ex 1) Arătați că

a) $\mathbb{Q}[x]/_{(x^2-1)} \simeq \mathbb{Q}\times\mathbb{Q}$

$x^2-1=0 \Rightarrow x_{1,2}=\pm 1$

Fie $f: \mathbb{Q}[x] \to \mathbb{Q}\times\mathbb{Q}$ - surj

$f(P(x)) = (P(-1), P(1))$

Q - surj : Fie $(a,b)\in \mathbb{Q}\times\mathbb{Q}$

caut P a.î. $(P(-1), P(1)) = (a,b)$

$P = mx+n \to P(-1) = -m+n = a$

$m,n\in\mathbb{Q}$ $\quad \underline{P(1) = m+n = b}$ (+)

$2n = a+b \Rightarrow n = \frac{a+b}{2}$

$m = a - \frac{a+b}{2} \Rightarrow m\ \frac{b-a}{2}$

$P = \frac{b-a}{2}\cdot x + \frac{a+b}{2} \Rightarrow f = \text{surj}$

(2) $\mathrm{Ker}\, f = \{P\in\mathbb{Q}[x] \mid f(P) = (0,0)\} = \{P\in\mathbb{Q}[x] \mid P(-1), P(1) = (0,0)\} = \{P\in\mathbb{Q}[x] \mid P = (x^2-1)\cdot g,\ g\in\mathbb{R}[x]\} = (x^2-1)$

$= \{P\in\mathbb{Q}[x] \mid \pm 1$ răd pt $P\}$

$\overset{(T.F.I)}{\Rightarrow} \mathbb{Q}[x]/_{x^2-1} \simeq \mathbb{Q}\times\mathbb{Q}$

b) $\underbrace{\mathbb{Z}[x]/_{(x^2-1)}}_{\text{2 elem idempotente}} \not\simeq \underbrace{\mathbb{Z}\times\mathbb{Z}}_{\text{4 elem idempotente}}$

dacă era = $\Rightarrow$ nu știam sigur că sunt izom.

c) $\mathbb{R}[x]/_{(x^2+1)} \simeq \mathbb{C} = \mathbb{R}[i]$

Sau: $\mathbb{Z}[x]/_{(x^2+1)} \simeq \mathbb{Z}[i]$

$x^2+1=0 \Rightarrow x = \pm i$

Fie $f: \mathbb{R}[x] \to \mathbb{C}$

$f(P) = P(i)$

1) Surj : Fie $a+bi \in \mathbb{C}$, $a,b\in\mathbb{N}$

Caut $P\in\mathbb{R}[x]$ a.î. $P(i) = a+bi$

$\Rightarrow P = a+bx$ - surj

2) $\mathrm{Ker}\, f = \{P\in\mathbb{R}[x] \mid i = \text{răd}\} = \{P\in\mathbb{R}[x] \mid (x^2+1)\cdot g = P,\ g\in\mathbb{R}[x]\} = (x^2+1)$ $\xrightarrow{T.F.I}$ $\mathbb{R}[x]/_{(x^2+1)} \simeq \mathbb{C}$

$\mathbb{Z}[x]/_{(x^2+1)} \simeq \mathbb{Z}[i] = \{a+bi \mid a,b\in\mathbb{Z}\}$

d) $\mathbb{C}[x]/_{(x^2+1)} \simeq \mathbb{C}\times\mathbb{C}$

$x^2+1=0 \Rightarrow x = \pm i$

$P: \mathbb{C}[x] \to \mathbb{C}\times\mathbb{C}$

$\varphi(P) = (\varphi(i), f(-i))$

surj : $(z,t)\in\mathbb{C}\times\mathbb{C}$

$\varphi(i) = P$

$\begin{cases} f(i) = z \\ f(-i) = i \end{cases}$

$f = mx + n \Rightarrow \begin{cases} mi + n = z \\ -mi + n = t \end{cases} \Big| \overset{(+)}{\Rightarrow} 2n = z + t \Rightarrow n = \frac{z+t}{2}$

$mi = z - \frac{z+t}{2} \Rightarrow mi = \frac{z-t}{2} \Rightarrow m = \frac{z-t}{2i} = \frac{(z-t)i}{-2} = \frac{t-z}{2} \cdot i$

$f = \frac{t-z}{2} \cdot i \cdot x + \frac{z+t}{2} \Rightarrow f$ - surj

$\text{Ker } g = \{ f \in \mathbb{C}[x] \mid g(f) = (0,0) \} = \{ f \in \mathbb{C}[x] \mid f$ - răd i și $-i \} = (x^2+1)$

T.F.I $\Rightarrow \mathbb{C}[x]/(x^2+1) \simeq \mathbb{C} \times \mathbb{C}$

L.C.R (Lema chineză a resturilor)

$\mathbb{C}[x]/(x^2+1) \simeq \mathbb{C}[x]/(x-i) \times \mathbb{C}[x]/(x+i)$

$\simeq \mathbb{C} \times \mathbb{C}$

2. Arătați că $J = (2, x) \subset \mathbb{Z}[x]$ nu este principal

Obs: 1) J - sm principal dc $J = (f)$

2) $(f_1, \ldots, f_n) = \gcd(f_1, \ldots, f_n)$

doar dacă: $K[x]$ - corp: $\mathbb{R}[x]$, $\mathbb{Q}[x]$, dar nu și $\mathbb{Z}[x]$

