Introduction to Quantum Computing

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1 Introduction

2 Section 2

2.1 Section 2.1

2.2 Single qubit gates

(Nielsen and Chang sections 1.3.1, 2.1.8, 4.2)
Principle of <u>time evolution</u>: the quantum state $|\psi\rangle$ at current time point t transitions to a new quantum state $|\psi'\rangle$ at a later time t' $\geq$ t.
The transition will always be described by a complex unitary matrix U.

$$|\psi'\rangle = U \cdot |\psi\rangle$$

Circuit notation:

$$|\psi\rangle \;\text{———}\boxed{U}\text{———}\; |\psi'\rangle$$

Notes:

- The circuit is read from left to right, but the matrix time vector (U $|\psi\rangle$) from right to left.
- U preserves normalisation.

Examples:

- quantum analogue of classical NOT-gate ($0 \leftrightarrow 1$) flip $|0\rangle \leftrightarrow |1\rangle \rightsquigarrow$ Pauli-X gate:

$$X \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(check: $x\ |0\rangle = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = |1\rangle$)
analogue for 1 to 0.

- Pauli-Y gate:

$$Y \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

- Pauli-Z gate:

$$Z \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Z leaves $|0\rangle$ unchanged, but flips the sign of the coefficient of $|1\rangle$.

Recall the block sphere representation of a general quantum state:

$$|\psi\rangle = cos(\frac{\varphi}{2})\,|0\rangle + ie^{i\varphi} sin(\frac{\varphi}{2})\,|1\rangle$$

Then:

$$Z\,|\psi\rangle = cos(\frac{\varphi}{2})\,|0\rangle - e^{i\varphi} sin(\frac{\varphi}{2})\,|1\rangle \quad (1)$$

$$= cos(\frac{\varphi}{2})\,|0\rangle + e^{i(\varphi+\pi)} sin(\frac{\varphi}{2})\,|1\rangle \qquad \text{(with } e^{i\pi} = -1\text{)} \quad (2)$$

$\rightsquigarrow$ new Block sphere angles: $\theta' = \theta,\ \varphi' = \varphi + \pi$
(rotation by $\pi = 180\,\text{deg}$ around z-axis)
x, y and z gates are called Pauli matrices.
The <u>Pauli vector</u> $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3) = (x, y, z)$ is a vector of 2x2 matrices.

- Hadamard gate:

$$H = \frac{1}{\Omega} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\alpha\,|0\rangle + \beta\,|1\rangle \ \text{———} \boxed{H} \text{———}\ \alpha\frac{|0\rangle + |1\rangle}{\sqrt{2}} + \beta\frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

- Phase gate:

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

- T gate:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix}$$

  Note: $T^2 = S(\text{since } (e^{i\frac{\pi}{4}})^2 = e^{i\frac{\pi}{2}} = i)$

Pauli matrices satisfy:

$\sigma_j^2 = identity.$

$\sigma_j \cdot \sigma_k = -\sigma_k \cdot \sigma_j$ for all $j \neq k$

Commutator: $[\sigma_j, \sigma_k] := \sigma_j\sigma_k - \sigma_k\sigma_j = 2i\ \sigma_l$ for (j, k, l) — which is a cyclic permutation of (1,2,3)

Matrix exponential and rotation gates $R_x(\theta)\ R_y(\theta), R_z(\theta)$ (pdf in Moodle) Z-Y decomposition of an arbitrary 2x2 matrix:
For any unitary matrix $U \in \mathbb{C}^{2x2}$ there exists real numbers $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that:

$$U = e^{i\alpha} \cdot R_z(\beta) \cdot R_y(\gamma) \cdot R_z(\delta) = e^{i\alpha} \begin{pmatrix} e^{-i\frac{\beta}{2}} & 0 \\ 0 & e^{i\frac{\beta}{2}} \end{pmatrix} \cdot \begin{pmatrix} \cos(\frac{\gamma}{2}) & -\sin(\frac{\gamma}{2}) \\ \sin(\frac{\gamma}{2}) & \cos(\frac{\gamma}{2}) \end{pmatrix} \cdot \begin{pmatrix} e^{-i\frac{\delta}{2}} & 0 \\ 0 & e^{i\frac{\delta}{2}} \end{pmatrix}$$

2.3 Multiple qubit

(Nielsen and Chang sections 1.2.1, 2.1.7)

So far: single qubits, superposition of basis states $|0\rangle$and $|1\rangle$.
For two qubits, this generalises to:
$|00\rangle, |01\rangle, |10\rangle, |11\rangle$
as computational basis states: all combinations (bitstrings) of 0s and 1s.
General two-qubit state:

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

with amplitudes $\alpha_{ij} \in \mathbb{C}$ such that:

$$|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1 \quad \text{(normalization)}$$

Can identify the basis states with unit vectors:

$$|00\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |01\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |10\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, |11\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Thus:

$$|\psi\rangle = \begin{pmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{pmatrix} \in \mathbb{C}^4$$

What happens if we measure only one qubit of a two-qubit state?
Say we measure the first qubit: obtain the result.

