* $\mathrm{Subobj}(e_{\mathcal{D}})$*. Then there are two induced functors $\hat{L} : \mathcal{C}_F \to \mathcal{D}_{F'}$ and $\hat{R} : \mathcal{D}_{F'} \to \mathcal{C}_F$ forming an adjunction.*

Proof. The functor R is a right adjoint, therefore it commutes with limits, and notably it commutes with the filter F'. For any $U \in F$, we have a morphism $U \to R \circ L(U)$, therefore $U \leq R \circ L(U)$, and the filter F contains $R(F')$. We obtain two induced functors $\hat{L}$, $\hat{R}$ by the previous lemma. Because the filter F' is generated by $L(F)$, $L(F)$ is dense in F'. Therefore, we have the following natural isomorphism:

$$\operatorname*{colim}_{U \in F'} \mathcal{D}(A \times U, B \times U) \cong \operatorname*{colim}_{U \in F} \mathcal{D}(A \times L(U), B \times L(U)).$$

We finally obtain the sequence of natural isomorphisms

$$\begin{aligned}
\mathcal{D}_{F'}(\hat{L}(A), B) &\cong \operatorname*{colim}_{U \in F'} \mathcal{D}(L(A) \times U, B \times U) \\
&\cong \operatorname*{colim}_{U \in F'} \mathcal{D}(L(A) \times U, B) \\
&\cong \operatorname*{colim}_{U \in F} \mathcal{D}(L(A) \times L(U), B) \\
&\cong \operatorname*{colim}_{U \in F} \mathcal{D}(L(A \times U), B) \\
&\cong \operatorname*{colim}_{U \in F} \mathcal{C}(A \times U, R(B)) \\
&\cong \operatorname*{colim}_{U \in F} \mathcal{C}(A \times U, R(B) \times U) \\
&\cong \mathcal{C}_F(A, \hat{R}(B))
\end{aligned}$$

hence we have an adjunction. □

2.2 Generalization

The filter quotient is useful in topos theory, but it is too restrictive in general. For example, it is always trivial in pointed categories. To solve this problem, we will see a generalization of the classical filter construction. This construction could be described as a quotient by a family of functors, forming a partial order similar to the functors $- \times U$ in the classical construction.

Definition 16 (Filter of functors)**.** A filter of functors is a family of endofunctors $F = (F_p)$ indexed by a poset P, which satisfies the following conditions:

1. P is downward directed, i.e. for any two elements $p, q \in P$, there exists an element $r \in P$ such that $r \leq p$ and $r \leq q$.
2. P has a maximal element 1, with $F_1 = \mathrm{id}_{\mathcal{C}}$.
3. For each pair $p \leq q$ in P, we have $F_p \circ F_q = F_p$.

Moreover, the filter is said to be ascending if for each pair $p \leq q$ in P, we have an associated natural transformation $\pi_{p,q} : F_p \to F_q$, called a projection, such that the following conditions are satisfied:

4. For each triplet $p \leq q \leq r$ in P, we have $\pi_{q,r} \circ \pi_{p,q} = \pi_{p,r}$.
5. For each element $p \in P$, any object $A \in \mathrm{Obj}\,\mathcal{C}$, $F_p(\pi_{p,1,A}) = \mathrm{id}_{F_p(A)}$.

The filter is said to be descending if for each pair $p \leq q$ in P, we have an associated natural transformation $i_{q,p} : F_q \to F_p$, called a coprojection, such that the following conditions are satisfied:

4'. For each triplet $p \leq q \leq r$ in P, we have $i_{q,p} \circ i_{r,q} = i_{r,p}$.

5'. For each element $p \in P$, any object $A \in \text{Obj}\,\mathcal{C}$, $F_p(i_{1,p,A}) = \text{id}_{F_p(A)}$.

For any filter of functors F on a category $\mathcal{C}$, we have an opposite filter of functors F^{op} on $\mathcal{C}^{\text{op}}$. A filter of functors is descending if and only if its opposite filter is ascending, hence every result about one of the two yields a dual result about the second one.

Example 17. Any filter on $\text{Subobj}(e)$ induces an ascending filter of functors, with the functors $- \times U : \mathcal{C} \to \mathcal{C}$.

Definition 18. Let F be a filter of functors. For any objects $A, B \in \text{Obj}\,\mathcal{C}$, we define the following relation: for $p, q \in P$, two morphisms

$$\begin{aligned} f &: F_p(A) \to F_p(B) \\ g &: F_q(A) \to F_q(B), \end{aligned}$$

we have $f \sim g$ if there exists an element $r \leq p, q$ such that $F_r(f) = F_r(g)$.

Lemma 19. *This relation is an equivalence relation, and for*

$$\begin{aligned} f &: F_p(A) \to F_p(B) \\ f' &: F_{p'}(A) \to F_{p'}(B) \\ g &: F_q(B) \to F_q(C) \\ g' &: F_{q'}(B) \to F_{q'}(C) \end{aligned}$$

such that $f \sim f'$ and $g \sim g'$, then for $r \leq p, q$ and $r' \leq p', q'$, we have

$$F_r(g) \circ F_r(f) \sim F_{r'}(g') \circ F_{r'}(f').$$

Proof. Reflexivity and symmetry are clear. This relation is transitive because for $f : F_p(A) \to F_p(B)$, $g : F_q(A) \to F_q(B)$, $h : F_r(A) \to F_r(B)$ such that $F_s(f) = F_s(g)$ and $F_t(g) = F_t(h)$, for any element $u \in P$ such that $u \leq s$ and $u \leq t$ we have

$$F_u(f) = F_u(F_s(f)) = F_u(F_s(g)) = F_u(F_t(g)) = F_u(F_t(h)) = F_u(h).$$

For the second part of the lemma, let $s \leq r, r'$ such that $F_s(f) = F_s(f')$ and $F_s(g) = F_s(g')$. Then,

$$F_s(F_r(g) \circ F_r(f)) = F_s(g) \circ F_s(f) = F_s(g') \circ F_s(f') = F_s(F_{r'}(g') \circ F_{r'}(f')).$$

$\square$

Definition 20 (Filter quotient)**.** We define the category $\mathcal{C}_F$ in the following way:

$$\begin{aligned} \text{Obj}\,\mathcal{C}_F &= \text{Obj}\,\mathcal{C} \\ \mathcal{C}_F(A, B) &= \Big(\bigsqcup_{p \in P} \mathcal{C}(F_p(A), F_p(B))\Big)_{\sim} \end{aligned}$$

and for $f : F_p(A) \to F_p(B)$, $g : F_q(B) \to F_q(C)$, with $r \le p, q$, the composition $[g] \circ [f]$ is defined as $[F_r(g) \circ F_r(f)]$. This is well-defined by the previous lemma. When we will want to clarify that the domain of a morphism $[f]$ is some object A and the codomain is some object B, we will write it as $[f]_{A \to B}$.

We also define the quotient functor $Q_F : \mathcal{C} \to \mathcal{C}_F$ as $Q_F(A) = A$ and $Q_F(f) = [f]$.

Proposition 21. *Suppose that F is an ascending filter of functors. The category $\mathcal{C}_F$, with the functor Q_F, is a localization at all the projections $\pi_{p,A} : F_p(A) \to A$.*
Dually, if F is a descending filter of functors, the category $\mathcal{C}_F$, with the functor Q_F, is a localization at all the coprojections $i_{p,A} : A \to F_p(A)$.

Proof. We prove the result for ascending filters. The image of a map $\pi = \pi_{p,A} : F_p(A) \to A$ is an isomorphism in $\mathcal{C}_F$, because by condition 5 we have

$$[\pi]_{F_p(A) \to A} \circ [\mathrm{id}_{F_p(A)}]_{A \to F_p(A)} = [\mathrm{id}_A]_{A \to A}$$

and

$$[\mathrm{id}_{F_p(A)}]_{A \to F_p(A)} \circ [\pi]_{F_p(A) \to A} = [\mathrm{id}_{F_p(A)}]_{F_p(A) \to F_p(A)}.$$

Moreover, for any morphism $f : F_p(A) \to F_p(B)$, by naturality of π_p, we have the following decomposition:

$$[f]_{A \to B} = [\pi_{p,B}]_{F_p(B) \to B} \circ [f]_{F_p(A) \to F_p(B)} \circ ([\pi_{p,A}]_{F_p(A) \to A})^{-1}.$$

So it is clear that for any functor $G : \mathcal{C} \to \mathcal{D}$ which sends any such projection on an isomorphism, if there exists a factorization $G = \tilde{G} \circ Q_F$, it is unique, because it must be defined as

$$\tilde{G}([f]_{A \to B}) = G(\pi_{p,B}) \circ G(f) \circ G(\pi_{p,A})^{-1}.$$

Now, to show that it exists, we must control that what we just wrote makes sense, so we need to check that if we have $f : F_p(A) \to F_p(B)$, $g : F_q(A) \to F_q(B)$, and $r \le p, q$, if $F_r(f) = F_r(g)$, then

$$G(\pi_{p,B}) \circ G(f) \circ G(\pi_{p,A})^{-1} = G(\pi_{q,B}) \circ G(g) \circ G(\pi_{q,A})^{-1}.$$

We know that the composition of the projections $F_r(A) \to F_p(A) \to A$ is equal to the projection $F_r(A) \to A$, and that the composition of the projections $F_r(A) \to F_q(A) \to A$ is equal to the projection $F_r(A) \to A$, hence we have

$$\begin{aligned}
&G(\pi_{p,B}) \circ G(f) \circ G(\pi_{p,A})^{-1} \\
&= G(\pi_{r,B}) \circ G(\pi_{r,p,B})^{-1} \circ G(f) \circ G(\pi_{r,p,A}) \circ G(\pi_{r,A})^{-1} \\
&= G(\pi_{r,B}) \circ G(F_r(f)) \circ G(\pi_{r,A})^{-1} \\
&= G(\pi_{r,B}) \circ G(F_r(g)) \circ G(\pi_{r,A})^{-1} \\
&= G(\pi_{r,B}) \circ G(\pi_{r,q,B})^{-1} \circ G(g) \circ G(\pi_{r,q,A}) \circ G(\pi_{r,A})^{-1} \\
&= G(\pi_{q,B}) \circ G(g) \circ G(\pi_{q,A})^{-1}.
\end{aligned}$$

Finally, we must show that $\tilde{G}$ preserves composition: for $f : F_p(A) \to F_p(B)$, $g : F_q(B) \to F_q(C)$, $r \leq p, q$, we have

$$\begin{aligned}
&\tilde{G}([g]_{B\to C}) \circ \tilde{G}([f]_{A\to B}) \\
&= G(\pi_{r,C}) \circ G(F_r(g)) \circ G(\pi_{r,B})^{-1} \circ G(\pi_{r,B}) \circ G(F_r(f)) \circ G(\pi_{r,A})^{-1} \\
&= G(\pi_{r,C}) \circ G(F_r(g)) \circ G(F_r(f)) \circ G(\pi_{r,A})^{-1} \\
&= G(\pi_{r,C}) \circ G(F_r(g) \circ F_r(f)) \circ G(\pi_{r,A})^{-1} \\
&= \tilde{G}([g]_{B\to C} \circ [f]_{A\to B}).
\end{aligned}$$

□

These projections form in fact a left calculus of fractions (and dually the coprojections form a right calculus of fractions), which is a concept defined by Peter Gabriel and Michel Zisman in [GZ67]. If we begin with a locally small cocomplete category, localizing with respect to a class of morphisms forming a left or right calculus of fractions gives a locally small category, and this category can be nicely described.

Lemma 22. *Let F be a filter of functors. Let $\mathcal{D}$ be a finite category and $G : \mathcal{D} \to \mathcal{C}_F$ a functor. Then there exists a functor $\tilde{G} : \mathcal{D} \to \mathcal{C}$ and a natural isomorphism $Q_F \circ \tilde{G} \to G$.*

Proof. Because $\mathcal{D}$ contains only finitely many morphisms, there exists an element $p \in P$ such that for any morphism $f : a \to b$ in $\mathcal{D}$ we can choose a representing element $f' : F_p(G(a)) \to F_p(G(b))$ for $G(f)$. Moreover, for each pair of composable morphisms f, g in $\mathcal{D}$, there exists an element $r \leq p$ such that $F_r(g' \circ f') = F_r((g \circ f)')$, and because there are only finitely many such pairs of composable morphisms, we can take an element r which works for all pairs, and we can define the functor $\tilde{G}(a) = F_r(G(a))$ and $\tilde{G}(f) = F_r(f')$. We then define the natural transformation $\alpha : Q_F \circ \tilde{G} \to G$ by

$$\alpha_a = [\mathrm{id}_{\tilde{G}(a)}]_{\tilde{G}(a)\to G(a)}$$

which has an inverse $\beta : G \to Q_F \circ \tilde{G}$ defined by

$$\beta_a = [\mathrm{id}_{\tilde{G}(a)}]_{G(a)\to \tilde{G}(a)}.$$

□

Proposition 23. *Suppose that F is an ascending filter. The category $\mathcal{C}_F$ has finite limits and Q_F preserves finite limits.*
Dually, if F is a descending filter, the category $\mathcal{C}_F$ has finite colimits and Q_F preserves finite colimits

This proposition is a direct consequence of the fact that the projections $\pi_{p,A}$ form a calculus of left fractions (and dually that the coprojections form a calculus of right fractions) [GZ67, Proposition 3.1]. We still give an elementary proof, for completeness.

Proof. We prove it for ascending filters. By Lemma 22, we only need to prove that Q_F preserves finite limits. Let $\mathcal{D}$ be a finite category, $G : \mathcal{D} \to \mathcal{C}$ a functor and $L = \lim G$, with the morphisms $f_a : L \to G(a)$ for any object $a \in \operatorname{Obj} \mathcal{D}$. Let $A \in \operatorname{Obj} \mathcal{C}$ be an object with a collection of morphisms $[g_a] : A \to G(a)$ in $\mathcal{C}_F$, such that for any morphism $h : a \to b$ in $\mathcal{D}$,

$$Q_F(G(h)) \circ [g_a] = [g_b].$$

Because this collection is finite and $\mathcal{D}$ contains only finitely many morphisms, there exists an element $p \in P$ such that we have representing elements $g_a : F_p(A) \to F_p(G(a))$ in $\mathcal{C}$ such that everything commutes in the obvious way. Composing with the projections $F_p(G(a)) \to G(a)$ induces a morphism $g : F_p(A) \to L$, and we have $[f_a] \circ [g] = [g_a]$ for any $a \in \operatorname{Obj} \mathcal{D}$. Now, suppose that there exists another morphism $g' : F_q(A) \to L$ such that $[f_a] \circ [g'] = [g_a]$, then there exists $r \leq p, q$ such that

$$F_r(f_a) \circ F_r(g') = F_r(f_a) \circ F_r(g) = F_r(g_a)$$

and therefore after taking composition with the projections we have the same induced morphisms $F_r(A) \to L$, hence

$$\pi_{r,L} \circ F_r(g') = \pi_{r,L} \circ F_r(g)$$

which implies

$$F_r(g') = F_r(\pi_{r,L}) \circ F_r(F_r(g')) = F_r(\pi_{r,L}) \circ F_r(F_r(g)) = F_r(g),$$

thus $[g] = [g']$. $\square$

Definition 24 (Dense in P)**.** We say that some subset $A \subseteq P$ is dense in P if for any $p \in P$, there exists $p' \in A$ such that $p' \leq p$.

Proposition 25. *Let F be a filter of functors and suppose that there exists a subset $A \subseteq P$ dense in P such that for any $p \in A$, F_p preserves finite colimits. Then $\mathcal{C}_F$ has finite colimits and Q_F preserves finite colimits.*
Dually, if there exists a subset $A \subseteq P$ dense in P such that for any $p \in A$, F_p preserves finite limits, then $\mathcal{C}_F$ has finite limits and Q_F preserves finite limits.

Proof. We prove it for finite colimits. By Lemma 22, we only need to prove that Q_F preserves finite colimits. Let $\mathcal{D}$ be a finite category, $G : \mathcal{D} \to \mathcal{C}$ a functor and $C = \operatorname{colim} G$, with the canonical morphisms $f_a : G(a) \to C$ for any $a \in \operatorname{Obj} \mathcal{D}$. Suppose that there exist an object A and a collection of morphisms $[g_a] : G(a) \to A$, such that it commutes in the obvious way. Because $\mathcal{D}$ is finite, there exists $p \in P$ such that we have representing elements $g_a : F_p(G(a)) \to F_p(A)$ that commute in the obvious way in $\mathcal{C}$. Without loss of generality, we can suppose that F_p preserves finite colimits. We then have an induced morphism $g : F_p(C) \to F_p(A)$ such that $F_p(f_a) \circ g_a = g$ for any $a \in \operatorname{Obj} \mathcal{D}$. Suppose that there exists another morphism $g' : F_q(C) \to F_q(A)$ such that $F_q(f_a) \circ g_a = g'$ for any a. Then we can find $r \leq p, q$ such that F_r preserves colimits, hence $F_r(g') = F_r(g)$ by the universal property of the colimit, and $[g] = [g']$. $\square$

Proposition 26. *Let $\mathcal{D}$ be a small category, then there is an induced filter of functors $F^{\mathcal{D}}$ on $\mathcal{C}^{\mathcal{D}}$. Moreover, this filter is ascending, respectively descending, if F is ascending, respectively descending. If $\mathcal{D}$ is finite, the obvious functor $(\mathcal{C}^{\mathcal{D}})_{F^{\mathcal{D}}} \to \mathcal{C}_F^{\mathcal{D}}$ is an equivalence of category.*

Proof. For any $p \in P$, the functor $F_p^{\mathcal{D}}$ is defined as $F_p^D(G) = G \circ F_p$ for any functor $G : \mathcal{D} \to \mathcal{C}$, and $F_p^{\mathcal{D}}(\alpha)_a = F_p(\alpha_a)$ for any natural transformation α, any object $a \in \mathrm{Obj}\,\mathcal{D}$. If F is ascending, for $p \leq q$, the natural transformation $\pi_{p,q} : F_p \to F_q$ induces a natural transformation $\pi_{p,q}^{\mathcal{D}} : F_p^{\mathcal{D}} \to F_q^{\mathcal{D}}$ by naturality, hence $F^{\mathcal{D}}$ is ascending, and dually if F is descending.

