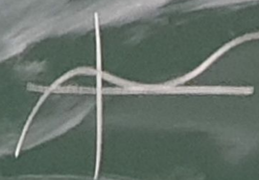
Tutoriat 7

① CS, CU pt $f_n:\mathbb{R}\to\mathbb{R}$, $f_n(x)=\dfrac{n^3x^2}{n^4+x^4}$, $\forall x\in\mathbb{R}$



Sol

CS: $\lim\limits_{n\to\infty} f_n(x) = \lim\limits_{n\to\infty}\dfrac{n^3 x^2}{n^4+x^4} = 0$, pt $x\neq 0$,

pt $x=0$ avem $f_n(0)=0$.

Luăm $f:\mathbb{R}\to\mathbb{R}$, $f(x)=0$ și spunem că $f_n \xrightarrow[\mathbb{R}]{s} f$

CU: $\lim\limits_{n\to\infty}\sup\limits_{x\in\mathbb{R}}|f_n(x)-f(x)|$.

Fixăm $n\in\mathbb{N}^*$

$$\sup_{x\in\mathbb{R}} \left|\frac{n^3x^2}{n^4+x^4}-0\right| = \sup_{x\in\mathbb{R}} \left|\frac{n^3x^2}{n^4+x^4}\right| \underset{x=n}{\geq} \frac{n^3\cdot n^2}{n^4+n^4} = \frac{n^5}{2n^4}=\frac{n}{2}$$

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}|f_n(x)-f(x)| \geq \lim_{n\to\infty}\frac{n}{2}=\infty\neq 0 \Rightarrow f_n \overset{u}{\underset{\mathbb{R}}{\not\rightarrow}} f$$

Tutoriat 7

① $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + 3xy^2 - 15x - 12y$, $\forall (x,y) \in \mathbb{R}^2$

$E \subseteq \underline{D_f \cup D_1 \cup D_2 \cup D_3 \cup D_4}$

Sol: f cont pe $\mathbb{R}^2 \Rightarrow \underline{D_f} = \{x \mid f$ nu e cont în $x\} = \emptyset$

Calculăm derivatele parțiale

$\frac{\partial f}{\partial x}(x,y) = (x^3 + 3xy^2 - 15x - 12y)'_x = 3x^2 + 3y^2 - 15$

$\frac{\partial f}{\partial y}(x,y) = (x^3 + 3xy^2 - 15x - 12y)'_y = 6xy - 12$

Obs că $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2$

$\mathbb{R}^2$ mulț deschisă

$\Rightarrow f$ e diferențiabilă pe $\mathbb{R}^2$

$\Downarrow$

$D_1 = \{x \mid f$ nu e diferenț$\} = \emptyset$

Vrem să det punctele critice ale lui f

$$\begin{cases} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0 \end{cases} \Longleftrightarrow \begin{cases} 3x^2+3y^2-15=0 \\ 6xy-12=0 \end{cases} \Longleftrightarrow \begin{cases} x^2+y^2-5=0 \\ xy=2 \end{cases} \xLeftrightarrow[y\neq 0]{x\neq 0} \begin{cases} x^2+y^2-5=0 \\ x=\frac{2}{y} \end{cases} \Longleftrightarrow \begin{cases} \frac{4}{y^2}+y^2-5=0 \,\big|\, \cdot y^2 \\ x=\frac{2}{y} \end{cases} \Longleftrightarrow$$

$$\Longleftrightarrow \begin{cases} 4+y^4-5y^2=0 \\ x=\frac{2}{y} \end{cases} \Longleftrightarrow \begin{cases} y^4-5y^2+4=0 \\ x=\frac{2}{y} \end{cases} \Longleftrightarrow \begin{cases} (y^2-4)(y^2-1)=0 \\ x=\frac{2}{y} \end{cases} \Longleftrightarrow \begin{cases} y_{1,2}=\pm 1;\ y_{3,4}=\pm 2 \\ x=\frac{2}{y} \end{cases} \Longrightarrow$$

$$\Longrightarrow C=\{(-2,-1),(2,1),(-1,-2),(1,2)\}$$ = mult pct critice.

