Funcții

$f: A \to B$ — A: domeniu, B: codomeniu

$x \longrightarrow f(x)$

$\operatorname{Im} f = \{f(x) \mid x \in A\}$

$A_1 \subseteq A : f(A_1) = \{f(x) \mid x \in A_1\}$

$B_1 \subseteq B : f^{-1}(B_1) = \{x \mid f(x) \in B_1\}$ (img inv / preimaginea lui B_1)

1) f - inj dacă
   1) $\forall x_1, x_2 \in A$ dacă $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
   2) $\forall x_1, x_2 \in A$ dacă $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
   3) $\forall$ paralelă la Ox printr-un pct din Codomeniu $\cap$ G_f în cel mult un pct

Obs: $\forall$ fct monotonă – injectivă

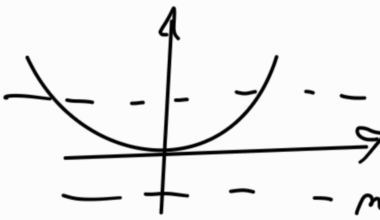
2) f = surjectivă
   1) $\forall y \in B, \exists x \in A$ a.î. $f(x) = y$
   2) $\operatorname{Im} f = B$
   3) $\forall$ paralela la Ox printr-un pct din codomeniu $\cap$ G_f în cel puțin un pct

3) f = bijectiva
   1) f = inj + surj
   2) $\forall y \in B, \exists!\, x \in A$ a.î. $f(x) = y$
   3) $\forall$ paralelă la Ox printr-un pct din codomeniu $\cap$ G_f într-un singur pct

f - bij $\Rightarrow \exists f^{-1}: B \to A$ a.î. $f \circ f^{-1} = 1_B$ - fct identică
$f^{-1} \circ f = 1_A$

Aplicații:

1) $f: \mathbb{R} \to \mathbb{R}, f(x) = x^2$



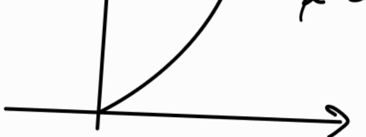
$f: \mathbb{R} \to \mathbb{R}, f(x) = ax^2 + bx + c$ (fct de gr. II)

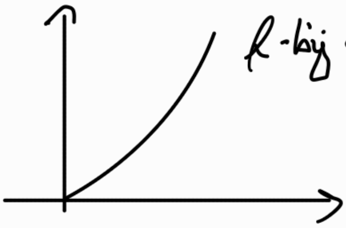
G_f - parabolă → dacă $a > 0$ $\cup$
→ dacă $a < 0$ $\cap$

$V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$

f-surj $\Leftrightarrow f: \mathbb{R} \to (0, \infty)$

f-inj $\Leftrightarrow f: [0, \infty) \to \mathbb{R}$





f - bij $\Leftrightarrow f:[0,\infty)\to[0,\infty)$

2) $f:\mathbb{R}\to\mathbb{R},\ f(x)=4x+3$

a) f - bij ? cu def

b) f - bij ? cu Graf

a) inj $f(x_1)=f(x_2)=x_1=x_2$

Fie $x_1,x_2\in\mathbb{R}$ a.î. $f(x_1)=f(x_2)\Rightarrow 4x_1-3=4x_2-3\Rightarrow x_1=x_2$ $\Rightarrow$ f - bij $\exists f^{-1}:\mathbb{R}\to\mathbb{R}$

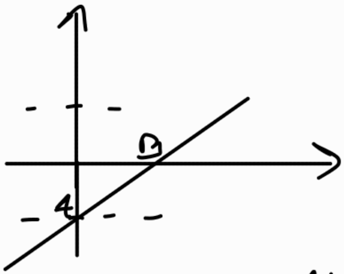
Surj Fie $y\in\mathbb{R}$ a.î. $f(x)=y$

$4x-3=y\Rightarrow 4x=y+3\Rightarrow x=\frac{y+3}{4}\in\mathbb{R}$

$f^{-1}(x)=\frac{x+3}{4}$

$f\circ f^{-1}(x)=f(f^{-1}(x))=f\left(\frac{x+3}{4}\right)=\cancel{4}\cdot\frac{x+3}{\cancel{4}}-3=x$

b)