Pp că $J = (f)$, $f \in \mathbb{Z}[x]$

$\Rightarrow (f) = (2, x)$

$\underbrace{2 \cdot F}_{\text{are termen liber par}} + \underbrace{x \cdot G}_{\text{nu are termen liber}}$

($\Rightarrow$ toți coef sunt pari 2·pol)

$x \in (f)$

I) grad $f = 0 \Rightarrow f$ - nr par

$\Rightarrow (f)$ pol cu coef par (F)

II grad $f = 1 \Rightarrow x = \pm x$

nr $2 \in (x)$ (F)

$\Rightarrow$ nu e polinom principal

$(2) = \{ 2 \cdot (a_n x^n + a_{n-1} \cdot x^{n-1} + \ldots + a_0) \}$

Aplicație L.C.R

$n_1, \ldots, n_t \in \mathbb{N}$

cu $(n_i, n_j) = 1$, $\forall i \neq j$

$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \vdots \\ x \equiv a_t \pmod{n_t} \end{cases}$ are sol unică $x_0 \bmod N = n_1 \cdot n_2 \cdot \ldots \cdot n_t$

sol in $\mathbb{Z}$: $S = \{ x_0 + N \cdot p \mid p \in \mathbb{Z} \}$

Alg de rez:

Pas 1: $N - n_i \ldots$

Pas 2: $m_i = \frac{N}{n_i}$

Pas 3: k_i = inversul lui m_i

[illegible] (mod n_i) (Alg lui Euclid pt nr mari)

$\hookrightarrow m_i \cdot k_i \equiv 1 (\bmod m_i)$ $\cdot a_t (\bmod N)$

Pas 4: $x_0 = k_1 \cdot m_1' \cdot a_1 + k_2 \cdot m_2' a_2 + \dots + k_t \cdot m_t' \cdot a_t (\bmod N)$

Pas 5: $\{x_0 + N \cdot p \mid p \in \mathbb{Z}\}$

4. Rez in $\mathbb{Z}$:

a) $\begin{cases} x \equiv 1 (\bmod 15) \\ x \equiv 5 (\bmod 8) \end{cases}$ $(15, 8) = 1$ - prime intre ele

$N = 15 \cdot 8 = 120 \Rightarrow \{0, 1, \dots, 119\}$

V_1: Verificare: $1, \cancel{16}, \cancel{31}, \cancel{46}, \boxed{61}$

$\Rightarrow x_0 = 61 \Rightarrow S = \{61 + 120 \cdot p \mid p \in \mathbb{Z}\}$

V_2: $m_1' = \frac{120}{15} = 8$, $m_2' = \frac{120}{8} = 15$

$8k_1 \equiv 1 (\bmod 15) \Rightarrow k_1 = 2$

$15k_2 \equiv 1 (\bmod 8) \Rightarrow k_2 = 7$

$x_0 = \underbrace{2 \cdot 8}_{16} \cdot 1 + \underbrace{7 \cdot 15}_{105} \cdot \underset{5}{5} = 16 + 5 \cdot 25 = 541 \equiv 61 (\bmod 120)$

b) $\begin{cases} x \equiv 4 (\bmod 20) \\ x \equiv 3 (\bmod 9) \end{cases}$

$N = 20 \cdot 9 = 180$

$\{0, 1, \dots, 179\}$

$\cancel{4}, \cancel{24}, \cancel{44}, \cancel{64}, \boxed{84} \Rightarrow x_0 = 84 \Rightarrow$

$\Rightarrow S = \{84 + 180 \cdot p \mid p \in \mathbb{Z}\}$

$N = 180$ $\gcd(9, 20) = 1$

$m_1' = \frac{180}{20} = 9$

$m_2' = \frac{180}{9} = 20$

$\begin{cases} 9k_1 \equiv 1 (\bmod 20) \\ 20k_1 \equiv 1 (\bmod 9) \end{cases} \Rightarrow \begin{cases} k_1 = 9 \\ k_2 = 5 \end{cases}$

$x_0 = 9 \cdot 9 \cdot 4 + 5 + 20 \cdot 3 = 624 (\bmod 180) = 84$

c) $\begin{cases} 2x \equiv 4 (\bmod 10) \\ x \equiv 1 (\bmod 21) \end{cases} \Rightarrow$ $2x : 10 = c$ rest 4

$2x = 10c + 4 \mid :2$

$x = 5c + 2$

$x \equiv 2 (\bmod 5)$

$\begin{cases} x \equiv 2 (\bmod 5) \\ x \equiv 1 (\bmod 21) \end{cases}$

$\gcd(15, 21) = 1$

$5 \cdot 21 = 105$

$x_0 = 22 \Rightarrow S = \{22 + 105 \cdot p \mid p \in \mathbb{Z}\}$

d) $\begin{cases} 2x \equiv 7 (\bmod 15) \\ x \equiv 2 (\bmod 8) \end{cases}$

$2x \equiv 7 \mid \cdot 8 \Rightarrow \begin{cases} x \equiv 11 (\bmod 15) \\ x \equiv 2 (\bmod 8) \end{cases} \Rightarrow x_0 = 26$

$S = \{26 + 120p \mid p \in \mathbb{Z}\}$