Studiem dif de ordin 2

$$\frac{\partial^2 f}{\partial x^2}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y)=(3x^2+3y^2-15)'_x=6x$$

$$\frac{\partial^2 f}{\partial y\,\partial x}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y)=(3x^2+3y^2-15)'_y=6y$$

$$\frac{\partial^2 f}{\partial x\,\partial y}(x,y)=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y)=(6xy-12)'_x=6y$$

$$\frac{\partial^2 f}{\partial y^2}(x,y)=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y)=(6xy-12)'_y=6x$$

① f

⇒ f

A

$H_f($

Tutorat 7

① $f:\mathbb{R}^2\to\mathbb{R}$, $f(x,y)=x^3+3xy^2-15x-12y$, $\forall (x,y)\in\mathbb{R}^2$

Obs $\frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y\partial x}, \frac{\partial^2 f}{\partial^2 y}$ cont pe $\mathbb{R}^2$ (mult. desch) $\Rightarrow$

$\Rightarrow$ f e dif de 2 ori $\Rightarrow D_2=\{X \mid f$ nu e dif de 2 ori în $X\}=\emptyset$

Aplicăm criteriul lui Sylvester în pctele critice în care f e dif de 2 ori

$$H_f(x,y)=\begin{pmatrix}\frac{\partial^2 f}{\partial^2 x}(x,y) & \frac{\partial^2 f}{\partial x\partial y}(x,y)\\ \frac{\partial^2 f}{\partial y\partial x}(x,y) & \frac{\partial^2 f}{\partial^2 y}(x,y)\end{pmatrix}=\begin{pmatrix}6x & 6y\\ 6y & 6x\end{pmatrix}$$

D_3 = mult. pct. în care criteriul nu se pronunță

$$H_f(-2,-1)=\begin{pmatrix}-12 & -6\\ -6 & -12\end{pmatrix};\quad \Delta_1=-12<0,\ \Delta_2=144-36>0$$
$\Rightarrow$ $(-2,-1)$ pct de maxim local $\Rightarrow (-2,-1)\in D_3$

$$H_f(2,1)=\begin{pmatrix}12 & 6\\ 6 & 12\end{pmatrix};\quad \Delta_1=12>0,\ \Delta_2=144-36>0$$
$\Rightarrow$ $(2,1)$ pct de minim local $\Rightarrow (2,1)\in D_3$

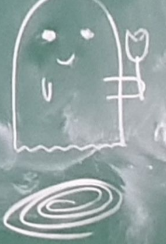
$C = \{(-2,-1),(2,1),(-1,-2),(1,2)\}$ - mult pct critice.

$H_f(-1,-2) = \begin{pmatrix} -6 & -12 \\ -12 & -6 \end{pmatrix}$; $\Delta_1 = -6 < 0$, $\Delta_2 = 36 - 144 < 0$ $\Rightarrow (-1,-2)$ nu e pct de ext local $\Rightarrow (-1,-2) \in D_3$

$H_f(1,2) = \begin{pmatrix} 6 & 12 \\ 12 & 6 \end{pmatrix}$; $\Delta_1 = 6 > 0$, $\Delta_2 = 36 - 144 < 0$ $\Rightarrow (1,2)$ nu e pct de ext local $\Rightarrow (1,2) \in D_3$

$\Rightarrow D_1 = \emptyset$ (D_4 = mult pct în care crit nu se pronunță)

$E \subseteq D_f \cup D_1 \cup D_2 \cup D_3 \cup D_4 \Rightarrow \boxed{E = \{(-2,-1),(2,1)\}.}$



