Internship Report

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1 The problem

First of all, I shall propose a differential equation and a set of boundary conditions that replicate Lord Talbot's experiment's features. This should define a function $u(t, x, y, z)$ that describes the Talbot effect. This effect appears when a plane wavefront, light in this case, is diffracted by an infinite set of equally spaced slits. As I am treating with wave optics, the differential equation must be the wave equation, which is

$$\partial_{tt}u = \partial_{xx}u + \partial_{yy}u + \partial_{zz}u, \tag{1.1}$$

where c is the speed of the wave, light in this case. I am going to stick to the ideal case of infinitely long gratings, which makes $u(t, x, y, z) = u(t, x, y', z)$, $\forall\, y, y' \in \mathbb{R}$. So the y coordinate can be disposed of: $u(t, x, y, z) \equiv u(t, x, z)$. I will also assume, without loss of generality, that the diffraction grating is set along the x axis, at $z = 0$.

Now, as there are infinitely many slits, the solution must be periodic, with a period d, equal to the distance between two slits, $u(t, x, z) = u(t, x + d, z)$. So I can restrict myself to studying the interval $0 \leq x \leq d$. Moreover, the wave must be equal at the left and the right side of every slit, so I must find an even function on x, $u(t, x, z) = u(t, -x, z)$. Having an even function means that

$$u(t, x, z) - u(t, -x, z) = 0 \implies \frac{u(t, +x_0, z) - u(t, -x_0, z)}{2x_0} = 0.$$

So, in the limit $x \to 0$ I find that

$$\partial_x u\big|_{x=0} \equiv \lim_{x_0 \to 0} \frac{u(t, x_0, z) - u(t, -x_0, z)}{2x_0} = 0.$$

Also, an even periodic function of period d can be expressed as a cosine Fourier series $\sum_n c_n \cos \frac{2\pi n}{d} x$. By considering that the Fourier series of the problem will have a *nice* behaviour, such that $\frac{\partial}{\partial x} \sum_n c_n \cos \frac{2\pi n}{d} x = \sum_n c_n \frac{\partial}{\partial x} \cos \frac{2\pi n}{d} x$, I find that

$$\partial_x u\big|_{x=0} = \partial_x u\big|_{x=d/2} = 0. \tag{1.2}$$

Knowing this, I can restrict myself to the region $0 \leq x \leq d/2$, $z \geq 0$, $t \geq 0$, as the complete solution may be recovered from the one in this region using the symmetries of the problem.

Now, I also know that the grating acts as a source, so its value at every instant of time should be known to us. If a perfect slit has a width w and is centred at $x = 0$, I will have $u(x, 0, t) = \sin(\omega t)\theta(t)$, $\forall\, x \in [0, w/2]$ in the case of monochromatic light, where A is the amplitude of the wave and ω is its angular frequency. Light is not able to traverse the interval between $w/2$ and $d/2$, so $u(x, 0, t) = 0$, $\forall\, x \in [w/2, d/2]$. Both expressions can be combined into only one by introducing the rectangular function, defined as

$$\chi(x) = \begin{cases} 1, & |x| < 1/2 \\ 0, & |x| > 1/2 \end{cases}.$$

I have then

$$u(t, x, 0) = f(t, x) = \sin(\omega t)\theta(t)\, \chi(x/w). \tag{1.3}$$

We can also treat the more general case $f(t, x) = \sin(\omega t)\theta(t)\, g(x)$ for some grating function $g(x)$. The final boundary condition I will take into account is the fact that

$$\lim_{z \to \infty} u(t, x, z) \text{is bounded}.$$

Finally, as we now want to understand the spreading of the wave starting with empty space. We thus set

$$u(t=0)=0, \quad \text{and}, \quad \partial_t u(t=0)=0. \tag{1.4}$$

With this last condition, the problem is complete and I may proceed to solve it. Figure 1 below summarises the content of this section.