0 with probability $|\alpha_{00}|^2 + |\alpha_{01}|^2$

1 with probability $|\alpha_{10}|^2 + |\alpha_{11}|^2$

Wavefunction directly after measurement:

if we measured 0: $|\psi'\rangle = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$

if we measured 1: $|\psi'\rangle = \frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{|\alpha_{10}|^2 + |\alpha_{11}|^2}}$

Mathematical formalism for constructing two-qubit states:
Tensor product of vector spaces can combine two (arbitrary) vector spaces V and W to form the tensor product $V \otimes W$.
(Look into the "Tensor products of vector spaces" — Moodle)

Generalisation to n qubits: 2^n computational basis states
$\{|0,\dots,0\rangle, |0,\dots,0,1\rangle, |0,\dots,1,0\rangle, \dots, |1,\dots,1\rangle\}$ (all bit strings of length n)
Thus, a general n-qubit quantum state, also denoted as "quantum register", is given by:

$$|\psi\rangle = \sum_{x_0=0}^{1} \sum_{x_1=0}^{1} \cdots \sum_{x_{n-1}=0}^{1} \alpha_{x_{n-1},\dots,x_1,x_0} |x_{n-1} \dots x_1 x_0\rangle \tag{3}$$

$$= \sum_{x=0}^{2^n-1} \alpha_x |x\rangle \qquad \text{binary representation} \tag{4}$$

with $\alpha_x \in \mathbb{C}$ for all $x \in \{0, \dots, 2^n - 1\}$ such that:

$$\| |\psi\rangle \|^2 = \sum_{x=0}^{2^n-1} |\alpha_x|^2 \stackrel{!}{=} 1 \quad \text{(normalisation)}$$

$\rightsquigarrow$ in general "hard" to simulate on a classical computer (for large n) due to this "curse of dimensionality".

Vector space as tensor products: $\underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{\text{n times}} = (\mathbb{C}^2)^{\otimes n} \approx \mathbb{C}^{(2^n)}$

2.4 Multiple qubit gates

(Nielsen and Chuang sections 1.3.2, 1.3.4, 2.17)

As for single qubits, an operation on multiple qubits is described by an unitary matrix U.
For n qubits: $U \in \mathbb{C}^{2^n \times 2^n}$

Example: <u>Controlled-NOT</u> gate (also denoted CNOT)
two qubits: control and target
target qubit gets flipped if control is 1:

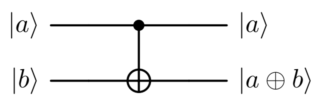
$$\underbrace{|00\rangle}_{control|target} \quad \mapsto \quad |00\rangle,$$

$$|01\rangle \mapsto |01\rangle,\ |10\rangle \mapsto |11\rangle,\ |11\rangle \mapsto |10\rangle$$

Can be expressed as:

$$|ab\rangle \mapsto |a, a \oplus b\rangle, \forall a, b \in \{0,1\} \qquad |a \oplus b \text{ defined as "addition modulo 2"}$$

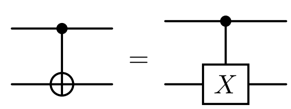
Circuit notation:



Matrix Representation:

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} Pauli - X$$

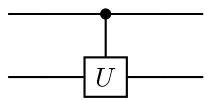
This matrix is unitary.
Alternative notation:



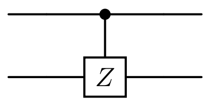
We can generalise Pauli-X to any unitary single-qubit gate U acting on the target qubit $\rightsquigarrow$ controlled U-gate:

$$|00\rangle \mapsto |00\rangle, |01\rangle \mapsto |01\rangle, |10\rangle \mapsto |1\rangle \otimes (U|0\rangle), |11\rangle \mapsto (|1\rangle \otimes (U|1\rangle)$$

Generalisation:

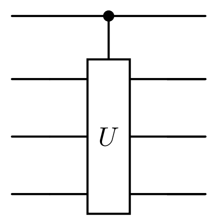
 $$\hat{=} \begin{pmatrix} 1 & & \\ & 1 & \\ & & U \end{pmatrix}$$

Example: controlled-z:

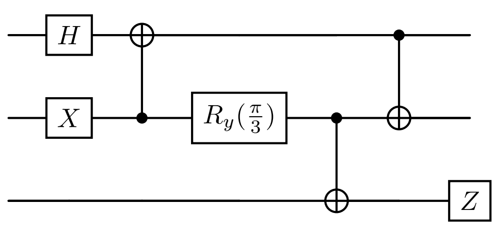
 $$\hat{=} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

Exercice: show that controlled-z gate is invariant when flipping control and target qubits.

Controlled-U gate for multiple target qubits:

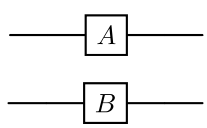


Note: single qubit and CNOT gates are universal: they can be used to implement an arbitrary unitary operation on n qubits.
(Quantum analogue of universality of classical NAND gate)
proof in Nielsen and Chang section 4.5
Example of a circuit consisting only of single qubit gates and CNOTs:



(time left to right)

Matrix Kronecker products: matrix representation of single qubit gates acting in parallel:



Operation on basis states: $a, b \in \{0, 1\}$

$$\underbrace{|a, b\rangle}_{|a\rangle \otimes |b\rangle} \mapsto (A\,|a\rangle) \otimes (B\,|b\rangle) = (A \otimes B)\,|a, b\rangle$$

Example: A = I (identity), B = Y:

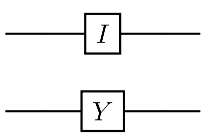
$$|00\rangle \mapsto |0\rangle \otimes \underbrace{(Y\,|0\rangle)}_{i|1\rangle} = i\,|01\rangle$$

$$|01\rangle \mapsto |0\rangle \otimes \underbrace{(Y\,|1\rangle)}_{-i|0\rangle} = -i\,|00\rangle$$

$$|10\rangle \mapsto |1\rangle \otimes (Y\,|0\rangle) = i\,|11\rangle$$

$$|11\rangle \mapsto |1\rangle \otimes (Y\,|1\rangle) = -i\,|10\rangle$$

Matrix Representation:
(ket on the left of mapto represents column and resulting one on the row)

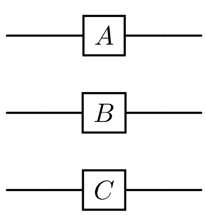


$$\triangleq \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} Pauli - Y = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix} = I \otimes Y$$

General formula: Kronecker product (matrix representation of tensor products of operators)

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{pmatrix} \in \mathbb{C}^{mp \times nq}$$

$\forall A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{p \times q}$ (NumPy np.kron(A, B))
Generalise to arbitrary number of tensor factors, e.g.



$$\triangleq A \otimes B \otimes C = (A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Basic properties:

(a) $(A \otimes B)^* = A^* \otimes B^*$ (elementwise complex conjugation)

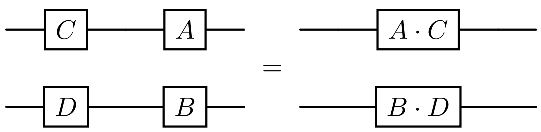
(b) $(A \otimes B)^T = A^T \otimes B^T$ (transposition)

(c) $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$

(d) $(A \otimes B) \otimes C = A \otimes (B \otimes C)$ (associative property)

(e) $\underbrace{(A \otimes B) \cdot (C \otimes D)}_{\text{matrix-matrix multiplication}} = (A \cdot B) \otimes (B \cdot D)$

for matrices of compatible dimensions.



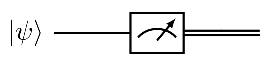
(side exchange because its read from left to right)

(f) Kronecker product of Hermitian matrices is Hermitian

(g) Kronecker product of unitary matrices is unitary (follows from (c) & (e))

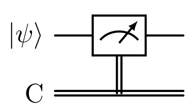
2.5 Quantum measurements

(Nielsen and Chuang sections 1.3.3, 2.2.3, 2.2.5)
Review: measurement of a single qubit $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$with respect to computational basis$|0\rangle$, $|1\rangle$:



classical data (measurement outcome 0 or 1)

alternative notation:



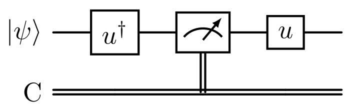
qubit after measurement (collapsed wavefunction)

measurement outcome

Unitary freedome of choice of measurement basis.