We have an obvious functor $i : (\mathcal{C}^{\mathcal{D}})_{F^{\mathcal{D}}} \to \mathcal{C}_F^{\mathcal{D}}$ given simply by $G \mapsto Q_F \circ G$, $[\alpha] \mapsto Q_F(\alpha)$, well-defined by the universal property of the localization. Now, if $\mathcal{D}$ is finite, any functor $G : \mathcal{D} \to \mathcal{C}_F$ has a lift $\tilde{G} : \mathcal{D} \to \mathcal{C}$, such that G is isomorphic to $Q_F \circ \tilde{G}$, hence i is essentially surjective. Moreover, any natural transformation $\alpha : G \to H$ is a finite family of morphisms, hence there exists an element $p \in P$ such that there is a lift $\tilde{\alpha} : F_p^{\mathcal{D}}(G) \to F_p^{\mathcal{D}}(H)$, therefore i is full. Finally, i is faithful because if for two functors $G, H : \mathcal{D} \to \mathcal{C}$, two natural transformations $\alpha, \beta : G \to H$, we have $F_p(\alpha_a) = F_p(\beta_a)$ for any $a \in \mathrm{Obj}\,\mathcal{D}$, then $F_p^{\mathcal{D}}(\alpha) = F_p^{\mathcal{D}}(\beta)$, hence $[\alpha] = [\beta]$ in $(\mathcal{C}^{\mathcal{D}})_{F^{\mathcal{D}}}$ if and only if $[\alpha_a] = [\beta_a]$ in $\mathcal{C}_F^{\mathcal{D}}$ for all $a \in \mathrm{Obj}\,\mathcal{D}$. □

Lemma 27. *A morphism $[f]$ is an isomorphism if and only if its equivalence class contains an isomorphism.*

Proof. The 'if' part is clear. Now, suppose that $[f] : A \to B$ is an isomorphism, with inverse $[g] : B \to A$, and representing elements $f : F_p(A) \to F_p(B)$ and $g : F_q(B) \to F_q(A)$. Then, there exists an element $r \leq p, q$ such that

$$F_r(f) \circ F_r(g) = F_r(\mathrm{id}_A) = \mathrm{id}_{F_r(A)}$$

and

$$F_r(g) \circ F_r(f) = F_r(\mathrm{id}_B) = \mathrm{id}_{F_r(B)},$$

hence $F_r(f)$ is an isomorphism. □

Lemma 28. *If the elements $p \in P$ such that F_p preserves pullbacks are dense in P, a morphism $[f]$ is a monomorphism if and only if its equivalence class contains a monomorphism.*
Dually, if the elements $p \in P$ such that F_p preserves pushouts are dense in P, a morphism $[f]$ is an epimorphism if and only if its equivalence class contains an epimorphism.

Proof. We will prove it for monomorphisms. In any category, a morphism $f : A \to B$ is mono if and only if there exists a commutative square

$$\begin{tikzcd} P \arrow[r, "r"] \arrow[d, "r"] & A \arrow[d, "f"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

that is a pullback square. Because Q_F preserves pullbacks, we obtain directly the 'if' part. Now, suppose that $[f]$ is a monomorphism, with a representing element $f : F_p(A) \to F_p(B)$. We then have the pullback diagram

$$\begin{tikzcd} P \arrow[r, "r_1"] \arrow[d, "r_2"] & F_p(A) \arrow[d, "f"] \\ F_p(A) \arrow[r, "f"] & F_p(B) \end{tikzcd}$$

in $\mathcal{C}$, and because $[f]$ is mono, $[r_1] = [r_2]$. Hence we can find $q \leq p$ such that F_q preserves pullbacks and $F_q(r_1) = F_q(r_2)$, so $F_q(f)$ is mono. □

Lemma 29. *Suppose that $\mathcal{C}$ is an additive category, and that the elements $p \in P$ such that F_p preserves biproducts are dense in P. Then, $\mathcal{C}_F$ is an additive category, and the isomorphism*

$$\hom_{\mathcal{C}_F}(A, B) \cong \operatorname*{colim}_{p \in P} \hom_{\mathcal{C}}(F_p(A), F_p(B))$$

is also an isomorphism of abelian groups for any objects $A, B \in \operatorname{Obj} \mathcal{C}$.

Proof. Let $A \subseteq P$ be the subset of elements $p \in P$ such that F_p preserves biproducts. Because A is dense in P, C_F has also biproducts. If a functor F_p preserves biproducts, it is an additive functor, therefore for any element $p \in A$, the map

$$\hom_{\mathcal{C}}(A, B) \to \hom_{\mathcal{C}}(F_p(A), F_p(B))$$

is also a morphism of abelian groups. The isomorphism

$$\hom_{\mathcal{C}_F}(A, B) \cong \operatorname*{colim}_{p \in A} \hom_{\mathcal{C}}(F_p(A), F_p(B))$$

is also an isomorphism of abelian groups, because the forgetful functor $\mathcal{A}b \to \mathcal{S}et$ preserves the filtered colimits. It is clear that the composition is bilinear. □

Proposition 30. *Suppose that $\mathcal{C}$ is an abelian category, and that the elements $p \in P$ such that F_p preserves finite limits are dense in P, and similarly for finite colimits. Then, $\mathcal{C}_F$ is abelian.*

Proof. By the previous lemma, we already know that $\mathcal{C}_F$ is additive. Moreover, the zero object is preserved by Proposition 25, and kernels and cokernels exist. Let $[f]$ be a monomorphism in $\mathcal{C}_F$. By Lemma 28, the equivalence class $[f]$ contains a monomorphism f. This monomorphism is regular, and because Q_F preserves finite limits, $[f]$ must be regular too. The proof for epimorphisms is dual. □

3 Filter Construction on Homotopical Categories

In this section we apply the filter construction to categories with additional, homotopical structure. We focus on two cases: *c*ategories of fibrant objects and *m*odel categories.

3.1 Category of fibrant objects

In this section we will suppose that $\mathcal{C}$ is equipped with the structure of a category of fibrant objects. We will give a sufficient condition such that C_F has also the structure of a category of fibrant objects. we will see that this condition is always verified when F is a filter on the subobjects of e.

Definition 31. Fibrations are said to be dense in the filter if there is a dense subset $A \subseteq P$ such that for any $p \in A$, F_p preserves fibrations and trivial fibrations.

Lemma 32. *If a functor F_p preserves trivial fibrations, it preserves weak equivalences.*

Proof. This is a direct consequence of Ken Brown's lemma, because all objects are fibrant. ☐

For the rest of this part, we will suppose that fibrations are dense in the filter.

Definition 33 (Induced fibrations and weak equivalences)**.** We define the induced classes of fibrations and weak equivalences in $\mathcal{C}_F$ in the following way: a morphism $[f]_A$ is a fibration if there exists an element $p \in P$ such that $F_p(f)$ is a fibration, and similarly for weak equivalences. It is well-defined for fibrations by compatibility with the filter, and it is well-defined for weak equivalences by Lemma 32.

Lemma 34. *The induced fibrations and weak equivalences are closed under composition.*

Proof. If $f : F_p(A) \to F_p(B)$, $g : F_q(B) \to F_q(C)$ are two fibrations, then there exists $r \leq p, q$ such that F_r preserves fibrations. Then, $[g] \circ [f] = [F_r(g) \circ F_r(f)]$ is an induced fibration. The proof for weak equivalences is *mutatis mutandis* the same. ☐

Lemma 35. *The induced weak equivalences respect the 2-out-of-3.*

Proof. The proof is very similar to the previous lemma. ☐

Lemma 36. *The induced weak equivalences and the induced fibrations contain the isomorphisms.*

Proof. It is a direct consequence of Lemma 27. ☐

Lemma 37. *Suppose that Q_F preserves finite limits. Fibrations and trivial fibrations are preserved under pullbacks.*

Proof. By the previous lemma and Lemma 22, we only need to show that pullbacks that factor through Q_F preserve fibrations and trivial fibrations, which is a consequence of the fact that Q_F preserves fibrations, trivial fibrations and finite limits. ☐

Lemma 38. *Suppose that Q_F preserves finite limits. All objects are fibrant in $\mathcal{C}_F$.*

Proof. Because the functor Q_F preserves finite limits, it preserves the terminal object, and the map $A \to e$ is a fibration directly from the definition. ☐

Lemma 39. *Every object has a path object in $\mathcal{C}_F$.*

Proof. Because Q_F preserves finite limits, fibrations and weak equivalences, we can just take the image of the path object in $\mathcal{C}$ by Q_F. $\square$

Theorem 40. *If Q_F preserves finite limits, the induced classes of fibrations and weak equivalences in $\mathcal{C}_F$ form the structure of a category of fibrant objects.*

Corollary 41. *For any filter F on the partial order* $\mathrm{Subobj}(e)$*, the quotient functor $Q_F : \mathcal{C} \to \mathcal{C}_F$ induces a structure of category of fibrant objects on $\mathcal{C}_F$.*

Proof. For any $U \in F$, the functor $- \times U$ preserves fibrations and trivial fibrations, hence fibrations are dense in F. Moreover, any filter on $\mathrm{Subobj}(e)$ is an ascending filter, therefore it preserves finite limits and it is a direct consequence of the previous theorem. $\square$

3.2 Model category

In this part, we will define what it means for a model structure to be dense in a filter of functors, and we will show that if it is the case, we obtain an induced model structure on the filter quotient. We will then show how different properties are preserved.

We will suppose that we have a structure of model category on $\mathcal{C}$. We don't require the weak factorization systems $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{WE})$ and $(\mathrm{Cof} \cap \mathrm{WE}, \mathrm{Fib})$ to be functorial, and we only require that $\mathcal{C}$ has finite limits and colimits. We will suppose that the quotient functor Q_F preserves finite limits and finite colimits.

Definition 42 (Dense structure)**.** We will say that cofibrations are dense in F if the subset of elements $p \in P$ such that F_p preserves cofibrations and weak equivalences is dense in P. Dually, fibrations are dense in F if the subset of elements $p \in P$ such that F_p preserves fibrations and weak equivalences is dense in P. The structure is dense in F if cofibrations and fibrations are dense in F. We will say the the structure is very dense in F if the subset of elements $p \in P$ such that F_p preserves cofibrations, fibrations and weak equivalences is dense in P.

Proposition 43. *If F is a filter on* $\mathrm{Subobj}(e)$*, the structure is very dense in the filter if and only if condition 1 is verified.*

Proof. A functor $- \times A : \mathcal{C} \to \mathcal{C}$ preserves fibrations for any object $A \in \mathrm{Obj}\,\mathcal{C}$. $\square$

Definition 44 (Pushout-product property)**.** A model category is said to have the pushout-product property if for any cofibrations $i : A \to B$, $j : C \to D$, the induced map

$$i \square j : (A \times D) \coprod_{A \times C} (B \times C) \to B \times D$$

is a cofibration, and trivial if any of the two cofibrations is trivial.

Proposition 45. *Suppose that F is a filter on* $\mathrm{Subobj}(e)$ *and that $\mathcal{C}$ has the pushout-product property. If the subset of cofibrant objects is dense in the filter, then the structure is very dense in the filter.*

Proof. If the category has the pushout-product property, for any cofibrant object U, the functor $- \times U$ preserves cofibrations and trivial cofibrations. Moreover, because this functor preserves trivial fibrations and any weak equivalence can be factorized as the composition of a trivial cofibration and a trivial fibration, this functor will also preserve weak equivalences. □

Definition 46. A filter F is said to be principal if P has a minimal element 0.

Proposition 47. *If the filter F is principal, the structure is dense in the filter if and only if the functor $F_0 : \mathcal{C} \to \mathcal{C}$ preserves cofibrations, fibrations and weak equivalences.*

Proof. If P has a minimal element 0, a subset $A \subseteq P$ is dense if and only if $0 \in A$. □

Definition 48 (Weakly induced model structure)**.** Let F be an ascending filter in which cofibrations are dense. For any element $p \in P$, we define the p-fibrations as the morphisms having the right lifting property with respect to all trivial cofibrations of the form $F_p(A) \to F_p(B)$. Then, in $\mathcal{C}_F$, a morphism $[f]$ is an induced cofibration or an induced weak equivalence if its equivalence class contains a cofibration or a weak equivalence, respectively. A morphism $[f]$ is an induced fibration if its equivalence class contains a p-fibration for some element $p \in P$. When these definitions give a model structure, it is called the weakly induced model structure.

Definition 49 (well-levelled filter)**.** We say that a filter is well-levelled if for any element $p \in P$, any objects $A, B \in \mathrm{Obj}\ \mathcal{C}$, for any morphism $f : F_p(A) \to F_p(B)$ we have $F_p(f) = f$.

Lemma 50. *Let F be an ascending well-levelled functor. Then we have the following properties:*

(1) For $p \leq q$, a q-fibration is a p-fibration.

(2) For α a p-fibration, $F_p(\alpha)$ is a p-fibration.

(3) The p-fibrations are closed under composition.

(4) The p-fibrations are closed under retracts.

Proof. The proof of (1) is a direct consequence of the definition, and (3) and (4) are classical properties of the right lifting property. We will prove property (2). Let $f : C \to D$ be a p-fibration, and $g : F_p(A) \to F_p(B)$ a trivial cofibration. If we have a commutative square

$$\begin{tikzcd}
F_p(A) \arrow[r] \arrow[d, "g"] & F_p(C) \arrow[d, "F_p(f)"] \\
F_p(B) \arrow[r] & F_p(D)
\end{tikzcd}$$

then we can extend it as

$$\begin{tikzcd}
F_p(A) \arrow[r] \arrow[d, "g"] & F_p(C) \arrow[r, "\pi_{p,C}"] \arrow[d, "F_p(f)"] & C \arrow[d, "f"] \\
F_p(B) \arrow[r] & F_p(D) \arrow[r, "\pi_{p,D}"] & D.
\end{tikzcd}$$

By definition, we have a lift $h : F_p(B) \to C$, and $F_p(h) : F_p(B) \to F_p(C)$ is then the lift that we want. $\square$

Definition 51. A p-fibration which is also a weak equivalence is called a trivial p-fibration. A trivial p-fibration which has the right lifting property with respect to all cofibrations of the form $F_p(A) \to F_p(B)$ is called a genuine trivial p-fibration.

Definition 52. We say that the trivial p-fibrations can be raised if for any trivial p-fibration $f : A \to B$, there exists a trivial fibration $f' : A' \to B'$ and two isomorphisms $\phi : F_p(A) \to F_p(A')$, $\psi : F_p(B) \to F_p(B')$ such that they form the commutative square

$$\begin{tikzcd}
F_p(A) \arrow[r, "\phi"] \arrow[d, "F_p(f)"] & F_p(A') \arrow[d, "F_p(f')"] \\
F_p(B) \arrow[r, "\psi"] & F_p(B').
\end{tikzcd}$$

Lemma 53. *Let F be an ascending well-levelled filter. Let $p \in P$ be an element such that F_p preserves weak equivalences, and suppose that the trivial p-fibrations can be raised. Then, for any p-fibration f which is also a weak equivalence, $F_p(f)$ is a genuine trivial p-fibration.*

Proof. Because the trivial p-fibrations can be raised, we need only to verify that for any trivial fibration f, $F_p(f)$ has the right lifting property with respect to cofibrations of the form $F_p(A) \to F_p(B)$. The proof of this is *mutatis mutandis* the same as the proof of property (2) in the previous lemma. $\square$

Theorem 54. *Let F be an ascending well-levelled filter whose trivial p-fibrations are genuine, and such that cofibrations are dense in the filter. Then, the weakly induced model structure exists on $\mathcal{C}_F$.*

Proof. Because cofibrations and weak equivalences are preserved by a dense subset of P, the induced cofibrations and weak equivalences will verify all the closure properties that we need. The factorization axiom is verified because we can take the factorization in $\mathcal{C}$. All the other properties are direct consequences of the previous lemmas. $\square$

This theorem is quite general, but it does not give us a lot of control on the properties of the obtained model category. Hence, we will prove another theorem based on different conditions. From now on, we will suppose that the structure is dense in F.

Definition 55 (Induced model structure)**.** We want to induce a new structure of model category on $\mathcal{C}_F$. We define $[f]_A$ to be a cofibration, a fibration or a weak equivalence if there exists $p \in P$ such that $F_p(f)$ is a cofibration, a fibration or a weak equivalence, respectively. By denseness of the structure, it is well-defined.