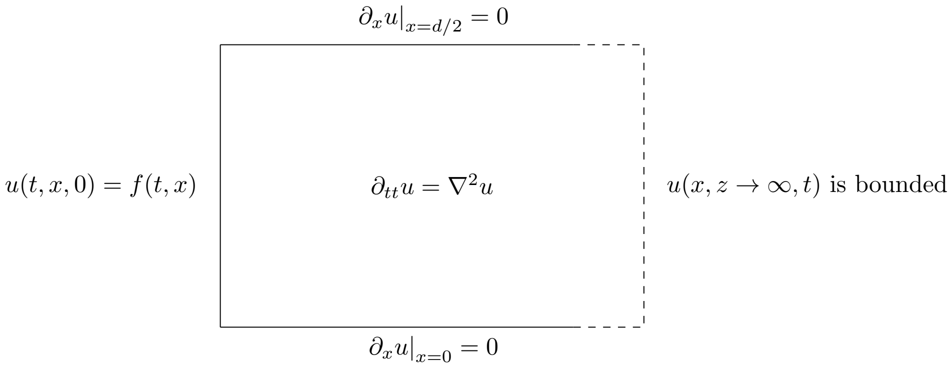

Figure 1: Sketch of the differential equation and boundary conditions obeyed by the system.

With a well-posed problem, we may try to solve it.

2 Reduction to a Klein-Gordon equation

We saw earlier that we can expand

$$u(t,x,z)=\sum_{n\in\mathrm{N}} c_n(t,z)\cos k_n x, \tag{2.1}$$

where $k_n=\frac{2\pi}{d}n$. By substituting this into the wave equation, we find

$$\sum_{n\in\mathrm{N}} \partial_{tt}c_n \cos k_n x = \sum_{n\in\mathrm{N}} \left(-k_n^2 c_n + \partial_{zz}c_n\right)\cos k_n x,$$

from which we can extract that the Fourier coefficients satisfy a 1D Klein-Gordon equation,

$$\partial_{tt}c_n - \partial_{zz}c_n + k_n^2 c_n = 0. \tag{2.2}$$

By combining equation (2.1) with the boundary conditions of u we can recover the boundary conditions that c_n must satisfy:

$$\begin{aligned} &c_n(z=0)=g_n \sin\omega t\,\theta(t), && c_n(z\to\infty) \text{ is bounded},\\ &c_n(t=0)=0, && \partial_t c_n(t=0)=0, \end{aligned}$$

where $g_n=\frac{4}{d}\int_0^{d/2} g(x)\cos k_n x\,dx$ is the Fourier coefficient of $g(x)$. For the original case $g(x)=\chi(x/w)$ we have

$$g_n=\chi_n=\begin{cases} 2\,w/d, & n=0\\ \dfrac{2}{n}\dfrac{\sin\frac{n\pi w}{d}}{\pi}, & n\neq 0 \end{cases}. \tag{2.3}$$

3 Laplace transform of the equation

We now take the Laplace transform of c_n in time, so we get

$$\begin{aligned} \mathcal{L}_t\left\{\partial_{tt}c_n - \partial_{zz}c_n + k_n^2 c_n\right\}(s) &= s^2\hat{c}_n - s c_n(t=0,z) - \partial_t c_n(t=0,z) - \partial_{zz}\hat{c}_n + k_n^2\hat{c}_n\\ &= s^2\hat{c}_n - \partial_{zz}\hat{c}_n + k_n^2\hat{c}_n, \end{aligned}$$

where $\hat{c}_n(s,z) \equiv \mathcal{L}_t\{c_n(t,z)\}(s) \equiv \int_0^\infty c_n(t,z)e^{-ts}\,dt$ is the Laplace transform of c_n. We thus have the equation

$$\partial_{zz}\hat{c}_n - (s^2+k_n^2)\hat{c}_n = 0, \tag{3.1}$$

with boundary conditions

$$\hat{c}_n(z=0) = g_n\frac{\omega}{\omega^2+s^2}, \qquad \hat{c}_n(z\to\infty) \text{ is bounded.} \tag{3.2}$$