Rezolvarea punctelor cu definiția

3. $f(x,y) = x^4 + y^3 - 4x^3 - 3y^2 + 3y$

$\Rightarrow C = \{(0,1), (3,1)\}$

$H_f(0,1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\Delta_1 = 0$, $\Delta_2 = 0$ $\Big| \Rightarrow (0,1)$ criteriul nu se pronunță $\Rightarrow (0,1) \in D_4$

$H_f(3,1) = \begin{pmatrix} 36 & 0 \\ 0 & 0 \end{pmatrix}$, $\Delta_1 = 36 > 0$, $\Delta_2 = 0$ $\Big| \Rightarrow (3,1)$ — " — $\Rightarrow (3,1) \in D_4$

Pentru punctele din D_4, verificăm cu definiția dacă sunt pcte de extrem !!

TEORIE

$f(x) \geq f(x_0)$, $\forall x \in D \cap V(x_0) \Rightarrow x_0$ e pct de min

$f(x) \leq f(x_0)$ maxim

$f(x) - f(x_0) \geq 0 \longrightarrow$ minim

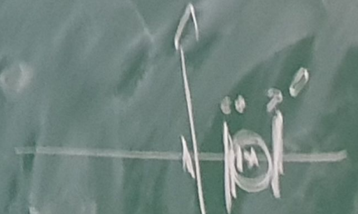
$\leq 0 \longrightarrow$ max

Evaluăm semnul diferenței $f(x,y) - f(0,1)$, când $(x,y) \in V(0,1)$

$$f(x,y) - f(0,1) = x^3(x-4) + y(y^2 - 3y + 3) - 1 \quad (\gtrless 0)$$

Luăm $\left(\frac{1}{n}, 1\right) \longrightarrow (0,1) \Rightarrow \frac{1}{n^3}\left(\frac{1}{n} - 4\right) + 1 - 1 = \underbrace{\frac{1}{n^3}}_{>0}\underbrace{\left(\frac{1}{n} - 4\right)}_{<0} < 0$

$\left(-\frac{1}{n}, 1\right) \longrightarrow (0,1) \Rightarrow f\left(-\frac{1}{n}, 1\right) = \underbrace{\frac{1}{n^4}}_{>0} + \underbrace{\frac{4}{n^3}}_{>0} > 0$



$\Rightarrow (0,1)$ nu e punct de extrem

Pt $(3,1)$, luăm șirurile $\left(3, 1 + \frac{1}{n}\right) > 0$

$\left(3, 1 - \frac{1}{n}\right) < 0$

$\Rightarrow (3,1)$ nu e pct de extrem

④ $f:\mathbb{R}^3 \to \mathbb{R}$, $f(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$

H.

$\left.\begin{array}{l} f \text{ continuă pe } \mathbb{R}^3 \\ \mathbb{R}^3 \text{ deschisă} \end{array}\right| \Rightarrow D_f = \emptyset$

$\left.\begin{array}{l} \text{Toate d.p. de ordin 1 sunt cont. pe } \mathbb{R}^3 \\ \mathbb{R}^3 \text{ deschisă} \end{array}\right| \Rightarrow D_p = \emptyset$

$\frac{\partial f}{\partial x}(x,y,z) = 2x - y + 1$

$\frac{\partial f}{\partial y}(x,y,z) = 2y - x$

$\frac{\partial f}{\partial z}(x,y,z) = 2z - 2$

mulțimea pct. critice

$$\begin{cases} \frac{\partial f}{\partial x}(x,y,z) = 0 \\ \frac{\partial f}{\partial y}(x,y,z) = 0 \\ \frac{\partial f}{\partial z}(x,y,z) = 0 \end{cases} \Leftrightarrow \begin{cases} 2x - y + 1 = 0 \\ 2y - x = 0 \\ 2z - 2 = 0 \end{cases} \Leftrightarrow \begin{cases} x = -2/3 \\ y = -1/3 \\ z = 1 \end{cases} \Rightarrow C = \left\{\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)\right\}$$