Lemma 56. *A morphism $[f] : A \to B$ is an induced trivial cofibration or a trivial fibration if and only if there exists an element $p \in P$ such that $F_p(f)$ is a trivial cofibration or a trivial fibration, respectively.*

Proof. Direct consequence of the denseness. $\square$

Lemma 57. *The induced model structure verifies the factorization axiom.*

Proof. Because any equivalence class containing a cofibration, a fibration or a weak equivalence is a cofibration, a fibration or weak equivalence, respectively, we can take the factorization in $\mathcal{C}$. $\square$

Lemma 58. *The induced cofibrations, fibrations and weak equivalences are closed under composition and under retract, and the induced weak equivalences verify the 2-out-of-3 axiom.*

Proof. Direct consequence of denseness. $\square$

Lemma 59. *Suppose that F is an ascending or descending filter. The induced model structure verifies the lifting axiom.*

Proof. We will prove it for ascending filters. Suppose that we have the following diagram in $\mathcal{C}_F$:

$$\begin{array}{ccc} A & \xrightarrow{[r]} & C \\ \big\downarrow{\scriptstyle [f]} & & \big\downarrow{\scriptstyle [g]} \\ B & \xrightarrow{[s]} & D \end{array}$$

with $[f]$ an induced cofibration and $[g]$ and induced trivial fibration. Then there exists an element $p \in P$ such that $F_p(g)$ is a trivial fibration and such that we have a lift of the diagram in $\mathcal{C}$:

$$\begin{array}{ccc} F_p(A) & \xrightarrow{F_p(r)} & F_p(C) \\ \big\downarrow{\scriptstyle F_p(f)} & & \big\downarrow{\scriptstyle F_p(g)} \\ F_p(B) & \xrightarrow{F_p(s)} & F_p(D) \end{array}$$

Then we take $q \leq p$ such that $F_q(f)$ is a cofibration, which gives us the following commutative diagram:

$$\begin{array}{ccccc} F_q(A) & \xrightarrow{F_q(r)} & F_q(C) & \xrightarrow{\pi_{q,p,C}} & F_p(C) \\ \downarrow{\scriptstyle F_q(f)} & & \downarrow{\scriptstyle F_q(g)} & & \downarrow{\scriptstyle F_p(g)} \\ F_q(B) & \xrightarrow{F_q(s)} & F_q(D) & \xrightarrow{\pi_{q,p,D}} & F_p(D) \end{array}$$

We then obtain a lift $F_q(B) \to F_p(C)$, and taking the image of this lift by F_q, we obtain a lift $F_q(B) \to F_q(C)$.

The proof for the case where $[f]$ is a trivial cofibration and $[g]$ a fibration is *mutatis mutandis* the same. □

Lemma 60. *Suppose that the structure is very dense in F. The induced model structure verifies the lifting axiom.*

Proof. Suppose that we have the following diagram in $\mathcal{C}_F$:

$$\begin{array}{ccc} A & \xrightarrow{[r]} & C \\ \downarrow{\scriptstyle [f]} & & \downarrow{\scriptstyle [g]} \\ B & \xrightarrow{[s]} & D \end{array}$$

with $[f]$ an induced cofibration and $[g]$ and induced trivial fibration. Then there exists an element $p \in P$ such that F_p preserves cofibrations, fibrations and weak equivalences and such that we have a lift of the diagram in $\mathcal{C}$:

$$\begin{array}{ccc} F_p(A) & \xrightarrow{F_p(r)} & F_p(C) \\ \downarrow{\scriptstyle F_p(f)} & & \downarrow{\scriptstyle F_p(g)} \\ F_p(B) & \xrightarrow{F_p(s)} & F_p(D) \end{array}$$

We then have a lift $F_p(B) \to F_p(C)$, hence a lift $B \to C$ in $\mathcal{C}_F$. The proof with $[f]$ an induced trivial cofibration and $[g]$ an induced fibration is *mutatis mutandis* the same. □

Theorem 61. *Suppose that F is an ascending or descending filter. The induced cofibrations, fibrations and weak equivalences form a structure of model category on $\mathcal{C}_F$.*

Theorem 62. *Suppose that the structure is very dense in F. The induced cofibrations, fibrations and weak equivalences form a structure of model category on $\mathcal{C}_F$.*

Definition 63 (Functorial factorization)**.** If we have a factorization of any morphism $f : A \to B$ in two morphisms $f_1 : A \to \text{Fact}(f)$, $f_2 : \text{Fact}(f) \to B$, $f_2 \circ f_1 = f$, we say that this factorization is functorial if for any commutative diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{k} & C \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ B & \xrightarrow{l} & D \end{array}$$

we have an associated morphism $H_{f,g}(k,l) : \mathrm{Fact}(f) \to \mathrm{Fact}(g)$ such that the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "f_1"] & C \arrow[d, "g_1"] \\
\mathrm{Fact}(f) \arrow[r, "H_{f,g}(k,l)"] \arrow[d, "f_2"] & \mathrm{Fact}(g) \arrow[d, "g_2"] \\
B \arrow[r, "l"] & D
\end{tikzcd}$$

and H is functorial in the following sense: for any commutative diagram

$$\begin{tikzcd}
A \arrow[r, "k"] \arrow[d, "f"] & C \arrow[r, "k'"] \arrow[d, "g"] & E \arrow[d, "h"] \\
B \arrow[r, "l"] & D \arrow[r, "l'"] & F
\end{tikzcd}$$

we have

$$H_{g,h}(k',l') \circ H_{f,g}(k,l) = H_{f,h}(k' \circ k, l' \circ l),$$

and for a diagram

$$\begin{tikzcd}
A \arrow[r, "\mathrm{id}_A"] \arrow[d, "f"] & A \arrow[d, "f"] \\
B \arrow[r, "\mathrm{id}_B"] & B
\end{tikzcd}$$

we have

$$H_{f,f}(\mathrm{id}_A, \mathrm{id}_B) = \mathrm{id}_{\mathrm{Fact}(f)}.$$

Definition 64 (functorial factorization compatible with the filter)**.** Let F be an ascending filter. For any morphism $f : A \to B$, we define A_f the set of elements $p \in P$ such that

$$\mathrm{Fact}(F_p(f)) = F_p(\mathrm{Fact}(f))$$
$$F_p(f)_1 = F_p(f_1)$$
$$F_p(f)_2 = F_p(f_2)$$

and such that the morphism associated with the commutative square

$$\begin{tikzcd}
F_p(A) \arrow[r, "\pi_{p,A}"] \arrow[d, "F_p(f)"] & A \arrow[d, "f"] \\
F_p(B) \arrow[r, "\pi_{p,B}"] & B,
\end{tikzcd}$$

is simply the projection, hence

$$H_{F_p(f),f}(\pi_{p,A}, \pi_{p,B}) = \pi_{p,\mathrm{Fact}(f)} : F_p(\mathrm{Fact}(f)) \to \mathrm{Fact}(f).$$

We say that the factorization is compatible with the filter F if for any morphism $[f]$ in $\mathcal{C}_F$, there is a representing element f such that A_f is dense in P.
The definition for a descending filter of functors is dual.

Lemma 65. *Suppose that F is an ascending or descending filter and that a functorial factorization* (Fact, H) *is compatible with F. Then, there is an induced functorial factorization on $\mathcal{C}_F$ such that for any morphism $[f]$, there exists a representing element g such that $[f]_1 = [g_1]$ and $[f]_2 = [g_2]$.*

Proof. We prove it for ascending filters. For any morphism $[f] : A \to B$ in $\mathcal{C}_F$, we choose a representing element $f : F_r(A) \to F_r(B)$ such that the set A_f is dense in P. We then define

$$\begin{aligned} \mathrm{Fact}([f]) &= \mathrm{Fact}(f) \\ [f]_1 &= [f_1] \\ [f]_2 &= [f_2]. \end{aligned}$$

We will now construct the morphisms associated with commutative squares. If for two morphisms $[f], [g]$ in $\mathcal{C}_F$, we have the chosen representing morphisms

$$\begin{aligned} f &: F_r(A) \to F_r(B) \\ g &: F_s(C) \to F_s(D) \end{aligned}$$

and a commutative square

$$\begin{tikzcd} A \arrow[r, "{[k]}"] \arrow[d, "{[f]}"] & C \arrow[d, "{[g]}"] \\ B \arrow[r, "{[l]}"] & D, \end{tikzcd}$$

we take an object $p \in A_g$, $p \leq r, s$, such that taking the image of the diagram with F_p gives a commutative diagram in $\mathcal{C}$, and $q \in A_f$ such that $q \leq p$, then with $m : \mathrm{Fact}(F_q(f)) \to \mathrm{Fact}(F_p(g))$ the morphism given by the exterior rectangle of the diagram

$$\begin{tikzcd} F_q(A) \arrow[r, "F_q(k)"] \arrow[d, "F_q(f)"] & F_q(C) \arrow[r, "\pi_{q,p,F_s(C)}"] \arrow[d, "F_q(g)"] & F_p(C) \arrow[d, "F_p(g)"] \\ F_q(B) \arrow[r, "F_q(l)"] & F_q(D) \arrow[r, "\pi_{q,p,F_s(D)}"] & F_p(D) \end{tikzcd} .$$

We can define

$$H_{[f],[g]}([k], [l]) = [m].$$

This morphism doesn't depend on the choice of p and q: if we take $p' \in A_g$, $q' \in A_f$ such that $p' \leq p$ and $q' \leq p', q$, we obtain a diagram

$$\begin{array}{ccccccc}
F_{q'}(A) & \xrightarrow{F_{q'}(k)} & F_{q'}(C) & \xrightarrow{\pi_{q',p',F_s(C)}} & F_{p'}(C) & \xrightarrow{\pi_{p',p,F_s(C)}} & F_p(C) \\
\downarrow{\scriptstyle F_{q'}(f)} & & \downarrow{\scriptstyle F_{q'}(g)} & & \downarrow{\scriptstyle F_{p'}(g)} & & \downarrow{\scriptstyle F_p(g)} \\
F_{q'}(B) & \xrightarrow{F_{q'}(l)} & F_{q'}(D) & \xrightarrow{\pi_{q',p',F_s(D)}} & F_{p'}(D) & \xrightarrow{\pi_{p',p,F_s(D)}} & F_p(D)
\end{array}$$

and three induced morphisms:

$$\begin{aligned}
m' &: \mathrm{Fact}(F_{q'}(f)) \to \mathrm{Fact}(F_p(g)) \\
m'' &: \mathrm{Fact}(F_{q'}(f)) \to \mathrm{Fact}(F_{p'}(g)) \\
m''' &: \mathrm{Fact}(F_{p'}(g)) \to \mathrm{Fact}(F_p(g)).
\end{aligned}$$

By functoriality we have

$$m''' \circ m'' = m'.$$

We also have the commutative diagram

$$\begin{array}{ccccc}
F_{p'}(C) & \xrightarrow{\pi_{p',p,F_s(C)}} & F_p(C) & \xrightarrow{\pi_{p,F_s(C)}} & F_s(C) \\
\downarrow{\scriptstyle F_{p'}(g)} & & \downarrow{\scriptstyle F_p(g)} & & \downarrow{\scriptstyle g} \\
F_{p'}(D) & \xrightarrow{\pi_{p',p,F_s(D)}} & F_p(D) & \xrightarrow{\pi_{p,F_s(D)}} & F_s(D)
\end{array}$$

from which we deduce

$$\pi_{p,\mathrm{Fact}(g)} \circ m''' = \pi_{p',\mathrm{Fact}(g)}$$

therefore $[m'''] = [\mathrm{id}_{\mathrm{Fact}(g)}]$ and therefore $[m''] = [m']$. Finally, we have the commutative diagram

$$\begin{array}{ccc}
F_{q'}(A) & \xrightarrow{\pi_{q',q,F_r(A)}} & F_q(A) \\
\downarrow{\scriptstyle F_{q'}(f)} & & \downarrow{\scriptstyle F_q(A)} \\
F_{q'}(B) & \xrightarrow{\pi_{q',q,F_r(B)}} & F_q(B)
\end{array}$$

from which we obtain a morphism

$$n : \mathrm{Fact}(F_{q'}(f)) \to \mathrm{Fact}(F_q(f)).$$

By functoriality of and naturality of the projections, we have

$$m \circ n = m'$$

and then by a argument similar as what we did before we conclude that $[n] = [\mathrm{id}_{\mathrm{Fact}(f)}]$, so $[m] = [m']$, thus $[m] = [m'']$.

This construction obviously preserves the identities. The only thing left is to check that it preserves compositions. Suppose that we have a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{[k]} & C & \xrightarrow{[k']} & E \\
\downarrow{\scriptstyle [f]} & & \downarrow{\scriptstyle [g]} & & \downarrow{\scriptstyle [h]} \\
B & \xrightarrow{[l]} & D & \xrightarrow{[l']} & F.
\end{array}$$

We take an element $p \in A_h$ such that taking the image of the diagram by F_p gives a commutative diagram in $\mathcal{C}$, and we take an object $q \in A_g$, an object $r \in A_f$ such that $r \leq q \leq p$. Then we have the induced morphisms $m : \mathrm{Fact}(F_r(f)) \to \mathrm{Fact}(F_q(g))$, $m' : \mathrm{Fact}(F_q(g)) \to \mathrm{Fact}(F_p(h))$, $m'' : \mathrm{Fact}(F_r(f)) \to \mathrm{Fact}(F_p(h))$, and by functoriality and naturality of the projections we have $m' \circ m = m''$, thus $[m'] \circ [m] = [m'']$. $\square$

Proposition 66. *Suppose that F is an ascending or descending filter and that the two weak factorization systems* $(\mathrm{Cof} \cap \mathrm{WE}, \mathrm{Fib})$ *and* $(\mathrm{Cof}, \mathrm{Fib} \cap \mathrm{WE})$ *are functorial. If they are compatible with F, then $\mathcal{C}_F$ has also functorial weak factorization systems.*

Proof. Direct consequence of the previous lemma and the fact that a morphism $[f]$ is a cofibration, a fibration or a weak equivalence if and only if it is an equivalence class containing a cofibration, a fibration or a weak equivalence, respectively. $\square$

Sadly, this condition seems to be rarely verified. Notably, the functorial factorizations obtained by small object argument don't work well with filter of functors.

Proposition 67. *If $\mathcal{C}$ has the pushout-product property, $\mathcal{C}_F$ has the pushout-product property too.*

Proof. For any morphisms $i, j \in \mathrm{Mor}\,\mathcal{C}$, $Q_F(i \square j) = Q_F(i) \square Q_F(j)$, because Q_F preserves finite limits and finite colimits, hence we can conclude by the fact that Q_F preserves cofibrations and weak equivalences. $\square$

Proposition 68. *If $\mathcal{C}$ is left or right proper, $\mathcal{C}_F$ is left or right proper, respectively.*

Proof. Very similar to the previous proposition. $\square$

Definition 69. A model category $\mathcal{C}$ is said to have cofibrant replacement of finite diagrams if for any finite category $\mathcal{D}$, any functor $G : \mathcal{D} \to \mathcal{C}$, there exists a functor $\tilde{G} : \mathcal{D} \to \mathcal{C}$ which is pointwise cofibrant and such that there exists a natural transformation $\alpha : \tilde{G} \to G$ which is a pointwise trivial fibration.

The definition of having fibrant replacement of finite diagrams is dual.

Proposition 70. *If $\mathcal{C}$ has cofibrant replacement of finite diagrams, then $\mathcal{C}_F$ too. Dually, if $\mathcal{C}$ has fibrant replacement of finite diagrams, then $\mathcal{C}_F$ too.*

Proof. We prove it for cofibrant replacements. Let $\mathcal{D}$ be a finite category, $G : \mathcal{D} \to \mathcal{C}_F$ a functor. Because the diagram is finite, we can choose an element $p \in P$ such that for any morphism $f : a \to b$ in $\mathcal{D}$, we can choose a representing morphism $f' : F_p(G(a)) \to F_p(G(b))$ of $G(f)$ and obtain a functor $G' : \mathcal{D} \to \mathcal{C}$. Then, there exists a functor $\tilde{G} : \mathcal{D} \to \mathcal{C}$ with a pointwise trivial fibration $\alpha : \tilde{G} \to G'$, which gives us a pointwise trivial cofibration $\alpha' : Q_f \circ \tilde{G} \to G$. Moreover, Q_F preserves the cofibrant objects, hence it is a cofibrant replacement. $\square$

Definition 71 (Homotopy limits and colimits)**.** For any model category $\mathcal{C}$, any small category $\mathcal{D}$, the category $\mathcal{C}^{\mathcal{D}}$ has induced weak equivalences, which are the pointwise weak equivalences. The homotopy limit is defined as the right derived functor of the functor $\lim : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$, when it exists. Dually, the homotopy colimit is defined as the left derived functor of the functor $\operatorname{colim} : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$, when it exists.

This definition is not very useful to us, because most of the constructions of these derived functors use functorial factorizations, and in this paper we don't assume that our weak factorization systems are functorial. We will therefore use the following definition, which indeed gives homotopy colimits when we have functorial cofibrant replacements:

Definition 72 (Projective homotopy colimit)**.** For $\mathcal{C}$ a model category, $\mathcal{D}$ a finite category such that the projective model structure exists on $\mathcal{C}^{\mathcal{D}}$. Then for any functor $G : \mathcal{D} \to \mathcal{C}$, any projective cofibrant replacement $\tilde{G}$, the projective homotopy colimit $\operatorname{hocolim} G$ is defined as the weak equivalence class of the colimit of $\tilde{G}$.