Given an orthonormal basis (ONB) $\{|u_1\rangle, |u_2\rangle\}$ can measure with respect to this orthonormal basis by performing a base change before and after the measurement:

$U = (|u_1\rangle\, |u_2\rangle) \in \mathbb{C}^{2\times 2}$ unitary



(measurement with respect to $\{|u_1, |u_2\rangle\rangle\}$, using a standard basis measurement)

Representing the qubit $|\psi\rangle = \alpha_1 |0\rangle + \alpha_2 |1\rangle$, $\alpha_1, \alpha_2 \in \mathbb{C}$

$$|\psi\rangle \;—\boxed{U^\dagger}—\; \rightsquigarrow U^\dagger |\psi\rangle = \alpha_1 |0\rangle + \alpha_2 |1\rangle \quad (U^\dagger |u_1\rangle = |0\rangle, U^\dagger |u_2\rangle = |1\rangle)$$

We will obtain measurement result 0 or 1 with the respective probabilities $|\alpha_1|^2$ and $|\alpha_2|^2$

After measuring and applying U: $|\psi\rangle$will be in the state $|u_1\rangle$ or $|u_2\rangle$.

Abstract, general defenition of quantum measurements

Quantum measurements are described by a collection $\{M_m\}$ of measurement operators acting on the quantum system, with the index m labelling possible measurement outcomes.

Denoting the quantum state before the measurement by $|\psi\rangle$, result m occurs with probability:

$$p(m) = \langle\psi| M_m^\dagger M_m |\psi\rangle = \|M_m |\psi\rangle\|^2,$$

state after the measurement is:

$$\frac{M_m |\psi\rangle}{\|M_m |\psi\rangle\|}$$

The measurement operators satisfy the completeness relation:

$$\sum_m M_m^\dagger M_m = I,$$

such that the probabilities sum to 1:

$$\sum_m p(m) = \sum_m \langle\psi| \underbrace{\sum_m M_m^\dagger M_m}_{I} |\psi\rangle = \langle\psi|\psi\rangle = 1$$

Example: measurement of a qubit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ with respect to computational basis $\{|0\rangle, |1\rangle\}$:

$$M_0 := \underbrace{\langle 0| |0\rangle}_{\text{outer product}} = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$$

$$M_1 := \langle 1| |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$$

$$\begin{aligned}\rightsquigarrow p(0) = \langle\psi| M_0^\dagger M_0 |\psi\rangle = \langle\psi| M_0 |\psi\rangle \quad &= \begin{pmatrix}\alpha^* & \beta^*\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}\\ &= |\alpha|^2\\ p(1) = \langle\psi| M_1^\dagger M_1 |\psi\rangle = \dots \quad &= |\beta|^2\end{aligned}$$

Projective measurements

$\rightsquigarrow$ see projection-operators (moodle)

Definition: a projective measurement is described by an observable M, a Hermitian operator acting on the quantum system spectral decomposition $\rightsquigarrow$

$$M = \sum_m \lambda_m P_m$$

with P_m: projection onto the eigenspace with eigenvalue λ_m.
The possible outcomes of the measurement correspond to the eigenvalues λ_m.
Probability of obtaining measurement result λ_m :

$$p(\lambda_m) = \langle\psi| P_m |\psi\rangle \quad |P_m^\dagger P_m = P_m \text{ since } P_m \text{ is a projection}$$

State of the quantum system after the measurement:

$$\frac{P_m|\psi\rangle}{\|P_m|\psi\rangle\|} = \frac{P_m|\psi\rangle}{\sqrt{p(\lambda_m)}}.$$

Properties

- **Normal matrix:**

  A is normal if it commutes with its conjugate transpose $A^\dagger$:

  $$A\ normal \Leftrightarrow A^\dagger A = AA^\dagger$$

- **Unitary matrix:**

  U is unitary if its matrix inverse U^-1 equals its conjugate transpose $U^\dagger$:

  $$U\ unitary \Leftrightarrow U^\dagger U = UU^\dagger = I$$

- **Hermitian matrix:**

  H is Hermitian if it's a square matrix that is equal to its conjugate transpose $H^\dagger$:

  $$H\ hermitian \Leftrightarrow H^\dagger = H$$

  For any hermitian matrix H of finite size, the following hold:

  - All eigenvalues are real.
  - Eigenvectors corresponding to distinct eigenvalues are orthogonal.
  - In real-valued matrices, "Hermitian" just means symmetric ($A = A^T$)