The solution is clearly of the form

$$\hat{c}_n(s,z) = A_n(s)e^{-\sqrt{s^2+k_n^2}z} + B_n(s)e^{\sqrt{s^2+k_n^2}z},$$

and by imposing the boundary conditions (3.2), we find

$$\hat{c}_n(s,z) = g_n\frac{\omega}{\omega^2+s^2}e^{-\sqrt{s^2+k_n^2}z}. \tag{3.3}$$

Thus, the solution to our Klein-Gordon problem is

$$c_n(t,z) = \mathcal{L}_s^{-1}\left\{\hat{c}_n(s,z)\right\}(t) = \mathcal{L}_s^{-1}\left\{g_n\frac{\omega}{\omega^2+s^2}e^{-\sqrt{s^2+k_n^2}z}\right\}(t). \tag{3.4}$$

We now need to devote ourselves to inverting this Laplace transform.

4 Inverting the Laplace transform

We know that

$$\mathcal{L}_t\{f(t)*g(t)\}(s) = F(s)G(s), \tag{4.1}$$

where $F(s) = \mathcal{L}_t\{f(t)\}(s)$ and $G(s) = \mathcal{L}_t\{g(t)\}(s)$, and where $*$ denotes the convolution operator

$$f(t)*g(t) = \int_0^t f(\tau)*g(t-\tau)\,d\tau. \tag{4.2}$$

In our case, we have

$$F(s) = g_n\frac{\omega}{\omega^2+s^2} \qquad G(s) = e^{-z\sqrt{k_n^2+s^2}}. \tag{4.3}$$

We then find

$$f(t) = g_n\sin\omega t\,\theta(t) \qquad g(t) = \delta(t-z) - \theta(t-z)k_nz\frac{J_1\left(k_n\sqrt{t^2-z^2}\right)}{\sqrt{t^2-z^2}}. \tag{4.4}$$

Where we have used equation (A.9) for $G(s)$. Thus

$$c_n(t,z) = g_n\int_0^t \sin\omega(t-\tau)\left(\delta(\tau-z) - \theta(\tau-z)k_nz\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\right)d\tau \tag{4.5}$$

$$= g_n\sin\omega(t-z)\theta(t-z) - g_n\int_0^t \sin\omega(t-\tau)\theta(\tau-z)k_nz\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\,d\tau \tag{4.6}$$

$$= g_n\sin\omega(t-z)\theta(t-z) - g_nk_nz\int_z^{\max(t,z)} \sin\omega(t-\tau)\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\,d\tau. \tag{4.7}$$

Putting everything together, we find that

$$u(t,x,z) = \sum_n g_n\left(\sin\omega(t-z) - k_nz\int_z^t \frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\sin\omega(t-\tau)\,d\tau\right)\theta(t-z)\cos k_nx. \tag{4.8}$$

5 Asymptotic behaviour at $t \to \infty$

Let us study the asymptotic behaviour of the solution when $t \to \infty$. For this, we will approximate

$$\int_z^t \sin\omega(t-\tau)\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\,d\tau \to \int_z^\infty \sin\omega(t-\tau)\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\,d\tau. \tag{5.1}$$

By making the substitution $r^2 = \tau^2 - z^2$ we get

$$\int_z^\infty \sin\omega(t-\tau)\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\,d\tau = \int_0^\infty \frac{\sin\omega(t-\sqrt{r^2+z^2})}{\sqrt{r^2+z^2}}J_1\left(k_n r\right)\,d\tau,$$

which can be rewritten into a Hankel integral:

$$\begin{aligned} I(t,z;k) := \int_0^\infty \frac{\sin\omega(t-\sqrt{r^2+z^2})}{\sqrt{r^2+z^2}}J_1\left(k_n r\right)\,d\tau &= \int_0^\infty \frac{\sin\omega(t-\sqrt{r^2+z^2})}{\sqrt{k_n r}\sqrt{r^2+z^2}}\sqrt{k_n r}J_1\left(k_n r\right)\,d\tau \\ &\equiv \mathcal{H}_1\left(\frac{\sin\omega(t-\sqrt{r^2+z^2})}{\sqrt{k_n r}\sqrt{r^2+z^2}};k_n\right). \end{aligned} \tag{5.2}$$