Lemma 73. *The projective homotopy colimit is well-defined.*

Proof. Two projective cofibrant replacements are weak equivalent. Moreover, we have the functor $\operatorname{colim} : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}$ has a right adjoint $\mathcal{C} \to \mathcal{C}^{\mathcal{D}}$ which associates to each object the constant functor at this object, and this right adjoint preserves fibrations and weak equivalences, hence colim preserves cofibrations and trivial cofibrations, thus by Ken Brown's lemma it preserves weak equivalences between cofibrant objects. □

Lemma 74. *If $\mathcal{C}$ is cofibrantly generated, $\mathcal{D}$ a finite category, $F^{\mathcal{D}}$ the induced filter of functors obtained by Proposition 26, if the elements $p \in P$ such that F_p preserves finite colimits and cofibrations are dense in P, then the projective model structure on $\mathcal{C}^{\mathcal{D}}$ is dense in $F^{\mathcal{D}}$.*

Proof. If a functor F_p preserves fibrations, $F_p^{\mathcal{D}}$ will also preserve fibrations, because fibrations are just pointwise fibrations in $\mathcal{C}^{\mathcal{D}}$. For the same reason, if F_p preserves weak equivalences, $F_p^{\mathcal{D}}$ will also preserve weak equivalences. Let I be a set of generating cofibrations in $\mathcal{C}$. If F_p preserves cofibrations, $F_p^{\mathcal{D}}$ will send free I-cell complexes on cofibrations, hence it will send the relative free I-cell complexes on cofibrations, and therefore cofibrations on cofibrations. Thus, the projective model structure is dense in $F^{\mathcal{D}}$. □

Proposition 75. *If $\mathcal{C}$ is cofibrantly generated, F ascending or descending and if the elements $p \in P$ such that F_p preserves colimits and cofibrations are dense in P, then Q_F preserves the finite projective homotopy colimits.*

Proof. Let $\mathcal{D}$ be a finite category. By the previous lemma and Proposition 26, the category $(\mathcal{C}^{\mathcal{D}})_{F^{\mathcal{D}}}$ has a model category structure and $\mathcal{C}_F^{\mathcal{D}}$ is equivalent to this category, so there is a model category structure on $\mathcal{C}_F^{\mathcal{D}}$. Moreover, by the commutative triangle

$$\begin{tikzcd}
\mathcal{C}^{\mathcal{D}} \arrow[r, "Q_F^{\mathcal{D}}"] \arrow[d, "Q_{F^{\mathcal{D}}}"] & \mathcal{C}_F^{\mathcal{D}} \\
(\mathcal{C}^{\mathcal{D}})_{F^{\mathcal{D}}}, \arrow[ur, "i"] &
\end{tikzcd}$$

it is clear that fibrations are the pointwise induced fibrations and that weak equivalences are the pointwise induced weak equivalences. Because the functor $Q_{F^{\mathcal{D}}}$ preserves the cofibrant replacements, Q_F preserves the projective homotopy colimits of $\mathcal{D}$-diagrams. □

Proposition 76. *Let $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be a Quillen pair and F be a filter on* Subobj(e) *respecting the conditions of Proposition 15, such that the model category structure of $\mathcal{C}$ is dense in F and the model category structure of $\mathcal{D}$ is dense in F'. Then, $\hat{L} : \mathcal{C}_F \rightleftarrows \mathcal{D}_{F'} : \hat{R}$ is a Quillen pair.*

Proof. Let $[f]$ be a cofibration in $\mathcal{C}_F$, then there exists U in F such that $f \times U$ is a cofibration in $\mathcal{C}$. Hence $L(f \times U) \cong L(f) \times L(U)$ is a cofibration in $\mathcal{D}$, and $\hat{L}([f]) = [L(f)]$ is a cofibration in $\mathcal{D}_{F'}$. Let $[g]$ be a fibration in $\mathcal{D}_{F'}$, there exists U in F' such that $g \times U$ is a fibration in $\mathcal{D}$, hence $R(g \times U) \cong R(g) \times R(U)$ is a fibration in $\mathcal{C}$, and $\hat{R}([g]) = [R(g)]$ is a fibration in $\mathcal{C}_F$. □

Proposition 77. *An element A is cofibrant in $\mathcal{C}_F$ if and only if there exists an element $p \in P$ such that the canonical map $F_p(\emptyset) \to F_p(A)$ is a cofibration.*

An element A is fibrant in $\mathcal{C}_F$ if and only if there exists an element $p \in P$ such that the canonical map $F_p(A) \to F_p(e)$ is a fibration.

Proof. It is a direct consequence of the definitions of cofibrations and fibrations in the induced model structure and of the fact that the initial and terminal objects are preserved by Q_F. □

Proposition 78. *Suppose that the elements $p \in P$ such that F_p preserves finite coproducts and weak equivalences are dense in P. For two morphisms $[f], [g] : A \to B$ in $\mathcal{C}$, $[f]$ is left homotopic to $[g]$ if and only if there exists an element $p \in P$ such that $F_p(f)$ is left homotopic to $F_p(g)$.*

Dually, if the elements $p \in P$ such that F_p preserves finite products and weak equivalences are dense in P, $[f]$ is right homotopic to $[g]$ if and only if there exists $p \in P$ such that $F_p(f)$ is right homotopic to $F_p(g)$.

Proof. We will prove it for the left homotopy. The proof of the 'if' part is very similar to the proof of Proposition 67 and doesn't need the condition given in the proposition. We will prove the 'only if' part. We can lift the left homotopy as the following commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd}
F_p(A) & \\
F_p(A \coprod A) \arrow[u] \arrow[d] \arrow[r] & F_p(\mathrm{Cyl}) \arrow[ul, "\sim"'] \arrow[dl] \\
F_p(B) &
\end{tikzcd}$$

We can ask without loss of generality that F_p preserves finite coproducts, that the map $F_p(A \coprod A) \to F_p(A)$ is the codiagonal map, hence we have a left homotopy in $\mathcal{C}$. □

Proposition 79. *Suppose that F is ascending or descending. Then there is a descending, respectively ascending filter of functors $\tilde{F}$ on* $\mathrm{Ho}\ \mathcal{C}$ *with a unique isomorphism* $\mathrm{Ho}\ \mathcal{C}_F \cong (\mathrm{Ho}\ \mathcal{C})_{\tilde{F}}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{C} \arrow[d] \arrow[dr] & \\
\mathrm{Ho}\ \mathcal{C}_F \arrow[r, "\cong"] & (\mathrm{Ho}\ \mathcal{C})_{\tilde{F}}.
\end{tikzcd}$$

Proof. Let $p \in P$ be an element such that F_p preserves weak equivalences. Then, we have an induced functor $\tilde{F}_p : \mathrm{Ho}\ \mathcal{C} \to \mathrm{Ho}\ \mathcal{C}$. Because these elements are dense in P, the associated functors form a filter of functors. If the filter is ascending, the projections between the functors preserving weak equivalences will also induce natural transformations between the induced functors, hence $\tilde{F}$ will also be ascending. If the filter is descending, the induced functor $\tilde{F}$ will be descending by the dual argument. The functors $\mathcal{C} \to (\mathrm{Ho}\ \mathcal{C})_{\tilde{F}}$ and $C \to \mathrm{Ho}\ \mathcal{C}_F$ correspond to the localization at all weak equivalences and all the projections, therefore there is a unique isomorphism $\mathrm{Ho}\ \mathcal{C}_F \cong (\mathrm{Ho}\ \mathcal{C})_{\tilde{F}}$ and we can conclude. □

4 Simplicial model category

In this section, we will define the notion of simplicial model category, and we will show that if the simpliciality is compatible with a filter on $\mathrm{Subobj}(e)$, it is preserved by the filter quotient. We will also define the notion of finite descent, and we will show that if the functor Q_F preserves finite colimits and $\mathcal{C}$ has finite descent, $\mathcal{C}_F$ has finite descent.

We will designate the category of simplicial sets by $\mathrm{s}\mathcal{S}\mathrm{et}$ and the category of finite simplicial sets by $\mathrm{fs}\mathcal{S}\mathrm{et}$. We will write $\emptyset$ to designate the initial object in $\mathcal{C}$, and $\emptyset_{ss}$ to designate the initial object in $\mathrm{s}\mathcal{S}\mathrm{et}$.

4.1 Simpliciality

In this part, we will define the notion of a simplicial model category. We will use a definition which is less restrictive than the one usually used, because we don't have infinite products and coproducts, but this notion will be sufficient to define finite homotopy limits and colimits and to have the full subcategory of bifibrant objects enriched on Kan complexes. We will show that the property of being simplicial is preserved by the filter quotient under some reasonable condition.

Definition 80 (Simplicial model category)**.** A model category $\mathcal{C}$ with a simplicial enrichment

$$\mathrm{Map} : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{s}\mathcal{S}\mathrm{et}$$

and with a tensor and a cotensor functors

$$- \otimes - : \mathrm{s}\mathcal{S}\mathrm{et} \times \mathcal{C} \to \mathcal{C}$$

$$-^{-} : \mathcal{C} \times \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}} \to \mathcal{C}$$

is said to be simplicial if the following conditions are verified:

1. For any $A, B \in \mathrm{Obj}\mathcal{C}$, any *finite* simplicial set K, we have the natural isomorphisms

$$\mathrm{Map}(K, \mathrm{Map}(A, B)) \cong \mathrm{Map}(K \otimes A, B) \cong \mathrm{Map}(A, B^K).$$

2. If $i : A \to B$ is a cofibration and $p : X \to Y$ is a fibration in $\mathcal{C}$, the induced morphism of simplicial sets

$$\mathrm{Map}(B, X) \to \mathrm{Map}(A, X) \times_{\mathrm{Map}(A,Y)} \mathrm{Map}(B, Y)$$

   is a Kan fibration, and it is a trivial fibration if i or p is also a weak equivalence.

This definition is less restrictive than the one usually used, in which the isomorphisms must be true for all simplicial sets, but it is sufficient to define finite homotopy limits and colimits and to have the full subcategory of bifibrant objects enriched on Kan complexes, and this definition still works in a context without infinite limits and colimits. In fact, it corresponds to the original definition that Daniel Quillen gave of a closed simplicial model category [Qui67, Definition 2.2.1.].

Lemma 81. *For any non-empty simplicial set K, any object A in $\mathcal{C}$, any subobject U of e, there exists a morphism $A \to U$ if and only if there exists a morphism $K \otimes A \to U$. We also have $\emptyset_{ss} \otimes U \cong \emptyset$.*

Proof. There is map $K \to \Delta[0]$, and a map $\Delta[0] \to K$, therefore there is a map $K \otimes A \to A$ and a map $A \to K \otimes A$.

For any objects A, B in $\mathcal{C}$, we have the bijection $\mathcal{C}(\emptyset_{ss} \otimes A, B) \cong \mathrm{s}\mathcal{S}\mathrm{et}(\emptyset_{ss}, \mathrm{Map}(A, B))$ which contains exactly one element, therefore $\emptyset_{ss} \otimes A \cong \emptyset$. ☐

Lemma 82. *If $\mathcal{C}$ is a simplicial model category, then for any non-empty simplicial set K, any subobject U of e, $U^K \cong U$ and $U^{\emptyset_{ss}} \cong e$.*

Proof. The functor $-^K$ is a right adjoint, therefore it preserves the terminal object and the monomorphisms, and U^K is a subobject of e. Moreover, there is a map $K \to \Delta[0]$ and a map $\Delta[0] \to K$ in sSet, therefore there is a map $U \to U^K$ and a map $U^K \to U$. We have the bijections $\mathcal{C}(A, U^{\emptyset_{ss}}) \cong \mathcal{C}(\emptyset_{ss} \otimes A, U) \cong \mathcal{C}(\emptyset, U)$ which contains exactly one element, therefore $U^{\emptyset_{ss}} \cong e$. $\square$

Definition 83 (Compatibility with simpliciality)**.** For F an ascending filter of functors and $\mathcal{C}$ a category with a model structure very dense in F, for any $p \in P$ such that F_p preserves cofibrations, fibrations and weak equivalences, we define $\mathcal{C}_p$ the full subcategory with objects the image of F_p. Because this category is equivalent to the category obtained by quotienting by F_p, it has an induced model structure. We will also define the functor $- \otimes_p - : \mathcal{C} \times \text{sSet} \to \mathcal{C}$ as

$$A \otimes_p K = F_p(A \otimes K),$$

the functor $-^{-_p} : \mathcal{C} \times \text{sSet} \to \mathcal{C}$ as

$$A^{K_p} = F_p(A^K)$$

and the functor $\text{Map}_p : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{sSet}$ as

$$\text{Map}_p(A, B)_n = \mathcal{C}_p(F_p(A \otimes \Delta[n]), F_p(B)),$$

with the degeneracy and boundary maps induced by $\Delta[n]$. For any objects $A, B \in \text{Obj}\,\mathcal{C}$, we obtain a natural map $r_{p,A,B} : \text{Map}(A, B) \to \text{Map}_p(A, B)$ because the morphism

$$\mathcal{C}(A \otimes \Delta[n], B) \to \mathcal{C}(F_p(A \otimes \Delta[n]), F_p(B))$$

which sends a morphism f on $F_p(f)$ commutes with the degeneracy and boundary maps. We say that p is compatible with simpliciality if the three following conditions are verified:

1. For any simplicial set K, any object $A \in \text{Obj}\,\mathcal{C}$, the morphism

$$F_p(K \otimes \pi_{p,A}) : F_p(K \otimes F_p(A)) \to F_p(K \otimes A)$$

is an isomorphism.

2. For any simplicial set K, any object $A \in \text{Obj}\,\mathcal{C}$, the morphism

$$F_p(\pi_{p,A}^K) : F_p(F_p(A)^K) \to F_p(A^K)$$

is an isomorphism.

3. The functor Map_p has a composition, which restricted to $\mathcal{C}_p \times \text{sSet}$ with the tensor and cotensor defined above gives a structure of simplicial model category on $\mathcal{C}_p$, and such that the composition commutes with $r_{A,B}$, meaning that for any $n \in N$, we have the commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}(A,B)\times\mathrm{Map}(B,C) & \longrightarrow & \mathrm{Map}(A,C) \\
\Big\downarrow{\scriptstyle r_{p,A,B}\times r_{B,C}} & & \Big\downarrow{\scriptstyle r_{p,A,C}} \\
\mathrm{Map}_p(A,B)\times Map_p(B,C) & \longrightarrow & \mathrm{Map}_p(A,C)
\end{array}$$

and $r_{A,A}$ sends the neutral element on the neutral element.

Then, the filter F is compatible with simpliciality if the elements in P which are compatible with simpliciality are dense. We say that a descending filter of functors on $\mathcal{C}$ is compatible with simpliciality if the opposite filter is compatible with simpliciality.
To simplify notations in this section, we will identify $F_p(K\otimes F_p(A))$ with $F_p(K\otimes A)$ and $F_p(K\otimes\pi_{p,A})$ with $\mathrm{id}_{F_p(K\otimes A)}$, and similarly for the cotensor.

For the rest of this section, we will suppose that the filter of functors F is ascending or descending and that it is compatible with simpliciality.

Lemma 84. *For any finite simplicial sets K, L, any object A in $\mathcal{C}$, there is a natural isomorphism $K\otimes(L\otimes A)\cong(K\times L)\otimes A$.*

Proof. If K and L are finite simplicial sets, $K\times L$ is also finite. For any objects X, Y in $\mathcal{C}$, we then have the natural isomorphisms

$$\begin{aligned}
\mathcal{C}((K\times L)\otimes X, Y) &\cong \mathrm{s}\mathcal{S}\mathrm{et}(K\times L, \mathrm{Map}(X,Y)) \\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}(K, \mathrm{Map}(L, \mathrm{Map}(X,Y))) \\
&\cong \mathrm{s}\mathcal{S}\mathrm{et}(K, \mathrm{Map}(L\otimes X, Y)) \\
&\cong \mathcal{C}(K\otimes(L\otimes X), Y)
\end{aligned}$$

and we conclude by the Yoneda lemma. □

Lemma 85. *For any morphisms $f: F_p(A)\to F_p(B)$, $g: F_q(A)\to F_q(B)$, if $[f]_A=[g]_A$, then for any simplicial set K, we have $[f\otimes K]_{A\otimes K}=[g\otimes K]_{A\otimes K}$, and $[f^K]_{A^K}=[g^K]_{A^K}$.*

Proof. Let $r\le p,q$ such that $F_r(f)=F_r(g)$. Then,

$$\begin{aligned}
F_r(f\otimes K) &= F_r(F_r(f)\otimes K) \\
&= F_r(F_r(g)\otimes K) \\
&= F_r(g\otimes K)
\end{aligned}$$

and

$$\begin{aligned}
F_r(f^K) &= F_r(F_r(f)^K) \\
&= F_r(F_r(g)^K) \\
&= F_r(g^K)
\end{aligned}$$

□

Proposition 86. *There are some induced functors* $\mathrm{Map}_F : \mathcal{C}_F^{\mathrm{op}} \times \mathcal{C}_F \to \mathrm{sSet}$*,* $- \otimes_F - : \mathcal{C}_F \times \mathrm{sSet} \to \mathcal{C}_F$*,* $-^- : \mathcal{C}_F \times \mathrm{sSet}^{\mathrm{op}} \to \mathcal{C}_F$*, with the isomorphisms for any* K *finite*

$$\mathrm{Map}(K, \mathrm{Map}_F(A, B)) \cong \mathrm{Map}_F(A \otimes K, B) \cong \mathrm{Map}_F(A, B^K)$$

natural in K, A *and* B*.*

Proof. We define $A \otimes_F K = A \otimes K$, $[f]_A \otimes_F K = [f \otimes K]_{A \otimes K}$, A^K defined as in $\mathcal{C}$, and $([f]_A)^K = [f^K]_{A^K}$. It is well-defined by Lemma 85. We also define $\mathrm{Map}_F(A, B)_n = \mathcal{C}_F(A \otimes_F \Delta[n], B)$.

