Seminar 3

I Algebră Liniară

→ Regula lui Laplace
→ Spații vectoriale

Fie $A \in M_n(K)$ $\quad p \in \mathbb{N}$, $1 \le p \le n$

K – corp comutativ

R.L. $\det A = \sum_{M} M \cdot M' = \sum \det_{ij} \cdot (-1)^{\sum_{r=1} i_r + j_r} \cdot \det A_{\bar{i}\bar{j}}$

C_n^p $\quad$ $I = \{1 \le i_1 \le i_2 \le \dots \le i_p \le n\}$ fixat

$J = \{1 \le j_1 \le j_2 \le \dots \le j_p \le n\}$

$\bar{I} = \{1, 2, \dots, n\} \setminus I$

$\bar{J} = \{1, 2, \dots, n\} \setminus J$

Ap! TL3 a) b) R.G → $\frac{L_3}{L_4}$

C.P. $p = 1 \rightarrow$ Dezv. linie / col.

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 3 & 4 \\ 2 & 5 & 1 & -1 \\ -1 & -2 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 5 \\ -1 & -2 \end{vmatrix} \cdot (-1)^{3+4+1+2} \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & -1 \\ -1 & 4 \end{vmatrix} \cdot (-1)^{3+4+1+4} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$+ \begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} \cdot (-1)^{3+4+1+3} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix}$$

$$+ \begin{vmatrix} 5 & 1 \\ -2 & 2 \end{vmatrix} (-1)^{3+4+2+3} \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} +$$

$$+ \begin{vmatrix} 5 & -1 \\ -2 & 4 \end{vmatrix} \cdot (-1)^{3+4+2+4} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} (-1)^{3+4+3+4} \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} =$$

$$= (-4+5)(8-9) + (8+1)(3-2) - (-4+1)(4-3) + (0+2)(4-3)$$

UNDEVA E GREŞIT (CRED)

Spaţii vectoriale

<u>Def:</u> Fie $V \neq \emptyset$, K - corp comutativ $+ : V \times V \to V$ op. internă (adunare vectorială)

$$(v_1, v_2) \to v_1 + v_2$$

$\cdot \; K \times V \to V$ op. externă (înmulţ. ext cu scalar)

$$(k, v) \to k \cdot v$$

I. grup abelian

II
1) $(k_1 + k_2) v = k_1 v + k_2 v$
2) $k(v_1 + v_2) = k v_1 + k v_2$
3) $(k_1 k_2) v = k_1 (k_2 v)$
4) $1 \cdot v = v$

$({}^{V}/_{K}, +, \cdot) \to$ sp. vectorial peste K

{ K - sp. vect }

V

$K = \mathbb{R} \to$ sp. vect real

$\mathbb{C} \to$ sp. vect complex

<u>Exp:</u> $K \to$ corp com

$({}^{K}/_{K}, +, \cdot)$ sp. vect. peste K, $H \subseteq K$ supcorp

C.p. ① $\mathbb{C}/\mathbb{C}$ $\quad \mathbb{R}/\mathbb{R}$ $\quad \mathbb{Q}/\mathbb{Q}$ $\quad \mathbb{Z}_p/\mathbb{Z}_p$ (p-prim)

$\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C} \Rightarrow \mathbb{C}/\mathbb{R},\ \mathbb{C}/\mathbb{Q},\ \mathbb{R}/\mathbb{Q}$

② Fie $V_1, V_2/K$ → două spații vectoriale peste K $\qquad V \overset{not.}{=} V_1\times V_2$

$+ : V\times V \to V$ op. internă $\qquad (v_1,v_2)+(w_1,w_2) \overset{def}{=} (v_1+w_1, v_2+w_2)$, $\forall (v_1,v_2),(w_1,w_2)\in V$

$\cdot : K\times V \to V$ op. externă $\qquad k(v_1,v_2) \overset{def}{=} (kv_1, kv_2)$, $\forall k\in K$, $(v_1,v_2)\in V$

$(V/K, +, \cdot) \to$ sp. vect. peste K $\qquad$ /GENERALIZARE:/

O PULĂ CA A ȘTERS!!!

C.P. $V_i/K = K/K\ (\forall)\ i=\overline{1,n} \to (K^n/K, +, \cdot)$ sp. vect. peste K

$K=\mathbb{R} \to \mathbb{R}^n/\mathbb{R}$ $\qquad \underset{2^n}{\mathbb{C}^n/\mathbb{R}}$ $\qquad \boxed{\mathbb{C}/\mathbb{R}}$ $\qquad \mathbb{C}\simeq\mathbb{R}\times\mathbb{R}=\mathbb{R}^2$

$\mathbb{C} \to \mathbb{C}^n/\mathbb{C}$ $\qquad \dim_{\mathbb{R}}\mathbb{C}=2$

③ $\left(M_{(m,n)}(K)/_K , + , \cdot \right) \to$ spațiu vect. peste K

$+$: ad. matrici

$\cdot$: înmulțirea matricelor cu scalari

④ $\left(K[X]/_K , + , \cdot \right) \to$ sp. vect. peste K

$+$: ad. polinoame

$\cdot$: înmulțirea pol. cu scalari

$$K_n[X] = \{ p \in K[X] \,/\, \text{grad } p \leq n \} \subset K[X]$$

$\left(K_n[X]/_K , +, \cdot \right)$ sp. vect. peste K

⑤ $E \neq \emptyset$

$V/_K$ – sp. vectorial $\quad V^E/_K = \{ f \,/\, f : E \to V \}$

$+ : V^E \times V^E \to V^E \qquad (f,g) \to f+g \qquad (f+g)(x) \overset{\text{def}}{=} f(x) + g(x),$
$\forall x \in E$