$\frac{\partial^2 f}{\partial x^2}(x,y,z) = 2.$ $\quad \frac{\partial^2 f}{\partial y^2}(x,y,z) = 2$ $\quad \frac{\partial^2 f}{\partial z^2}(x,y,z) = 2$

$\frac{\partial^2 f}{\partial x \partial y}(x,y,z) = -1 = \frac{\partial^2 f}{\partial y \partial x}(x,y,z)$ (cont. T. Schwartz)

$\frac{\partial^2 f}{\partial x \partial z}(x,y,z) = 0 = \frac{\partial^2 f}{\partial z \partial x}(x,y,z)$ (—,—)

$\frac{\partial^2 f}{\partial y \partial z}(x,y,z) = 0 = \frac{\partial^2 f}{\partial z \partial y}(x,y,z)$ (—,—)

Toate dp. de ordin 2 sunt cont. pe $\mathbb{R}^3$ | $\mathbb{R}^3$ deschis

$\Rightarrow \Delta_2 = \phi$

$$H_f(x,y,z) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial x \partial z} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial y \partial z} \\ \frac{\partial^2 f}{\partial z \partial x} & \frac{\partial^2 f}{\partial z \partial y} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad \forall (x,y,z) \in \mathbb{R}^3$$

$f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = x^2 + y^2 + z^2 - xy + x - 2z$.

$$H_f\left(-\frac{2}{3}, -\frac{1}{3}, 1\right) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\Delta_1 = 2 > 0$

$\Delta_2 = 4 - 1 = 3 > 0$

$\Delta_3 = 8 + 0 + 0 - (0 + 0 + 2) = 6 > 0$

$\Rightarrow \left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$ punct de minim l...

$\in D_3$.

$D_4 = \emptyset$.

$E \subseteq D_f \cup D_1 \cup D_2 \cup D_3 \cup D_4$

$E = \left\{\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)\right\}$

① $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = -x^4 - y^4$,

$D_f = \emptyset$

$\frac{\partial f}{\partial x}(x,y) = -4x^3$ $\quad \frac{\partial f}{\partial y}(x,y) = -4y^3$ $\Rightarrow D_1 = \emptyset$

$$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases} \Leftrightarrow \begin{cases} -4x^3 = 0 \\ -4y^3 = 0 \end{cases} \Leftrightarrow \begin{cases} x = 0 \\ y = 0 \end{cases} \Rightarrow C = \{(0,0)\}$$

$\frac{\partial^2 f}{\partial x^2}(x,y) = -12x^2$ $\quad \frac{\partial^2 f}{\partial y^2}(x,y) = -12y^2$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = 0 = \frac{\partial^2 f}{\partial y \partial x}(x,y)$ (conf. T. Schwartz) $\Rightarrow D_2 = \emptyset$

$$H_f(x,y) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} -12x^2 & 0 \\ 0 & -12y^2 \end{pmatrix}, \quad \forall (x,y) \in \mathbb{R}^2$$

$$H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = 0 \end{array}\right\} \Rightarrow \text{Crit nu decide} \Rightarrow D_4 = \{(0,0)\}$$

$$D_3 = \emptyset$$

$$\left.\begin{array}{l} f(0,0) = 0 \\ f(x,y) \leq 0 \end{array}\right| \Rightarrow f(x,y) \leq f(0,0) \ \forall (x,y) \in \mathbb{R}^2 \Rightarrow (0,0) \text{ punct de maxim local}$$

$$E = \{(0,0)\}$$

Ex din examene (seria 13)

* $f:(0,\infty)\times\mathbb{R}\longrightarrow\mathbb{R}$

$$f(x,y)=x\ln(x^2+y^2)$$

* $f:\mathbb{R}^2\longrightarrow\mathbb{R}$

$$f(x,y)=xye^{x+y}$$

* $f:(0,\infty)\times(0,\infty)\longrightarrow\mathbb{R}$

$$f(x,y)=\frac{1}{x}+\frac{1}{y}+xy+3$$