Because F is a filtered category, Map_F is defined as a filtered colimit, because we have

$$\mathrm{Map}_F(A, B)_n \cong \operatorname*{colim}_{p \in P} \mathcal{C}(F_p(A \otimes \Delta[n]), F_p(B)).$$

Let $A \subseteq P$ be the subset of elements compatible with simpliciality. Using the fact that any finite simplicial set is compact and Lemma 84, we obtain

$$\begin{aligned}
\mathrm{Map}(K, \mathrm{Map}_F(A, B))_n &\cong \mathrm{sSet}(\Delta[n] \times K, \mathrm{Map}_F(A, B)) \\
&\cong \operatorname*{colim}_{p \in A} \mathrm{sSet}(\Delta[n] \times K, \mathrm{Map}_p(A, B)) \\
&\cong \operatorname*{colim}_{p \in A} \mathrm{sSet}(\Delta[n] \times K, \mathrm{Map}_p(F_p(A), F_p(B)) \\
&\cong \operatorname*{colim}_{p \in A} \mathcal{C}(F_p(A) \otimes_p (\Delta[n] \times K), F_p(B)) \\
&\cong \operatorname*{colim}_{p \in A} \mathcal{C}((F_p(A) \otimes_p K) \otimes_p \Delta[n], F_p(B)) \\
&\cong \operatorname*{colim}_{p \in A} \mathcal{C}(F_p((A \otimes K) \otimes \Delta[n]), F_p(B)) \\
&\cong \mathcal{C}_F((A \otimes K) \otimes \Delta[n], B) \\
&\cong \mathrm{Map}(A \otimes_F K, B)_n
\end{aligned}$$

where the isomorphisms are natural with respect to $\Delta[n]$, therefore they commute with the boundary and degeneracy maps, hence it is an isomorphism of simplicial sets. We also have, for any simplicial set K,

$$\begin{aligned}
\mathrm{Map}_F(A \otimes_F K, B) &\cong \operatorname*{colim}_{p \in p} \mathrm{Map}(F_p(A \otimes K), F_p(B)) \\
&\cong \operatorname*{colim}_{p \in A} \mathrm{Map}_p(A \otimes_p K, B) \\
&\cong \operatorname*{colim}_{p \in A} \mathrm{Map}_p(A, B^{K_p}) \\
&\cong \operatorname*{colim}_{p \in P} \mathrm{Map}(F_p(A), F_p(B^K)) \\
&\cong \mathrm{Map}_F(A, B^K)
\end{aligned}$$

$\square$

Proposition 87. *If $[i] : A \to B$ is a cofibration and $[p] : X \to Y$ a fibration in $\mathcal{C}_F$, the induced morphism of simplicial sets*

$$\mathrm{Map}_F(B, X) \to \mathrm{Map}_F(A, X) \times_{\mathrm{Map}_F(A,Y)} \mathrm{Map}_F(B, Y)$$

is a Kan fibration, and it is a trivial fibration if $[i]$ or $[p]$ is also a weak equivalence.

Proof. The filtered colimits commute with finite limits in $\mathcal{S}\mathrm{et}$, therefore they commute in any presheaf category and notably in $\mathrm{s}\mathcal{S}\mathrm{et}$. Moreover, any subset $A \subseteq P$ which is dense is also filtered as a full subcategory, therefore we only need to show that a filtered colimit of Kan fibrations, respectively of trivial Kan fibrations, is a Kan fibration, respectively a trivial Kan fibration. But this is a direct consequence of the fact that the class of morphisms which have the right lifting property with respect to a morphism $C \to D$ between compact objects is closed under filtered colimits. □

Proposition 88. *The functor Map_p has an induced composition which gives a simplicial enrichment on $\mathcal{C}_F$.*

Proof. It is a direct consequence of the condition 3 of the definition of compatibility with simpliciality: for $f \in \mathrm{Map}(A, B)_n$, $g \in \mathrm{Map}(B, C)_n$, which can be described as morphisms $A \otimes \Delta[n] \to B$ and $B \otimes \Delta[n] \to C$, for any element $p \in P$ which is compatible with simpliciality, we have $F_p(f) \in \mathrm{Map}_p(A, B)_n$, $F_p(g) \in \mathrm{Map}_p(B, C)_n$, and $F_p(g \circ f) = F_p(g) \circ F_p(f)$, hence Map_F has a well-defined composition. □

Theorem 89. *The category $\mathcal{C}_F$ is a simplicial model category.*

4.2 Homotopy limits and descent

In this part, we will define the finite homotopy limits and colimits and the notion of finite descent. We will show that everything is preserved by the filter quotient under reasonable conditions.

Definition 90 (Finite homotopy colimits)**.** Let $\mathcal{C}$ have cofibrant replacement of finite diagrams, let $\mathcal{D}$ be a finite category and $H : \mathcal{D} \to \mathcal{C}$ a functor. We choose a cofibrant replacement G of H, and the homotopy colimit $\operatorname{hocolim} H$ is defined as the colimit of the diagram

$$\coprod_{(f:a\to b)\in \mathrm{Mor}\,\mathcal{D}} \mathrm{N}(b \downarrow \mathcal{D})^{\mathrm{op}} \otimes G(a) \underset{\psi}{\overset{\phi}{\rightrightarrows}} \coprod_{a\in \mathrm{Obj}\,\mathcal{D}} \mathrm{N}(a \downarrow \mathcal{D})^{\mathrm{op}} \otimes G(a)$$

where the morphism ϕ on the summand $f : a \to b$ is induced by the morphism

$$\mathrm{N}(b \downarrow \mathcal{D})^{\mathrm{op}} \otimes f : \mathrm{N}(b \downarrow \mathcal{D})^{\mathrm{op}} \otimes G(a) \to \mathrm{N}(b \downarrow \mathcal{D})^{\mathrm{op}} \otimes G(b)$$

and the morphism ψ on the summand $f : a \to b$ is induced by the morphism

$$\mathrm{N}(f^{\mathrm{op}}) \otimes G(a) : \mathrm{N}(b \downarrow \mathcal{D})^{\mathrm{op}} \otimes G(a) \to \mathrm{N}(a \downarrow \mathcal{D})^{\mathrm{op}} \otimes G(a)$$

Definition 91 (Finite homotopy limits)**.** Let $\mathcal{C}$ have fibrant replacement of finite diagrams, $\mathcal{D}$ be a finite category and $H : \mathcal{D} \to \mathcal{C}$ a functor. We choose a fibrant replacement G of H and the homotopy limit $\operatorname{holim} H$ is defined as the limit of the diagram

$$\prod_{a \in \mathrm{Obj}\, \mathcal{D}} G(a)^{\mathrm{N}(\mathcal{D}\downarrow a)} \overset{\phi}{\underset{\psi}{\rightrightarrows}} \prod_{(f:a\to b)\in \mathrm{Mor}\, \mathcal{D}} G(b)^{\mathrm{N}(\mathcal{D}\downarrow a)}$$

where the morphism ϕ on the factor $f : a \to b$ is defined as the composition of the projection with the morphism

$$f^{\mathrm{N}(\mathcal{D}\downarrow a)} : G(a)^{\mathrm{N}(\mathcal{D}\downarrow a)} \to G(b)^{\mathrm{N}(\mathcal{D}\downarrow a)}$$

and the morphism ψ on the factor $f : a \to b$ is defined as the composition of the projection with the morphism

$$G(b)^{\mathrm{N}(f)} : G(b)^{\mathrm{N}(\mathcal{D}\downarrow b)} \to G(b)^{\mathrm{N}(\mathcal{D}\downarrow a)}$$

This construction is homotopy invariant, hence the choice of replacement matters only up to weak equivalence (see [Hir03, 18.5]).

Proposition 92. *If Q_F preserves finite limits, it preserves the finite homotopy limits. Dually, if Q_F preserves finite colimits, it preserves the finite homotopy colimits.*

Proof. The functor Q_F preserves finite limits, the cotensor functor and the fibrant replacements, therefore it also preserves the finite homotopy limits, and dually for finite homotopy colimits. $\square$

We will now define the notion of finite descent, and we will show that if it is easily preserved. The property of descent was first introduced by Charles Rezk [Rez10, 4.5].

Definition 93 (Finite descent)**.** For $\mathcal{D}$ a finite category, $G, H : \mathcal{D} \to \mathcal{C}$ two functors, a natural transformation $\alpha : G \to H$ is called equifibered if for any morphism $f : a \to b$ in $\mathcal{D}$, the square

$$\begin{tikzcd} G(a) \arrow[r, "G(f)"] \arrow[d, "\alpha_a"] & G(b) \arrow[d, "\alpha_b"] \\ H(a) \arrow[r, "H(f)"] & H(b) \end{tikzcd}$$

is a homotopy pullback square.
A model category $\mathcal{C}$ is said to have finite descent if it verifies the two following conditions:

(P1) Let $\mathcal{D}$ be a finite category, $H : \mathcal{D} \to \mathcal{C}$ a functor, $\tilde{H} = \operatorname{hocolim} H$, and $f : \tilde{G} \to \tilde{H}$ in $\mathcal{C}$ a morphism. We define a functor $G : \mathcal{D} \to \mathcal{C}$ as

$$G(a) = H(a) \times^h_{\tilde{H}} \tilde{G}$$

for any $a \in \mathrm{Obj}\, \mathcal{D}$. Then, the induced morphism $\operatorname{hocolim} G \to \tilde{G}$ is a weak equivalence.

(P2) Let $\mathcal{D}$ be a finite category, $G, H : \mathcal{D} \to \mathcal{C}$ two functors and $\alpha : G \to H$ an equifibered natural transformation. We define $\tilde{G} = \operatorname{hocolim} G$ and $\tilde{H} = \operatorname{hocolim} H$, and we have an induced morphism $\tilde{G} \to \tilde{H}$. Then the induced morphism $G(a) \to H(a) \times^h_{\tilde{H}} \tilde{G}$ is a weak equivalence for any object $a \in \operatorname{Obj} \mathcal{D}$.

Theorem 94. *Suppose that $\mathcal{C}$ has finite descent. If Q_F preserves finite limits and colimits, then $\mathcal{C}_F$ has finite descent.*

Proof. Any finite diagram in $\mathcal{C}_F$ is isomorphic to a diagram which factors through Q_F, hence we only need to show the property of finite descent for functors of the form $Q_F \circ G$ and $Q_F \circ H$. But the functor Q_F preserves weak equivalences, and it preserves finite homotopy limits and colimits by Proposition 92, hence $\mathcal{C}_F$ has finite descent. $\square$

5 Categories of functors

In this section we will see how to define the ascending partial order associated with a small category $\mathcal{D}$, and that filters on this partial order induces ascending filters of functors on categories of the form $\mathcal{C}^{\mathcal{D}}$. We will give conditions to have an induced model structure on the category when we quotient out by these filters for special cases: the injective model structure, the Reedy model structure when $\mathcal{D}$ is a Reedy category and the projective model structure when $\mathcal{C}$ is cofibrantly generated.

5.1 Partial order associated with a small category

In this part, we will define the notion of ascending partial order associated with a small category $\mathcal{D}$, and we will define the ascending filter of functors induced on a category $\mathcal{C}^{\mathcal{D}}$. We will show that if $\mathcal{C}$ has a strict initial object, the injective model structure induces a model structure on the quotiented category. We will define the notion of a Reedy connected filter, and we will show that there is an induced model structure on the quotiented category if F is Reedy connected.

Definition 95 (The ascending partial order associated with a small category)**.** Let $\mathcal{D}$ be a small category. A subset $A \subseteq \operatorname{Obj} \mathcal{D}$ is said to be left closed under morphisms if for any $a \in A$, any b in $\mathcal{D}$, if there exists a morphism $f : a \to b$, then $b \in A$. We define $P(\mathcal{D})$ the ascending partial order associated with $\mathcal{D}$ as the set of subsets left closed under morphisms, with the order given by the inclusion.

For any $p \in P(\mathcal{D})$, we have an induced functor $F_A : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$ defined as $F_A(G)(a) = \emptyset$ if $a \in p$ and $F_A(G)(a) = G(a)$ otherwise, for any functor G, and it is defined in the obvious way for the morphisms. For F be a filter on $P(\mathcal{D})$, this gives an induced ascending filter of functors on $\mathcal{C}^{\mathcal{D}}$, that we will also write F to simplify notations.

Definition 96 (well-initialized category)**.** A category $\mathcal{C}$ is called well-initialized if its initial object is strict and distinct from the terminal object. In this situation, for a small category $\mathcal{D}$, we have an inclusion $i : P(\mathcal{D}) \to \operatorname{Subobj}(c_e)$, where $c_e : \mathcal{D} \to \mathcal{C}$ is the constant functor at e. This inclusion is defined as $i(A)(a) = \emptyset$ if $a \in A$, and $i(A)(a) = e$ otherwise.

Proposition 97. *Let $\mathcal{C}$ be a well-initialized category with finite limits, and* $\mathrm{Subobj}(e) = \{\emptyset, e\}$*. Then, for any small category $\mathcal{D}$, the canonical inclusion $i : P(\mathcal{D}) \to \mathrm{Subobj}(c_e)$ is an isomorphism.*

Proof. Limits in $\mathcal{C}^{\mathcal{D}}$ are computed point-wise, and in any category, a morphism $f : A \to B$ is a monomorphism if and only if the diagram

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}_A} & A \\ \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle f} \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback square, hence a morphism in $\mathcal{C}^{\mathcal{D}}$ is a monomorphism if and only if it is a point-wise monomorphism. So for any functor F which is a subobject of c_e, for any object d in $\mathcal{D}$, either $F(d) = e$ or $F(d) = \emptyset$. Moreover, because $\emptyset < e$, there is no morphism from e to $\emptyset$, and the set $A = \{d \in \mathrm{Obj}\ \mathcal{D} \mid F(d) = e\}$ must be closed under morphisms, hence $A \in P(\mathcal{D})$, and $F = i(A)$. For the same reason, for any $A, B \in P(\mathcal{D})$, $i(A) \leq i(B)$ if and only if $A \leq B$. $\square$

Definition 98 (Fibrantly regular model structure)**.** For $\mathcal{D}$ a small category, a model structure on $\mathcal{C}^{\mathcal{D}}$ is said to be fibrantly regular if weak equivalences are exactly the pointwise weak equivalences and fibrations must be pointwise fibrations.

Lemma 99. *Suppose that $\mathcal{C}$ has all infinite coproducts. Let $\mathcal{D}$ be a small category, and suppose that there is a fibrantly regular model structure on $\mathcal{C}^{\mathcal{D}}$. Then for any element $p \in P(\mathcal{D})$, any object $a \in p$, any p-fibration α, α_a is a fibration.*

Proof. For any object $a \in \mathrm{Obj}\mathcal{D}$, any object $A \in \mathrm{Obj}\mathcal{C}$, we define the functor $\mathcal{D}(a, -) \times A$ as

$$\mathcal{D}(a, b) \times A = \coprod_{\mathcal{D}(a,b)} A$$

with the obvious maps between the objects. Then, for a morphism $f : A \to B$ in $\mathcal{C}$, the free cell generated by f at a, $\mathcal{D}(a, -) \times f : \mathcal{D}(a, -) \times A \to \mathcal{D}(a, -) \times B$, is point-wise defined as

$$\mathcal{D}(a, b) \times f = \coprod_{\mathcal{D}(a,b)} f : \coprod_{\mathcal{D}(a,b)} B \to \coprod_{\mathcal{D}(a,b)} C.$$

A natural transformation β has the right lift property with respect to the free cell $\mathcal{D}(a, -) \times f$ if and only if β_a as the right lifting property with respect to f. Therefore, if f is a trivial cofibration, $\mathcal{D}(a, -) \times f$ has the left lifting property with respect to all natural transformations which are fibration at point a, and notably with respect to all pointwise fibrations. Therefore, $\mathcal{D}(a, -) \times f$ is a trivial cofibration. Thus, if $a \in p$, any p-fibration must be a fibration at point a. $\square$

Definition 100 (Left spiky filter)**.** Let $\mathcal{D}$ be a partial order, seen as a category. An element $p \in P(\mathcal{D})$ is said to be left spiky if any connected component of p has a least element. A filter F on $P(\mathcal{D})$ is said to be left spiky if the left spiky elements are dense in F.