$K \times V^E \to V^E \qquad (kf)(x) \overset{\text{def}}{=} k \cdot f, \forall$
$(k,f) \to kf$

$\left(V^E/_K , + , \cdot \right)$ sp. vect. peste K

$\left(\mathbb{R}^{\mathbb{R}}/_{\mathbb{R}} , \underset{??}{+} , \cdot \right) \to$ sp. vect. real

Subspațiu vectorial

Def: Fie $(V/_K, +, \cdot)$ sp. vect.

$\emptyset \neq U \subseteq V$ $\quad U$ s.n. subspațiu vectorial dacă:

$$\begin{cases} 1) \ (\forall) u_1, u_2 \in U \Rightarrow u_1 + u_2 \in U \\ 2) \ \forall k \in K, u \in U \Rightarrow ku \in U \end{cases} \Leftarrow \boxed{\not{P}} U \subseteq V/_K$$

subsp. vect.

U ssp. vect al lui $V \Leftrightarrow \{ (\forall) u_1, u_2 \in U \Rightarrow k_1 u_1 + k_2 u_2 \in U$

$k_1, k_2 \in K$

Exp:

① $V/_K$ $\{0_V\}$, $V \in V \rightarrow$ subsp. v. triviale!!

② $K^m \leq K^n$, $m \leq n$, $m, n \in \mathbb{N}$

$K^m = \{ (x_1, \dots, x_m, \underset{m+1}{0}, \dots, \underset{n}{x_n}) / x_i \in K, \forall i = \overline{1,m} \} \subset K^n$

③ $K_n[x] \leq K[x]$

④ $\{ f / f : \mathbb{R} \to \mathbb{R} \} \subset \mathbb{R}^{\mathbb{R}}$ $\qquad G \leq \mathbb{R}^{\mathbb{R}}$

ssp. vect.

$\mathcal{R}_{[a,b]} \subseteq \boxed{R}$ sp. fct. integrabile Riemann

ssp. vect

APL. Stabiliți dacă urm. mulț. sunt ssp. vect. în sp. vect. indicate:

a) $U = \{(x,y,z) \in \mathbb{R}^3 / x+2y+z=0\} \subseteq \mathbb{R}^3$

$(\mathbb{R}^3 /_{\mathbb{R}}, +, \cdot)$ sp. vect. real

$+ : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \quad (x_1, y_1, z_1) + (x_2, y_2, z_2) \stackrel{def}{=} (x_1+x_2, y_1+y_2, z_1+z_3)$

op. int. (ad. vect.)

$\forall (x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$

$\cdot : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$

op. ext (înmul. cu scal.)

$k \cdot (x,y,z) \stackrel{def}{=} (kx, ky, kz), \forall k \in \mathbb{R}$

$(x,y,z) \in \mathbb{R}^3$

Fie $u_1 = (x_1, y_1, z_1)$, $u_2 = (x_2, y_2, z_2) \in U \Rightarrow k_1 u_1 + k_2 u_2 \in U$?

$k_1, k_2 \in \mathbb{R}$

$k_1 u_1 + k_2 u_2 = k_1 (x_1, y_1, z_2) + k_2 (x_2, y_2 + z_2)$

$x_1 + 2y_1 + z_1 = 0$

$x_2 + 2y_2 + z_2 = 0$

$= (\underbrace{k_1x_1 + k_2x_2}) + 2(k_1y_1 + k_2y_2) + (k_1z_1 + k_2z_2) = 0$

$= k_1(x_1 + 2y_1 + z_1) + k_2(x_2 + 2y_2 + z_2) = 0$

$\Rightarrow k_1u_1 + k_2u_2 \in U \Rightarrow \underset{\text{ssp. vect.}}{U \leq \mathbb{R}^3}$

I. 6. $\underset{\text{plan vect. } (\ni \text{ originea})}{\underline{U \leq \mathbb{R}^3}}$

ⓟ Fie $V/_K$ sp. vect. Dacă: $\underset{\text{ssp. vect.}}{U \leq V} \Rightarrow 0_V \in U$

NU MAI VREAAAU!!!

<u>TEMĂ:</u>

c) $\mathcal{S} = \{A \in \mathcal{M}_2(\mathbb{R}) \mid \underset{\text{m. simetrice}}{{}^tA = A}\} \subset \mathcal{M}_2(\mathbb{R})/_{\mathbb{R}}$

$\mathcal{A} = \{B \in \mathcal{M}_2(\mathbb{R}) \mid \underset{\text{m. antisimetrice}}{{}^tB = -B}\} \subset$ —— // ——

$\mathcal{U} = \{ C \in \mathcal{M}_2(\mathbb{R}) \mid \underset{\substack{\shortparallel \\ \text{urma} \\ (\text{suma } I_2)}}{Tr}\, C = 0 \}$ C —"—

$\mathcal{D} = \{ D \in \mathcal{M}_2(\mathbb{R}) \mid (\exists)\, \lambda \in \mathbb{R} \text{ a.î. } D = \lambda I_2 \}$ C —"—

m. diagonal