$x=0\Rightarrow f(0)=-3\Rightarrow A(0,-3)$ $\Rightarrow$ f - bij

$f(x)=0\Rightarrow x=\frac{3}{4}\Rightarrow B\left(\frac{3}{4},0\right)$

Q: $f:\mathbb{Z}\to\mathbb{Z}$, b= bij

$f(x)=4x-3$

pt $y=2$: $\frac{5}{4}\notin\mathbb{Z}\Rightarrow$ NU

3. $f:\mathbb{R}\to\mathbb{R},\ f(x)=e^x$

f - bij

$f\neq$ surj deoarece $e^x\in(0,+\infty)$

Temă: f - bij

1) $f:\mathbb{R}\to\mathbb{R},\ f(x)=x^3$

2) $g:\mathbb{R}\to\mathbb{R},\ g(x)=5x+2$

3) $h:\mathbb{Z}\to\mathbb{Z},\ h(x)=5x+2$

4) $k:[0,\infty)\to[0,\infty),\ k(x)=x^2+6x+5$

$\cdot A, B$ – mult, $|A| = a$, $|B| = b$

1) Câte fct $f: A \to B$ există?

A are a elem: $i_1, \dots, i_a$

$\underbrace{b \cdot b \cdot \ldots \cdot b}_{a\text{-ori}}$ $\left\{ \begin{array}{l} f(i_1) \to b\text{-var} \\ \vdots \\ f(i_a) \to b\text{-var} \end{array} \right.$ $\Rightarrow b^a$ - fct

2) Câte fct surj $f: A \to B$?

3) Câte fct inj $f: A \to B$? $(a \le b)$

$f(i_1) \to b$

$f(i_2) \to b-1$

$\vdots$

$f(i_a) \to b-(a-1) = b-a+1$

$\rightarrow A_b^a = \frac{b!}{(b-a)!}$

4) Câte fct bij $f: A \to B$, $(a=b)$

$P_a = P_b = a!$

Principiul includerii și al excluderii

$|A \cup B| = |A| + |B| - |A \cap B|$

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

$\vdots$

$|A_1 \cup A_2 \ldots \cup A_m| = \sum_{i=1}^{m} |A_i| - \sum_{1 \le i < j \le m} |A_i \cap A_j| + \sum_{1 \le i < j < k \le m} |A_i \cap A_j \cap A_k| - \ldots (-1)^{m+1} |A_1 \cap \ldots \cap A_m|$

2) Câte fct surj $f: A \to B$?

Q: Câte fct nesurj $f: A \to B$

$f \ne$ surj dacă $\exists b \in B$ a.î. $f(x) \ne b$, $\forall x \in A$

$B = \{1, 2, \dots, b\}$

$B_1 = \{f: A \to B \mid f(x) \ne 1, \forall x \in A\}$

$B_2 = \{f: A \to B \mid f(x) \ne 2, \forall x \in A\}$

$\vdots$

$B_m = \{f: A \to B \mid f(x) \ne b, \forall x \in A\}$

$|B_1 \cup B_2 \cup \ldots \cup B_b|$

$|B_i| = (b-1)^a$

$|B_i \cap B_j| = (b-2)^a$

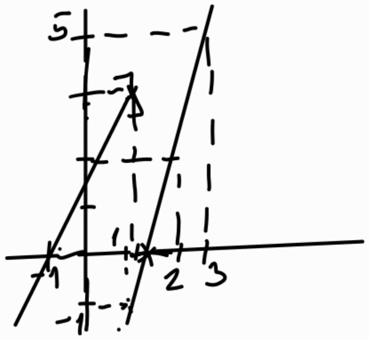
$|B_{i_1} \cap B_{i_2} \ldots B_{i_k}| = (b-k)^a$

$N = b \cdot (b-1)^a - C_b^2 (b-2)^a + \ldots + (-1)^{a+1} \cdot 0 = \sum_{i=1}^{b} C_b^i (-1)^{i+1} (b-i)^a$

Fct surj. $= b^a - N$

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 2x+1, & x \le 1 \\ 3x-4, & x > 1 \end{cases}$

a) f - bij
$x=0 \to f(0)=1$ | $x=1 \to -1$
$x=1) \to f(1)=3$ | $x=2 \to f(2)=2$



b) $f([2,3)) = [2,5]$
$f([1,3]) = (-1,5]$

Temă: $f:\mathbb{R}\to\mathbb{R}, f(x) = \begin{cases} x+1, & x\le 2 \\ 2x+1, & x>2 \end{cases}$

a) f - bij
b) $f([-1,3])$

Temă:

1) a) $f:\mathbb{R}\to\mathbb{R}, f(x)=x^3$
Fie $x_1,x_2\in A$ a.î. $f(x_1)=f(x_2)$
$f(x_1)=f(x_2) \Leftrightarrow x_1^3=x_2^3 \;/\sqrt[3]{\;} \Rightarrow x_1=x_2$ - inj $\Rightarrow f$ - bij
Fie $y\in\mathbb{R}$ a.î. $f(x)=y$
$f(x)=y \Rightarrow x^3=y \Leftrightarrow x=\sqrt[3]{y}\in\mathbb{R}$

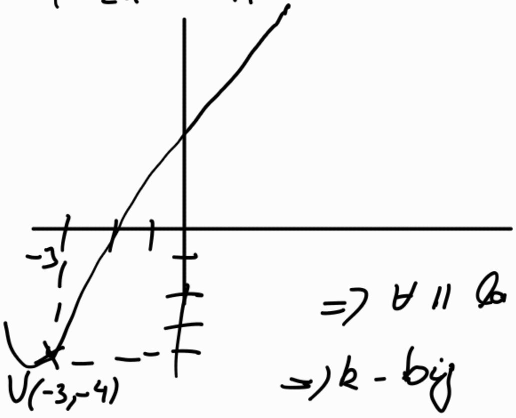
b) $g:\mathbb{R}\to\mathbb{R}, g(x)=5x+2$
Fie $x_1,x_2\in\mathbb{R}$ a.î. $g(x_1)=g(x_2) \Leftrightarrow 5x_1+1=5x_2+2 \Leftrightarrow x_1=x_2$ - g - inj $\Rightarrow g$ - bij
Fie $y\in\mathbb{R}$ a.î. $g(x)=y$
$g(x)=y \Rightarrow 5x+2=y \Rightarrow x=\frac{y-2}{5}\in\mathbb{R}$ - g - surj

c) $h:\mathbb{Z}\to\mathbb{Z}, h(x)=5x+2$
Fie $x_1,x_2\in\mathbb{R}$ a.î. $h(x_1)=h(x_2) \Leftrightarrow 5x_1+2=5x_2+2 \Rightarrow x_1=x_2$ - h - inj
Fie $y\in\mathbb{Z}$ a.î. $h(x)=y \Rightarrow 5x+2=y \Rightarrow x=\frac{y-2}{5}\notin\mathbb{Z}$ - h - nesurj

d) $k:(0,\infty)\to[0,\infty), k(x)=x^2+6x+5$
Fie $x_1,x_2\in(0,\infty)$ a.î. $k(x_1)=k(x_2) \Leftrightarrow$
$x_1^2+6x_1+5=x_2^2+6x_2+5 \Rightarrow x_1(x_1+6)=x_2(x_2+6) \Leftrightarrow x_1=x_2$ - k - inj
Fie $y\in[0,\infty)$ a.î. $k(x)=y$
$k(x)=y \Leftrightarrow x^2+6x+5=y$

Grafic

$V\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right) \Rightarrow V = (-3, -4)$



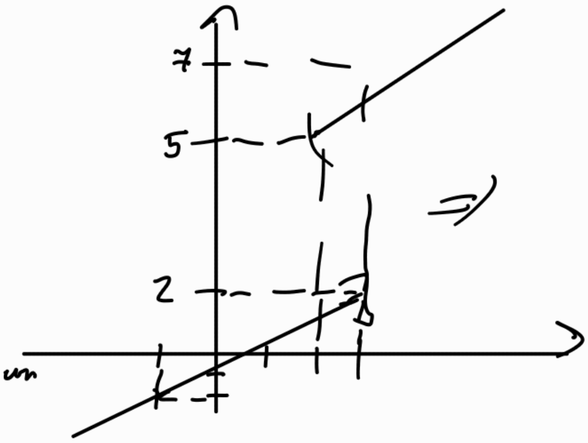
$\Rightarrow \forall \parallel$ la Ox dusă prin $(0, \infty) \cap G_f$ în fix un pct $\Rightarrow$

$\Rightarrow f$ - bij

2. $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x+1, & x \le 2 \\ 2x+1, & x > 2 \end{cases}$

a) $f(0) = 1$ | $x = 2 \to 5$

$f(2) = 3$ | $x = 3 \Rightarrow f(3) = 7$



$\Rightarrow$

$\Rightarrow f$ nu e bij deoarece $\forall \parallel$ la Ox $\cancel{\cap}\, G_f$ în fix un pct

b) $f[-1,3])$

$f(-1) = 0$

$f(2) = -1)$

$f([-1,3]) = [-2,2]$