Theorem 101. *Suppose that $\mathcal{C}$ has all infinite products and coproducts. Let $\mathcal{D}$ be a partial order, seen as a category, F a left spiky filter on $P(\mathcal{D})$, and suppose that there is a fibrantly regular model structure on $\mathcal{C}^{\mathcal{D}}$ such that cofibrations are dense in F. Then the weakly induced model structure exists on $(\mathcal{C}^{\mathcal{D}})_F$.*

Proof. We will show that we verify the conditions of Theorem 54. The filter is ascending and well-levelled, hence we only need to verify that we can raise the trivial p-fibrations, for any p in the filter. Let α be a p-fibration which is also a weak equivalence. We will suppose without loss of generality that $F_p(\alpha) = \alpha : G \to H$, to simplify notations. For any object $a \in \operatorname{Obj} \mathcal{D}$ not in p, we define the element $p_a \in P(\mathcal{D})$ as the set of objects in p greater than a. Let C_a be the set of connected components of p_a, and for any connected component $C \in C_a$, let $c_C \in C$ be its least element. We define the functor G' as

$$G'(a) = \begin{cases} \prod_{C \in C_a} c_C & \text{if } a \notin p \\ G(a) & \text{otherwise} \end{cases}$$

and for any objects $a, b \notin p$, any morphism $f : a \to b$, $G(f)$ is given by the universal property of the product; for any objects $a \notin p$, $b \in p$, any morphism $f : a \to b$, let C be the connected component of p_a containing b, then $G(f)$ is defined as the composition of the projection $G'(a) \to G(c_C)$ with the map $G(c_C) \to G(b)$. We define H' in the same way. We define $\alpha' : G' \to H'$ as the obvious natural transformation. By Lemma 99, for any a not in p, the map α'_a is a product of trivial fibrations, therefore it is a trivial fibrations and α' is a pointwise weak equivalence, hence a weak equivalence. Then, for any trivial cofibration $\beta : K \to L$, any commutative square

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\beta"] & G' \arrow[d, "\alpha'"] \\ L \arrow[r] & H' \end{tikzcd}$$

induces a commutative square

$$\begin{tikzcd} F_p(K) \arrow[r] \arrow[d, "F_p(\beta)"] & G \arrow[d, "\alpha"] \\ F_p(L) \arrow[r] & H. \end{tikzcd}$$

By definition, there is a lift $\gamma : F_p(L) \to G$, which we can extend to a lift $\gamma' : L \to G'$ by the universal property of the limit. Hence α' is a fibration. We conclude that α' is a trivial fibration. $\square$

Corollary 102. *Suppose that $\mathcal{C}$ has all infinite products and coproducts. Let $\mathcal{D}$ be a partial order, seen as a category, such that the injective model structure exists on $\mathcal{C}^{\mathcal{D}}$. Then, for any left spiky filter F on $P(\mathcal{D})$, the category $(\mathcal{C}^{\mathcal{D}})_F$ has the weakly induced model structure.*

Proof. The free cells generated by cofibrations of $\mathcal{C}$ are cofibrations in the injective model structure, hence fibrations must be pointwise fibrations. It is then a direct consequence of the previous theorem. $\square$

Theorem 103. *Suppose that for any small category $\mathcal{D}$, the injective model structure exists on $\mathcal{C}^{\mathcal{D}}$. Then, for any small category $\mathcal{D}$ with the injective model structure on $\mathcal{C}^{\mathcal{D}}$, any filter F on $P(\mathcal{D})$, the weakly induced model structure exists on $(\mathcal{C}^{\mathcal{D}})_F$.*

Proof. We will use Theorem 54. We will show that the trivial p-fibrations are genuine trivial p-fibrations. For any natural transformations α, α is a genuine trivial p-fibration if and only if $F_p(\alpha)$ is a genuine trivial p-fibration too. Let $\mathcal{D}_p$ be the full subcategory of $\mathcal{D}$ whose objects are exactly the ones contained in p. We define the functor $G : \mathcal{C}^{\mathcal{D}_p} \to \mathcal{C}^{\mathcal{D}}$ defined, for any functor $H : \mathcal{D}_p \to \mathcal{C}$, by

$$G(H)(a) = \begin{cases} H(a) & \text{if } a \in p \\ \emptyset & \text{otherwise} \end{cases}$$

The category $\mathcal{C}^{\mathcal{D}_p}$ is equivalent by this functor to the full subcategory of $\mathcal{C}^{\mathcal{D}}$ with objects of the form $F_p(A)$. Moreover, $\mathcal{C}^{\mathcal{D}_p}$ has the injective model structure, and G creates a correspondence between fibrations in $\mathcal{C}^{\mathcal{D}_p}$ and p-fibrations of the form $F_p(A) \to F_p(B)$. Also, because a weak equivalence in the injective model structure is simply a pointwise weak equivalence, we see that a trivial fibration in $\mathcal{C}^{\mathcal{D}_p}$ is send on a trivial p-fibration in $\mathcal{C}^{\mathcal{D}}$, which then must have the right lifting property with respect to all cofibrations of the form $F_p(A) \to F_p(B)$. We conclude that the trivial p-fibrations are genuine trivial p-fibrations. $\square$

Theorem 104. *Suppose that the initial object in $\mathcal{C}$ is strict. Let $\mathcal{D}$ be a small category with a model structure on $\mathcal{C}^{\mathcal{D}}$, and F a filter on $P(\mathcal{D})$ such that cofibrations are dense in F. Then, there is an induced model structure on $(\mathcal{C}^{\mathcal{D}})_F$.*

Proof. We will use Theorem 61. Let suppose that $\mathcal{C}$ is not a trivial category. Then $\mathcal{C}$ is a well-initialized category, therefore any element in $P(\mathcal{D})$ corresponds to a subobject of the terminal object in $\mathcal{C}^{\mathcal{D}}$, hence F generates a filter F' on the subobjects of the terminal object, and we can conclude with Proposition 43. $\square$

Corollary 105. *Suppose that the initial object in $\mathcal{C}$ is strict. Let $\mathcal{D}$ be a small category such that the injective model structure exists on $\mathcal{C}^{\mathcal{D}}$, and F a filter on $P(\mathcal{D})$. Then, there is an induced model structure on $(\mathcal{C}^{\mathcal{D}})_F$.*

Proof. Direct consequence of the previous theorem. $\square$

We will now explore the case when $\mathcal{D}$ is a Reedy category. We will fix some definitions and notations first.

Definition 106. Let $\mathcal{D}$ be a Reedy category. For any object $a \in \operatorname{Obj} \mathcal{D}$, the latching category $\partial(\overrightarrow{\mathcal{D}} \downarrow a)$ at a is the full subcategory of $(\overrightarrow{\mathcal{D}} \downarrow a)$ containing all the objects except the identity map at a. For any functor $G : \mathcal{D} \to \mathcal{C}$, we define the functor

$$\mathrm{L}G_a : \partial(\overrightarrow{\mathcal{D}} \downarrow a) \to \mathcal{C}$$

which sends an object $(f : b \to a)$ on $G(b)$, and defined in the obvious way on the morphisms. The latching object $\mathrm{L_a}\, G$ at a is defined as $\mathrm{L_a}\, G = \operatorname{colim} \mathrm{L}G_a$.

Definition 107 (Reedy connected filter)**.** Let $\mathcal{D}$ be a Reedy category. An element $p \in P(\mathcal{D})$ is said to be left Reedy connected if for any object $a \in p$, one of the two following conditions is verified:

1. For any object $b \in \operatorname{Obj} \mathcal{D}$ distinct from a, any morphism $(f : b \to a) \in \overrightarrow{\mathcal{D}}$, the category which has as objects the commutative triangles of the form

$$\begin{tikzcd} & a \\ c \arrow[ur, "h"] & b \arrow[u, "f"'] \arrow[l, "g"] \end{tikzcd}$$

   with $g, h \in \overrightarrow{\mathcal{D}}$, and as morphisms the commutative diagrams of the form

$$\begin{tikzcd} & a & \\ c \arrow[ur] \arrow[rr, bend right, "k"'] & b \arrow[u, "f"'] \arrow[l] \arrow[r] & d \arrow[ul] \end{tikzcd}$$

   with $k \in \overrightarrow{\mathcal{D}}$, with the obvious composition, is non-empty and connected as a graph.

2. Any object $b \in p$ distinct from a has no morphism to a in $\overrightarrow{\mathcal{D}}$.

A filter F on $P(\mathcal{D})$ is said to be left Reedy connected if the set of left Reedy connected elements in F is dense.

Lemma 108. *Let $\mathcal{D}$ be a Reedy category, $p \in P(\mathcal{D})$ a left Reedy connected element, and $a \in p$ an object such that condition 1 is verified. Let $\partial(\overrightarrow{\mathcal{D}} \downarrow a)_p$ be the full subcategory of $\partial(\overrightarrow{\mathcal{D}} \downarrow a)$ containing as objects the morphisms between objects of $\mathcal{D}$ contained in p, and $i : \partial(\overrightarrow{\mathcal{D}} \downarrow a)_p \to \partial(\overrightarrow{\mathcal{D}} \downarrow a)$ the inclusion. Then, for any functor $G : \partial(\overrightarrow{\mathcal{D}} \downarrow a)$, we have the canonical morphism* $\operatorname{colim} G \circ i \to \operatorname{colim} G$ *is an isomorphism.*

Proof. The condition 1 is exactly asking that the inclusion i is a right cofinal functor, as defined in [Hir03, Definition 14.2.1], and it is proved that the precomposition by a right cofinal functor preserves colimits in [Hir03, Theorem 14.2.5]. $\square$

Theorem 109. *Suppose that $\mathcal{C}$ is complete and cocomplete and that the initial object is strict. Let $\mathcal{D}$ be a Reedy category and F a left Reedy connected filter in $P(\mathcal{D})$. Then there is an induced model structure on $(\mathcal{C}^{\mathcal{D}})_F$.*

Proof. We will use Theorem 61. Because weak equivalences are defined pointwise, for any element $p \in P$, the functor F_p preserves weak equivalences. Because the initial object is strict, F_p preserves also fibrations. We will show that if p is left Reedy connected, F_p preserves also cofibrations. A natural transformation $\alpha : G \to H$ is a cofibration if for every object $a \in \operatorname{Obj} \mathcal{D}$, the relative latching map

$$G(a) \coprod_{\mathrm{L}_a G} \mathrm{L}_a H \to H(a)$$

is a cofibration in $\mathcal{C}$. Suppose that α is a cofibration. For $a \in \operatorname{Obj} \mathcal{D}$, there are three cases that we need to verify:

- If $a \in p$ and p verifies the condition 1 of left Reedy connected at a, we have a natural isomorphism $\mathrm{L}_a F_p(K) \cong \mathrm{L}_a K$ for any functor K by the previous lemma, hence the relative latching map of $F_p(\alpha)$ at a is equal to the relative latching map of α at a, and it is a cofibration.

- If $a \in p$ and p verifies the condition 2 at a, $\mathrm{L}_a F_p(K)$ is defined as an empty colimit, hence $\mathrm{L}_a F_p(K) \cong \emptyset$, and the relative latching map of $F_p(\alpha)$ at a is simply α_a, which is a cofibration (for a proof, see [Hir03, Proposition 15.3.11]).

- If $a \notin p$, then the relative latching map of $F_p(\alpha)$ at a is $\emptyset \to \emptyset$, which is a cofibration.

Thus, cofibrations are preserved. $\square$

5.2 Projective model structures with cofibrantly generated model categories

We will define the notion of left closed filters. We will show that if $\mathcal{C} = \mathcal{S}\mathrm{et}$ with the injective-surjective model structure, and $\mathcal{D}$ is a small category without non-epic endomorphisms, then for any filter F on $P(\mathcal{D})$, if the projective model structure on $\mathcal{C}^{\mathcal{D}}$ is dense in F, F must be left closed. Then, we prove that if $\mathcal{C}$ is cofibrantly generated, for any small category $\mathcal{D}$, any filter F on $P(\mathcal{D})$, the projective model structure on $\mathcal{C}^{\mathcal{D}}$ is dense in F, hence $(\mathcal{C}^{\mathcal{D}})_F$ has a induced model structure.

Definition 110 (Injective-surjective model structure)**.** The injective-surjective model structure on $\mathcal{S}\mathrm{et}$ has injections as cofibrations, surjections as fibrations and all maps as weak equivalences.

Lemma 111. *The injective-surjective model structure on $\mathcal{S}\mathrm{et}$ is a combinatorial model structure. Notably, the injective and projective model structures always exist on presheaves categories.*

Proof. The proof that being combinatorial implies that the injective and projective model structures exist can be found in the book of Jacob Lurie [Lur09, Proposition A.2.8.2.]. The generating set, for cofibrations and for trivial cofibrations, is just $I = \{\emptyset \to \{*\}\}$. The I-injective morphisms are the surjections, and a map has the left lifting property with respect to all surjections if and only if it is an injection. Moreover, any injection can be constructed as a transfinite composition of pushouts of $\emptyset \to \{*\}$, and $\emptyset$ is small relative to any class of maps, thus we have a cofibrantly generated model category.

Every set can be written as the colimit of the partial order of its finite subsets, therefore if we take a set A containing a set of each finite cardinality, A contains only compact objects and every other object of $\mathcal{S}\mathrm{et}$ can be written as a filtered colimit over objects in A.

$\square$

Definition 112 (Left boundary)**.** Let $\mathcal{D}$ be a small category and $F : \mathcal{D} \to \mathcal{S}\mathrm{et}$ a functor. For an object a in $\mathcal{D}$, an element $x \in G(a)$, and an object b in $\mathcal{D}$, an element $y \in G(b)$, we say that (b, y) is a left boundary of (a, x) if for any object c, any element $z \in G(c)$ and any morphism $f : c \to a$ such that $G(f)(z) = x$, there exists a unique morphism $g : b \to c$ such that $G(g)(y) = z$.

Lemma 113. *If (b, y) is a left boundary of (a, x) and there exists an object c in $\mathcal{D}$, an element $z \in G(c)$ and a morphism $f : c \to a$ such that $G(f)(z) = x$, then (b, y) is a left boundary of (c, z).*

Proof. Suppose that we have another pair (c', z') and a morphism $f' : c' \to c$ such that $f'(z') = z$. Then, $f \circ f'(z') = x$, therefore there exists a unique morphism $g : b \to z'$ such that $G(g)(y) = z'$. $\square$

Lemma 114. *Let $\mathcal{D}$ be a small category without non-epic endomorphisms, and we equip $\mathcal{S}\mathrm{et}$ with the injective-surjective model structure. Then, a functor $G : \mathcal{D} \to \mathcal{S}\mathrm{et}$ is cofibrant in the projective model structure if and only if for any object a in $\mathcal{D}$, any element $x \in G(a)$, the pair (a, x) has a left boundary.*

Proof. A functor verifies this condition if and only if it is isomorphic to a free cell complex, with the generating set $I = \{\emptyset \to \{*\}\}$. Because the cofibrant objects are retracts of free cell complexes, we just need to show that this condition is preserved under retracts. Suppose that we have the diagram

$$\begin{tikzcd} H \arrow[r, "\alpha"] \arrow[rr, bend left, "\mathrm{id}_H"] & G \arrow[r, "\beta"] & H \end{tikzcd}$$

with G verifying this condition. For any object a in $\mathcal{D}$, any element $x \in H(a)$, there exist an object b in $\mathcal{D}$ and an object $y \in G(b)$ such that (b, y) is a left boundary of $(a, \alpha_a(x))$. We then want to show that $(b, \beta_b(y))$ is a left boundary of (a, x). For any object c in $\mathcal{D}$, any element $z \in H(c)$ and any morphism $f : c \to a$ such that $H(f)(z) = x$, then $G(f)(\alpha_c(z)) = \alpha_a(f(z))$, hence there exists a morphism $g : b \to c$ such that $G(g)(y) = \alpha_c(z)$, and therefore $H(g)(\beta_b(y)) = z$. Let $g' : b \to c$ be a morphism such that $H(g')(\beta_b(y)) = z$. The element $\alpha_b \circ \beta_b(y)$ is send on $\alpha_c(z)$ by $G(g)$, hence there is a morphism $r : b \to b$ such that $G(r)(y) = \alpha_c(z)$, and this morphism is epic by hypothesis. But then $G(g' \circ r)$ and $G(g \circ r)$ send y on $\alpha_c(z)$, so $g' \circ = g \circ r$, thus $g' = g$. $\square$

Definition 115 (left closed filters)**.** Let $\mathcal{D}$ be a small category. For any element $A \in P(\mathcal{D})$, any object a in $\mathcal{D}$, we can define a preorder on

$$P_a^A = \bigsqcup_{b \in A} \mathcal{D}(a, b)$$

where for two morphisms $f : a \to b$, $g : a \to c$, $f \leq g$ if there exists $h : b \to c$ such that $h \circ f = g$. An element $A \in P(\mathcal{D})$ is said to be left closed if for any object a in $\mathcal{D}$, any connected component of P_a^A has an epic least element.

A filter F in $P(\mathcal{D})$ is said to be left closed if the set of left closed objects is dense in F.

Proposition 116. *Let $\mathcal{P}$ be a partial order, seen as a category. An element $A \in P(\mathcal{P})$ is left closed if and only if for any connected subset $B \subseteq A$, if B has a lower bound in $\mathcal{P}$, it has a lower bound in A. Notably, if an element is left spiky, it is left closed.*

Proof. Because any morphism in $\mathcal{P}$ is epic, the condition of the proposition implies that A is left closed. Now, if A is left closed, for $B \subseteq A$, if there exists an object x in $\mathcal{P}$ which is smaller than or equal to any element in B, the subset $\{x \to b \mid b \in B\}$ is connected in P_x^A by the connectedness of B, hence there is a connected component C of P_a^A containing this subset, with a least element $x \to c$. This element c must then be smaller than or equal to any element in B. $\square$

Proposition 117. *Let $\mathcal{D}$ be a small category without non-epic endomorphism and F a filter on $P(\mathcal{D})$. If the projective model structure on $\mathsf{Set}^{\mathcal{D}}$ with respect to the injective-surjective model structure on Set is dense in F, then F is left closed.*

Proof. We define the functor

$$G = \coprod_{a \in \mathrm{Obj}} \mathcal{D}(a, -)$$

which is a cofibrant object, because it is a coproduct of free cells. We will show that for any element $A \in F$, if $F_A(G)$ is cofibrant, then A is left closed, and therefore that the left closed elements must be dense in F. Suppose that $F_A(G)$ is cofibrant. Let a be an object of $\mathcal{D}$, a connected component $C \in C(P_a^A)$, and a morphism $(f : a \to b) \in C$. This morphism is also an element of $G(b)$. By Lemma 114, there exists an object $c \in p$ and a morphism $g : a \to c$ such that (c, g) is a left boundary of (b, f). We will show that for any morphism $(f' : a \to b') \in C$, (c, g) is a left boundary of (b', f'). If $f' \leq f$, there exists a morphism $h : b' \to b$ such that $h \circ f' = f$ by definition, hence the map $G(h) : \mathcal{D}(a, b') \to \mathcal{D}(a, b)$ sends f' on f, and (c, g) is a left boundary of (b', f'). If $f \leq f'$, the pair (b', f') has a left boundary (c', g'), and by what we just said, it is also a left boundary of (b, f), hence $g' \leq f$, so (c, g) is a left boundary of (c', g'), and therefore it is a left boundary of (b', f'). Because C is connected, (c, g) is a left boundary for any pair (b', f') with $f' \in C$. Then, g is a least element of C, and we only need to show that it is epic. Suppose that there exists $f_1, f_2 : c \to b$ such that $f_1 \circ g = f_2 \circ g$. Because $f_1 \circ g$ and $f_2 \circ g$ are in C, (c, g) is a left boundary of $(b, f_1 \circ g)$, therefore by definition there exists a unique morphism $h : c \to b$ such that $G(h)(g) = h \circ g = f_1 \circ g$, thus $f_1 = f_2$. $\square$

Theorem 118. *Let $\mathcal{D}$ be a small category and F a filter on $P(\mathcal{D})$. If F is left closed, then for any cofibrantly generated model category $\mathcal{C}$, the projective model structure on $\mathcal{C}^{\mathcal{D}}$ is dense in F. Notably, there is an induced model structure on $(\mathcal{C}^{\mathcal{D}})_F$.*

Proof. We will use Theorem 61. The set I will designate the set of generating cofibrations in $\mathcal{C}$. First, we show that the condition is sufficient. Suppose that $\mathcal{D}$, F and $\mathcal{C}$ verify these conditions. Because weak equivalences and fibrations are point-wise defined in $\mathcal{C}^{\mathcal{D}}$ and $\emptyset \to \emptyset$ is a weak equivalence and a fibration, for any $A \in P(\mathcal{D})$, the functor $F_A : \mathcal{C}^{\mathcal{D}} \to \mathcal{C}^{\mathcal{D}}$ preserves weak equivalences and fibrations. We will show that for a left closed element $A \in P(\mathcal{D})$, the functor F_A sends free I-cells on coproduct of free I-cells. Because this functor preserves colimits, it will also preserve the relative free cell complexes, and therefore all cofibrations. For an object a in $\mathcal{D}$, an object B in $\mathcal{C}$, we define the functor $\mathcal{D}(a, -) \times B$ as

$$\mathcal{D}(a, b) \times B = \coprod_{\mathcal{D}(a,b)} B$$

with the obvious maps between the objects. Then, for a morphism $f : B \to C$ in I, the I-cell generated by f at a, $\mathcal{D}(a, -) \times f : \mathcal{D}(a, -) \times B \to \mathcal{D}(a, -) \times C$, is point-wise defined as

$$\mathcal{D}(a, b) \times f = \coprod_{\mathcal{D}(a,b)} f : \coprod_{\mathcal{D}(a,b)} B \to \coprod_{\mathcal{D}(a,b)} C.$$

Let $C(P_a^A)$ be the set of connected components of the preorder P_a^A, and for each $C \in C(P_a^A)$, we choose an epic least element $g_C : a \to b_C$. Then, for any $b \in A$, there is a bijection

$$\bigsqcup_{C \in C(P_a^A)} \mathcal{D}(b_C, b) \cong \mathcal{D}(a, b)$$

given by $(h : b_C \to b) \mapsto h \circ g_C$, and we have

$$F_A(\mathcal{D}(a, -) \times f) = \coprod_{C \in C(P_a^A)} \mathcal{D}(b_C, -) \times f.$$

$\square$

Corollary 119. *Let $\mathcal{D}$ be a small category without non-epic endomorphisms, and F a filter on $P(\mathcal{D})$. Then, F is left closed if and only if for any cofibrantly generated model category $\mathcal{C}$, the projective model structure on $\mathcal{C}^{\mathcal{D}}$ is dense in F.*

Proof. Direct consequence of Theorem 118 and Proposition 117. □

Corollary 120. *Let $\mathcal{C}$ be a simplicial cofibrantly generated model category, $\mathcal{D}$ a small category, and F a left closed filter on $P(\mathcal{D})$. Then $(\mathcal{C}^{\mathcal{D}})_F$ is a simplicial model category.*

Proof. The induced cofibrantly generated model structure on $\mathcal{C}^{\mathcal{D}}$ is simplicial, with for F, G two functors, K a simplicial set, $A \in \operatorname{Obj} \mathcal{C}$, $n \in \mathbb{N}$:

- $(F \otimes K)(A) = F(A) \otimes K$
- $F^K(A) = F(A)^K$
- $\operatorname{Map}(F, G)_n = \mathcal{C}^{\mathcal{D}}(F \otimes \Delta[n], G)$

(for a proof, see [Hir03, Theorem 11.7.3]), and the composition of Map is defined pointwise. For any element $p \in P(\mathcal{D})$ which is left closed, we can see p as a full subcategory, and the image of F_A as a full subcategory is equivalent to $\mathcal{C}^p$. If we give to $\mathcal{C}^p$ the simplicial structure described above, we obtain that F_p preserves the tensor, cotensor and Map functors, and because p is left closed, F_p preserves cofibrations, fibrations and weak equivalences, hence p is compatible with simpliciality. By definition, the left closed elements are dense in F, hence F is compatible with simpliciality. We can conclude with Theorem 89. □

6 Examples

In this section, we will see applications of the results of the previous sections to some model categories, first on categories of functors, and then on classical model structures on chain complexes and on sequential pre-spectra. At the end, we will also give examples where the induced model structure doesn't exist.

6.1 Filter product

One nice example of filter quotient is the filter product: let $(M_i)_{i \in I}$ be an indexed set of model categories. The category $\mathcal{C} = \prod_{i \in I} M_i$ has a model structure with cofibrations being cofibrations componentwise, fibrations being fibrations componentwise and weak equivalences being weak equivalences componentwise. The partial order of the subobjects of the terminal element $\operatorname{SubObj}(e)$ is simply the product of the partial orders for each component, and for F a filter on I, we obtain a filter of functors defined as for $A \in F$, $F_A(x)_i = x_i$ if $i \in A$, and $F_A(x)_i = \emptyset$ otherwise. Taking the filter quotient with respect to F gives the category $\mathcal{C}_F$, which has the same objects as $\mathcal{C}$, and for two objects x, y, the morphisms $\mathcal{C}_F(x, y)$ are of the form $f : F_A(x) \to F_a(y)$, for some subset $A \in F$, and two morphisms are identified if they are equal when restricted on the components of some subset in F. Then $\mathcal{C}_F$ has an induced structure of model category by Theorem 61, because each F_A preserves cofibrations, fibrations and weak equivalences. Moreover, if each M_i is a simplicial model category, $\mathcal{C}$ is a simplicial model category, and $\mathcal{C}_F$ too by Theorem 89. Taking $I = \mathbb{N}$, $M_i = \mathcal{S}\mathrm{et}$ for all $i \in \mathbb{N}$, and F the filter containing all cofinite subsets, we obtain an example where $\mathcal{C}_F$ is neither complete nor cocomplete. First, we show that it is not cocomplete. Suppose that $(x_i)_{i\in\mathbb{N}} = \coprod_{n\in\mathbb{N}}[(e)_{i\in\mathbb{N}}]$ exists, and let $(x_i)_{i\in\mathbb{N}}$, and $[r_n] : (e)_{i\in\mathbb{N}} \to x$ the n'th inclusion of the terminal element, with $r_n : (e)_{i\in\mathbb{N}} \times U_n \to x$ a representing element. Because any finite coproduct of copies of the terminal element has a monomorphism to x, for any integer k, there is an integer N_k such that $r_0, r_1, \dots, r_k$ have all non-empty disjoint images in coordinates N_k and higher. We will also suppose that $N_k > k$. Let $y = (\{0,1\})_{i\in\mathbb{N}}$, and $[f] : x \to y$ be any map in $\mathcal{C}_F$, we will show that there exists a different map $[g] : x \to y$ but whose composition with any $[r_n]$ is the same. Let $(f_i)_{i\in\mathbb{N}} : x \times V \to y$ a representing element of $[f]$. We define, for all $k \in \mathbb{N}$, $g_{N_k}(a) = 1 - f_{N_k}(a)$ for any a in the image of r_{k,N_k}, and $g_n(a) = f_n(a)$ otherwise. Therefore, f and g are different on all coordinates of the form N_k, so they are different for infinitely many coordinates, and $[f] \neq [g]$. But $f_n \circ r_{k,n} = g_n \circ r_{k,n}$ for all $n > N_k$, therefore $[f] \circ [r_k] = [g] \circ [r_k]$ for all $k \in \mathbb{N}$, which is a contradiction to the universal property of the coproduct. The proof that $\prod_{n\in\mathbb{N}}(\{0,1\})_{i\in\mathbb{N}}$ doesn't exist in $\mathcal{C}_F$ is very similar.

Doing the same construction, but with sSet instead of $\mathcal{S}$et also gives an example where a coproduct of infinitely many copies of the terminal object doesn't exist, which implies that the resulting category is not a simplicial model category in the usual sense, but it is still a simplicial model category for the definition used in this paper, see the section on simplicial model categories.

6.2 Examples on categories of functors

Example 121. Let $\mathcal{D}$ be any model category and $\mathbb{N}$ the positive integers with the usual order, seen as a category. The objects of $\mathcal{C} = \mathcal{D}^{\mathbb{N}}$ are therefore of the form

$$A_0 \longrightarrow A_1 \longrightarrow A_2 \longrightarrow A_3 \longrightarrow \cdots$$

We can then define a filter of functors on this category, with for any $i \in \mathbb{N}$, the functor F_i defined as $F_i(B)_n = B_n$ if $n > i$, and $F_i(B)_n = \emptyset$ otherwise, where $\emptyset$ is the initial object. The category $\mathcal{C}_F$ constructed by filter quotient has the same objects as $\mathcal{C}$, and the set of morphisms from some object A to some object B are natural transformations of the form $F_n(A) \to F_n(B)$ for any integer n, and we identify morphisms which are equal from some degree onward.
Because $\mathbb{N}$ has a structure of Reedy category, we can put on $\mathcal{C}$ the Reedy model structure, which in this case can be explicitly described:

- A Reedy weak equivalence is a degreewise weak equivalence.
- A Reedy fibration is a degreewise fibration.
- A Reedy cofibration is a natural transformation $\alpha : A \to B$ such that for any $i \in \mathbb{N}$, α_i and the induced map

$$A_{i+1} \coprod_{A_i} B_i \to B_{i+1}$$

  are cofibrations.

As shown in the proof of Theorem 109, the functors F_p preserve cofibrations and weak equivalences, and fibrations are preserved because they are defined pointwise, hence by Theorem 61, we have an induced model structure on $\mathcal{C}_F$ defined as:

- A weak equivalence is a degreewise weak equivalence from some degree onward.
- A fibration is a degreewise fibration from some degree onward.
- A cofibration is a morphism which verifies the condition of Reedy cofibration from some degree onward.

Another possible model structure on $\mathcal{C}$ is the injective model structure, described as:

- A weak equivalence is a degreewise weak equivalence.
- A cofibration is a degreewise cofibration.

If this model structure exists, then by Corollary 102, there is another model structure on $\mathcal{C}_F$ given by:

- A weak equivalence is a degreewise weak equivalence from some degree onward.
- A cofibration is a degreewise cofibration from some degree onward.

The Corollary 102 has the condition that $\mathcal{C}$ has all infinite products and coproducts, but in this particular case it is not necessary.

Example 122. Let $\mathcal{D}$ be a cofibrantly generated model category and $\mathbb{R}$ be the reals with the usual order, seen as a category. The category $\mathcal{C} = \mathcal{D}^{\mathbb{R}}$ has then a cofibrantly generated model structure induced by the model structure of $\mathcal{D}$, defined as:

- Weak equivalences are pointwise weak equivalences.
- Fibrations are pointwise fibrations.

For every integer $n \in \mathbb{N}$, we define the functor $F_n : \mathcal{C} \to \mathcal{C}$ as for any object $A \in \operatorname{Obj} \mathcal{C}$, any real $r \in \mathbb{R}$, $F_n(A)_r = \emptyset$ if $r < n$, and $F_n(A)_r = A_r$ otherwise. These functors form a filter of functors F, and quotienting by this functor gives the category $\mathcal{C}_F$, with the same objects as $\mathcal{C}$, and the set of morphisms from some object A to some object B are the natural transformations of the form $F_n(A) \to F_n(B)$ for any integer $n \in \mathbb{N}$, and we identify the morphisms which are equal for all sufficiently great integers. By Theorem 118, there is an induced model structure on $\mathcal{C}_F$ where weak equivalences are the natural transformations which are pointwise weak equivalence for sufficiently great integers, and fibrations are the natural transformations which are pointwise fibrations for sufficiently great integers.

6.3 Chain complexes of modules over PID's

Example 123. Let R be a PID, and $\mathcal{C}$ the category of non-negatively graded chain complexes of R-modules. There is a model structure, called the projective model structure, on $\mathcal{C}$, defined as:

- Weak equivalences are the quasi-isomorphisms.
- Fibrations are the degreewise epimorphisms on all strictly positive degrees.
- Cofibrations are the degreewise monomorphisms with projective cokernels.

A proof can be found in [DI10]. We would want to have a new category with the same objects, but where we only retain the properties "at the limit". Notably, the morphisms should be defined only for sufficiently great levels, and two morphisms which are equal for sufficiently great levels should be equal. Moreover, we would like to have an induced model structure on this category. Here's how to do it in with a filter quotient: for every integer $n \in \mathbb{N}$, we define the functor $F_n : \mathcal{C} \to \mathcal{C}$, called the canonical truncation, in the following way: a chain complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \xrightarrow{d^C_{n-1}} C_{n-2} \longrightarrow \dots$$

is send on the chain subcomplex

$$\dots \longrightarrow C_{n+1} \longrightarrow \ker d^C_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

It is defined in the obvious way on the chain maps. It is clear that F_n preserves weak equivalences, and we will show that it preserves cofibrations. It clearly preserves the degreewise monomorphisms, and for $f : C \to D$ a cofibration, we want to show that $\ker d^D_n / f_n(\ker d^C_n)$ is projective. It is a submodule of $D_n/f_n(C_n)$, because for any $c \in C_n$, if $f_n(c) \in \ker d^D_n$, $f_{n-1} \circ d^C_n(c) = 0$, and because f_{n-1} is injective, $d^C_n(c) = 0$, hence $c \in \ker d^C_n$. A submodule of a projective module over a PID is also projective, hence cofibrations are preserved.
But fibrations are not preserved by these functors, so we need to work a little bit more. For every integer $n \in \mathbb{N}$, we define the functor $F'_n : \mathcal{C} \to \mathcal{C}$ in the following way: a chain complex

$$\dots \longrightarrow C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \xrightarrow{d^C_{n-1}} C_{n-2} \longrightarrow \dots$$

is send on the chain subcomplex

$$\dots \longrightarrow C_{n+1} \longrightarrow \operatorname{im} d^C_{n+1} \longrightarrow 0 \longrightarrow 0 \longrightarrow \dots$$

It is defined in the obvious way on the chain maps. It is clear that F'_n preserves weak equivalences and fibrations. The F_n's and the F'_n's form an ascending filter of functors F, hence the category $\mathcal{C}_F$ has finite limits. Moreover, for any $n \in \mathbb{N}$, we can define the functor G_n in the following way: a chain complex

$$\ldots \longrightarrow C_{n+1} \xrightarrow{d^C_{n+1}} C_n \xrightarrow{d^C_n} C_{n-1} \xrightarrow{d^C_{n-1}} C_{n-2} \longrightarrow \ldots$$

is send on the chain complex

$$\ldots \longrightarrow C_{n+1} \longrightarrow C_n \longrightarrow 0 \longrightarrow 0 \longrightarrow \ldots$$

The G_n's form a descending filter of functors G, and $\mathcal{C}_F$ is equivalent to $\mathcal{C}_G$, therefore $\mathcal{C}_F$ has finite colimits. And by Theorem 61, $\mathcal{C}_F$ has an induced structure of model category.

Moreover, we can prove that any morphism can be functorially factorized as the composition of a trivial cofibration with a fibration, because we can define such a factorization on $\mathcal{C}$ in a way compatible with the filter. For any module M, any strictly positive integer n, we define the n-disk chain complex $D^n(M)$ as

$$D^n(M)_k = \begin{cases} M & \text{if } k \in \{n, n-1\} \\ 0 & \text{otherwise} \end{cases}$$

and $d_n = \mathrm{id}_M$. For any chain complex C, we define the projective chain complex

$$P(C) = \coprod_{\substack{n>0 \\ a \in C_n \setminus \operatorname{im} d_{n+1}}} D^n(R)$$

and we obtain a canonical map $P(C) \to C$ which is surjective on all strictly positive degrees. Then, any chain map $f : B \to C$ can be factorized as $B \to B \oplus P(C) \to C$, and the canonical inclusion $B \to B \oplus P(C)$ is a trivial cofibration because all homology groups of $P(C)$ are trivial. Because each summand of $P(C)$ corresponds to a unique element in a module in C, the functoriality is easy to deduce. For any positive integer m, $F'_m(P(C)) \cong P(F'_m(C))$, hence this factorization is compatible with the filter, and we have an induced functorial factorization in $\mathcal{C}_F$ by Lemma 65.

By Proposition 30, C_F is an abelian category.

Example 124. This one is almost dual to the previous example. Let R be a PID, and $\mathcal{C}$ the category of non-negatively graded cochain complexes of R-modules. There is a model structure, called the injective model structure, on $\mathcal{C}$, defined as:

- Weak equivalences are the quasi-isomorphisms.
- Fibrations are the degreewise epimorphisms with injective kernels
- Cofibrations are the degreewise monomorphisms on strictly positive degrees.

Similarly to the previous example, we would want to have a new category which preserves only the properties "at the limit", with an induced model structure. For every integer $n \in \mathbb{N}$, we define the functor $F_n : \mathcal{C} \to \mathcal{C}$ in the following way: a cochain complex

$$\dots \longrightarrow C_{n-2} \xrightarrow{d_C^{n-2}} C_{n-1} \xrightarrow{d_C^{n-1}} C_n \xrightarrow{d_C^n} C_{n+1} \longrightarrow \dots$$

is send on the cochain complex

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow \operatorname{coker} d_C^{n-1} \longrightarrow C_{n+1} \longrightarrow \dots$$

It is defined in the obvious way on the chain maps. It is clear that F_n preserves weak equivalences, and we will show that it preserves fibrations. It clearly preserves the degreewise epimorphisms, and for $f : C \to D$ a fibration, we want to show that $\ker F_n(f)_n$ is an injective module. The induced morphism $\ker f_n \to \ker F_n(f)_n$ is surjective, because for any element $[c] \in \ker F_n(f)_n$ and a representing element $c \in C_n$, $f_n(c) \in \operatorname{im} d_D^{n-1}$, therefore there exists $d \in D_{n-1}$ such that $d_D^{n-1}(d) = f_n(c)$. But f_{n-1} is surjective, hence we have an element $\tilde{d} \in C_{n-1}$ such that $f_n \circ d_C^{n-1}(\tilde{d}) = f_n(c)$, so $c' = c - d_C^{n-1}(\tilde{d}) \in \ker f_n$, and $[c'] = [c]$. But the quotient of an injective module is an injective module over a PID, thus $\ker F_n(f)_n$ is an injective module.
But cofibrations are not preserved by these functors, so we need to work a little bit more. For every integer $n \in \mathbb{N}$, we define the functor $F'_n : \mathcal{C} \to \mathcal{C}$ in the following way: a cochain complex

$$\dots \longrightarrow C_{n-2} \xrightarrow{d_C^{n-2}} C_{n-1} \xrightarrow{d_C^{n-1}} C_n \xrightarrow{d_C^n} C_{n+1} \longrightarrow \dots$$

is send on the cochain complex

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow C_n / \ker d_C^n \longrightarrow C_{n+1} \longrightarrow \dots$$

It is defined in the obvious way on the chain maps. It is clear that F'_n preserves weak equivalences and cofibrations. The F_n's and the F'_n's form a descending filter of functors F, hence the category $\mathcal{C}_F$ has finite colimits. Moreover, for any $n \in \mathbb{N}$, we can define the functor G_n in the following way: a cochain complex

$$\dots \longrightarrow C_{n-2} \xrightarrow{d_C^{n-2}} C_{n-1} \xrightarrow{d_C^{n-1}} C_n \xrightarrow{d_C^n} C_{n+1} \longrightarrow \dots$$

is send on the cochain subcomplex

$$\dots \longrightarrow 0 \longrightarrow 0 \longrightarrow C_n \longrightarrow C_{n+1} \longrightarrow \dots$$

It is defined in the obvious way on the chain maps. The G_n's form an ascending filter of functors G, and $\mathcal{C}_F$ is equivalent to $\mathcal{C}_G$, therefore $\mathcal{C}_F$ has finite limits. And by Theorem 61, $\mathcal{C}_F$ has an induced structure of model category.

By Proposition 30, C_F is an abelian category.

6.4 Sequential pre-spectra in topological spaces

Example 125. Let $\mathcal{C}$ be the category of sequential pre-spectra in topological spaces, defined as follows:

- A sequential pre-spectrum X is a collection of $\mathbb{N}$-graded pointed compactly generated topological spaces, with for any $n \in \mathbb{N}$, a map $\sigma_n^X : S^1 \wedge X_n \to X_{n+1}$, called a structure map, where $\wedge$ is the smash product.
- A morphism $f : X \to Y$ is a collection of morphisms $f_n : X_n \to Y_n$ such that all the diagrams

$$\begin{array}{ccc} S^1 \wedge X_n & \xrightarrow{\sigma_n^X} & X_{n+1} \\ \big\downarrow{\scriptstyle S^1 \wedge f_n} & & \big\downarrow{\scriptstyle f_{n+1}} \\ S^1 \wedge Y_n & \xrightarrow{\sigma_n^Y} & Y_{n+1} \end{array}$$

  commute.

There is a model structure on this category, called the strict model structure, first introduced by Bousfield and Friedlander [BF78] on sequential pre-spectrum of simplicial sets, which is defined in the following way:

- A weak equivalence is a pointwise weak equivalence with respect to the Serre model structure on the topological spaces.
- A fibration is a pointwise fibration with respect to the Serre model structure.
- A cofibration is a pointwise cofibration and all the maps

$$X_{n+1} \coprod_{S^1 \wedge X^n} (S^1 \wedge Y^n) \to Y_{n+1}$$

  are cofibrations, with respect to the Serre model structure.

As before, we would like to keep only the properties of the objects "at the limit", and to have an induced model structure. We define $F_n : \mathcal{C} \to \mathcal{C}$ the n-truncation, as

$$F_n(X)_m = \begin{cases} \{*\} & \text{if } m < n \\ X_m & \text{otherwise} \end{cases}$$

and it is defined in the obvious way on the structure maps and on the morphisms. These functors form an ascending filter of functors F, and each F_n preserves finite limits and colimits because limits and colimits are computed pointwise. Moreover, they preserve fibrations and weak equivalences, because they are defined pointwise, and they preserve cofibrations because

$$X_{n+1} \coprod_{\{*\}} \{*\} \to Y_{n+1}$$

is simply the map $X_{n+1} \to Y_{n+1}$. Therefore, $\mathcal{C}_F$ has all finite limits and colimits, and $\mathcal{C}_F$ has an induced model structure.

There is another model structure on sequential pre-spectra, called the stable homotopy structure. It would be interesting to know if this model structure is also dense in the filter, and hence if it induces another model structure on $\mathcal{C}_F$.

6.5 Counterexamples

We will next see some examples which show how things can go wrong when we don't have the denseness condition in the filter. We will concentrate on filters on the subobjects of the terminal object, because these filters work better than the general case. In this setting, we can't keep the same definition of weak equivalences and of cofibrations, because it is possible that there are f and g which are equivalent and there exists F_p in the filter such that $F_p(f)$ is a cofibration or a weak equivalence, but there exists no F_q in the filter such that $F_q(g)$ is a cofibration or a weak equivalence. We will therefore call "induced weak equivalences", respectively "induced cofibrations" the morphisms $[f]$ such that there exist $g \in [f]$ and F_p in the filter such that $F_p(g)$ is a cofibration, respectively a weak equivalence.

Example 126. In this example, the induced cofibrations behave very badly, but the induced fibrations and weak equivalences form a trivial model structure. Let $\mathcal{C} = \mathcal{S}\text{et} \times \mathcal{S}\text{et}$, (f, g) is a cofibration if f and g are injective and g is not the map $\emptyset \to \emptyset$, or if f and g are isomorphisms; (f, g) is a fibration if f and g are surjective or if g is the map $\emptyset \to \emptyset$; all maps are weak equivalences. We take the principal filter $F = \{e, (*, \emptyset)\}$ in $\mathrm{Subobj}(e)$, and the category $\mathcal{C}_F$ can be described as having pairs of sets (A_1, A_2) as objects, and morphisms $\hat{f} : (A_1, A_2) \to (B_1, B_2)$ are simply morphisms $f : A_1 \to B_1$ on the first coordinate. The induced weak equivalences and the induced fibrations are all maps, a morphism $\hat{f} : (A_1, A_2) \to (B_1, B_2)$ is an induced cofibration if $f : A_1 \to B_1$ is an isomorphism or if it is an injection and $B_2 \neq \emptyset$. The lifting axiom is not verified: the inclusion $i : (\{0\}, \{0\}) \to (\{0, 1\}, \{0\})$ is a trivial cofibration and a trivial fibration, but it doesn't have the lifting property with respect to itself. Cofibrations are not closed by composition with an isomorphism, hence cofibrations are not closed by composition and are not closed by retracts: the map $(\emptyset, \{*\}) \to (\{*\}, \{*\})$ is a cofibration, the map $(\{*\}, \{*\}) \to (\{*\}, \emptyset)$ is an isomorphism, but the map $(\emptyset, \{*\}) \to (\{*\}, \emptyset)$ is not a cofibration. However, the induced fibrations and weak equivalences form a model structure on $\mathcal{C}_F$ with isomorphisms as cofibrations.

If we wanted to, we could extend this category and this filter by taking products with copies of $\mathcal{S}\text{et}$, and we would have an example of quotient by a non-principal filter where the same properties fail.

Example 127. In this example, it is now weak equivalences which behave very badly, but the induced cofibrations and fibrations form a model structure. Let $\mathcal{C} = \mathcal{S}\text{et} \times \mathcal{S}\text{et}$, cofibrations are all maps, fibrations are all maps of the form $(\emptyset \to A, \emptyset \to B)$ and the isomorphisms, and weak equivalences are all maps which are not of the form $(\emptyset \to A, \emptyset \to B)$, excepted the identity map $\mathrm{id}_{(\emptyset,\emptyset)}$. We take the same principal filter as in the previous example, $F = \{e, (*, \emptyset)\}$. The induced cofibrations are all maps, the induced weak equivalences are maps $\hat{f} : (A, B) \to (C, D)$ such that A or 2 is not $\emptyset$ or $\hat{f} = \mathrm{id}_{\emptyset,\emptyset}$, the induced fibrations are maps of the form $(\emptyset \to A, B \to C)$. The induced weak equivalences are not closed by composition with an isomorphism, hence they are not closed by composition and are not closed by retracts: the map $(\emptyset, \{*\}) \to (\{*\}, \{*\})$ is a weak equivalence, the map $(\{*\}, \{*\}) \to (\{*\}, \emptyset)$ is an isomorphism, but the map $(\emptyset, \{*\}) \to (\{*\}, \emptyset)$ is not a weak equivalence.

However, there is a model structure on $\mathcal{C}_F$ with the induced cofibrations and the induced fibrations: we define weak equivalences as the maps which are not of the form $(\emptyset \to A, B \to C)$ excepted the isomorphisms.

Example 128. In this example, the induced cofibrations, fibrations and weak equivalences don't form a model structure, but the induced fibrations and weak equivalences do induce a model structure which is non-trivial. Let $P = \mathbb{N} \cup \{\omega\}$, with the partial order induced by $n \leq n+2$, $n \leq \omega$ for all $n \in \mathbb{N}$, and $0 \leq 1$. Then, the category $\mathcal{C} = \mathcal{S}\mathrm{et}^P$ has objects of the form

$$\begin{tikzcd}
 & F_1 \arrow[r] & F_3 \arrow[r] & F_5 \arrow[r] & \dots \arrow[dr] & \\
F_0 \arrow[ur] \arrow[dr] & & & & & F_\omega \\
 & F_2 \arrow[r] & F_4 \arrow[r] & F_6 \arrow[r] & \dots \arrow[ur] &
\end{tikzcd}$$

We can give it the projective model structure, with fibrations which are point wise surjections, and weak equivalences which are all maps. We define a filter F as all the functors $G : P \to \mathcal{S}\mathrm{et}$ such that $G(p) = \emptyset$ for finitely many $p \in P$, and $G(p) = \{*\}$ otherwise. Suppose that the induced cofibrations, fibrations and weak equivalences form a structure of model category on $\mathcal{C}_F$. Fibrations and trivial fibrations are in this case the equivalence classes of maps which are surjective on all objects but finitely many. Then, the terminal element E with $E(p) = \{*\}$ for all $p \in P$ is cofibrant, because it was cofibrant in $\mathcal{C}$. We define the object R with $R(0) = \emptyset$, $R(n) = \{*\}$ for all $n > 0$, $R(\omega) = \{0, 1\}$, $R(n \to \omega)(*) = 0$ for n even, and $R(n \to \omega)(*) = 1$ for n odd. The equivalence class of the unique map $\alpha : R \to E$ in $\mathcal{C}$ is a trivial fibration in $\mathcal{C}_F$, because α_p is surjective for all $p \in P$ except $p = 0$, and so by the lifting property there should be a map $\beta : E \times U \to R$, for some $U \in F$. But then there should be $m \in \mathbb{N}$ such that $E \times U(m)$ and $E \times U(m+1)$ are not empty, and so $R(m \to \omega) \circ \beta_m$ having disjoint image from $R(m+1 \to \omega) \circ \beta_{m+1}$ is in contradiction with the fact that $\beta_\omega \circ (E \times U)(m \to \omega) = \beta_\omega \circ (E \times U)(m+1 \to \omega)$

However, there is a model structure on $\mathcal{C}_F$ with the induced fibrations and weak equivalences, but not the induced cofibrations. Let $Q = P \setminus \{0\}$ with the same order as P, and $\mathcal{D} = \mathcal{S}\mathrm{et}^Q$. We can define a filter F' also with functors with only finitely many empty images, and it is clear that $\mathcal{C}_F$ is equivalent to $\mathcal{D}_{F'}$, and that the induced fibrations and the induced weak equivalences are the same. But in D, we claim the product with an element U in the filter preserves cofibrations, thus the structure in $\mathcal{D}$ is dense in the filter, and the induced cofibrations, fibrations and weak equivalences form a structure of model category in $\mathcal{D}_{F'}$. Let's prove the claim: let's suppose that we have a cofibration $\alpha : G \to H$, such that $\alpha \times U$ is not cofibration. Therefore there is a trivial fibration $\beta : R \to S$ and a commutative square

$$\begin{tikzcd} G \times U \arrow[r] \arrow[d, "\alpha \times U"] & R \arrow[d, "\beta"] \\ H \times U \arrow[r] & S \end{tikzcd}$$

without lift $H \times U \to R$. Then $\alpha \times U$ shouldn't have the left lifting property with respect to $\beta \times U$ either, and we can therefore suppose that $R \times U = R$ and $S \times U = S$. Let M, N be the greatest integers such that $R(2M+1) \neq \emptyset$ and $R(2N) \neq \emptyset$. We can then define $\tilde{R}(k) = R(k)$ if $R(k)$ is non-empty and otherwise $\tilde{R}(k) = R(2M+1)$ if k odd and $\tilde{R}(k) = R(2N)$ if k even, the maps in between being just identities. We define $\tilde{S}$ in exactly the same way, and there is a unique way to obtain extended maps $G \to \tilde{R}$, $H \to \tilde{S}$, and $\tilde{\beta} : R \to S$. $\tilde{\beta}$ is also surjective, therefore it is a trivial fibration, and we have a commutative square

$$\begin{tikzcd} G \arrow[r] \arrow[d, "\alpha"] & \tilde{R} \arrow[d, "\tilde{\beta}"] \\ H \arrow[r] & \tilde{S} \end{tikzcd}$$

But the existence of a lift in this diagram implies the existence of a lift in the previous diagram, and we have a contradiction.

Example 129. This example, very similar in principle to the previous one, but a bit more complicated, demonstrates that the induced fibrations and weak equivalences don't need to generate a structure of model category, even when they induce a model structure on the full subcategory $\mathcal{C} \times U$ for any object U in the filter. Let $A = \{0, 1\}^{<\omega}$ be the set of finite sequences of 0 and 1, also containing the empty sequence ( ). We will define the partial order $P = A \cup \{\omega\}$, with $\nu \leq \mu$ if ν is a prefix of μ, and $\nu \leq \omega$ for any $\nu \in P$. Let $\mathcal{C} = \mathcal{S}\mathrm{et}^P$, with the projective model structure. The objects in this category are of the form

$$
\begin{tikzcd}
 & & & \cdots \arrow[ddddr] & \\
 & & F_{(00)} \arrow[ur] \arrow[r] & \cdots \arrow[dddr] & \\
 & F_{(0)} \arrow[ur] \arrow[dr] & & \cdots \arrow[ddr] & \\
 & & F_{(01)} \arrow[ur] \arrow[r] & \cdots \arrow[dr] & \\
F_{(\,)} \arrow[uur] \arrow[ddr] & & & & F_{\omega} \\
 & & F_{(10)} \arrow[r] \arrow[dr] & \cdots \arrow[ur] & \\
 & F_{(1)} \arrow[ur] \arrow[dr] & & \cdots \arrow[uur] & \\
 & & F_{(11)} \arrow[r] \arrow[dr] & \cdots \arrow[uuur] & \\
 & & & \cdots \arrow[uuuur] &
\end{tikzcd}
$$

We choose the filter F of functors G with $G(p) = \emptyset$ for only finitely many objects of P, and $G(p) = \{*\}$ otherwise. Then, the induced fibrations and weak equivalences in $\mathcal{C}_F$ don't form a model structure. If it was the case, there should exist a cofibrant object G such that $G(\nu) \neq \emptyset$ for some sequence $\nu \in A$. Let ν_0 be the sequence ν with an additional 0 at the end, ν_1 with an additional 1. Let l be the length of ν. We define $H : P \to \mathcal{S}\mathrm{et}$ as $H(\nu) = \emptyset$ for any $\nu \in A$ of length smaller than or equal to l, $H(\nu) = \{*\}$ for any $\nu \in A$ of length greater than l, $H(\omega) = \{0, 1\}$, $H(\mu \to \omega)(*) = 0$ for any μ which is an extension of ν_0, and $H(\mu \to \omega)(*) = 1$ otherwise. The unique map $H \to e$ is then a trivial fibration in $\mathcal{C}_F$. We will show that the map $\emptyset \to G$ doesn't have the left lifting property with respect to $H \to e$ in $\mathcal{C}_F$. Suppose that there was a subobject U in the filter and a morphism $\alpha : G \times U \to H$ such that the following diagram commutes in $\mathcal{C}$

$$
\begin{tikzcd}
 & H \arrow[d] \\
G \times U \arrow[ur, "\alpha"] \arrow[r] & e
\end{tikzcd}
$$

But because $U(p)$ is empty for only finitely many elements $p \in P$, we should have a sequence μ_0 which is an extension of ν_0, and a sequence μ_1 which is an extension of

ν_1, such that $(G \times U)(\mu_0)$ and $(G \times U)(\mu_1)$ are not empty. Then $H(\mu_0 \to \omega) \circ \alpha_{\mu_0}$ and $H(\mu_1 \to \omega) \circ \alpha_{\mu_1}$ have disjoint images in $H(\omega)$, while $\alpha_\omega \circ (G \times U)(\mu_0 \to \omega) = \alpha_\omega \circ (G \times U)(\mu_1 \to \omega)$, thus we have a contradiction.

On the other hand, for any object U in F, the full subcategory $\mathcal{C} \times U$ is equivalent to a presheaves category, with the new diagram corresponding to the previous diagram, but truncated. In this category the projective model structure exists, therefore fibrations and weak equivalences induce a model structure in each full subcategory $\mathcal{C} \times U$.

If we had begun with the injective model structure on $\mathcal{C}$, we would in fact have an induced model structure on $\mathcal{C}_F$, by Corollary 105. So there is a model structure on $\mathcal{C}_F$ with cofibrations and weak equivalences defined pointwise, but not with fibrations and weak equivalences defined pointwise.

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