Now we use the fact that $\sin\omega(t-\sqrt{r^2+z^2}) = \sin\omega t\,\cos\omega\sqrt{r^2+z^2} - \cos\omega t\,\sin\omega\sqrt{r^2+z^2}$ and the following tabulated transforms from [1]

$f(r)$	$\mathcal{H}_1\left(f(r);k\right) \equiv \int_0^\infty f(r)\sqrt{k_n r}J_1\left(k_n r\right)\,d\tau$
$\dfrac{\sin\omega\sqrt{r^2+z^2}}{\sqrt{r}\sqrt{r^2+z^2}}$	$\begin{cases} \frac{1}{z\sqrt{k}}\left[\sin\omega z - \sin z\sqrt{\omega^2-k^2}\right] & : 0<k<\omega \\ \frac{\sin\omega z}{z\sqrt{k}} & : \omega<k<\infty \end{cases}$
$\dfrac{\cos\omega\sqrt{r^2+z^2}}{\sqrt{r}\sqrt{r^2+z^2}}$	$\begin{cases} \frac{1}{z\sqrt{k}}\left[\cos\omega z - \cos z\sqrt{\omega^2-k^2}\right] & : 0<k<\omega \\ \frac{1}{z\sqrt{k}}\left[\cos\omega z - e^{-z\sqrt{k^2-\omega^2}}\right] & : \omega<k<\infty \end{cases}$

So we find that

$$\begin{aligned} I(t,z;k_n) &= \sin\omega t\,\mathcal{H}_1\left(\frac{\cos\omega(\sqrt{r^2+z^2})}{\sqrt{r}\sqrt{r^2+z^2}};k_n\right) - \cos\omega t\,\mathcal{H}_1\left(\frac{\sin\omega(\sqrt{r^2+z^2})}{\sqrt{r}\sqrt{r^2+z^2}};k_n\right) \\ &= \frac{1}{k_n z}\begin{cases} \sin\omega t\,\left[\cos\omega z - \cos z\sqrt{\omega^2-k_n^2}\right] - \cos\omega t\,\left[\sin\omega z - \sin z\sqrt{\omega^2-k_n^2}\right] & : 0<k_n<\omega \\ \sin\omega t\,\left[\cos\omega z - e^{-z\sqrt{k_n^2-\omega^2}}\right] - \cos\omega t\,\sin\omega z & : \omega<k_n<\infty \end{cases} \\ &= \frac{1}{k_n z}\begin{cases} \sin\omega(t-z) - \sin\left(\omega t - z\sqrt{\omega^2-k_n^2}\right) & : 0<k_n<\omega \\ \sin\omega(t-z) - \sin\omega t\,e^{-z\sqrt{k_n^2-\omega^2}} & : \omega<k_n<\infty \end{cases} \end{aligned} \tag{5.3}$$

Thus we find that the asymptotic behaviour of our solution as $t \to \infty$ is

$$\begin{aligned} u(t,x,z) &= \sum_n g_n\left(\sin\omega(t-z) - k_n z\int_z^{t\to\infty}\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\sin\omega(t-\tau)\,d\tau\right)\theta(t-z)\cos k_n x \\ &= \sum_n g_n\cos k_n x \times \begin{cases} \sin\left(\omega t - z\sqrt{\omega^2-k_n^2}\right) & : 0<k_n<\omega \\ \sin\omega t\,e^{-z\sqrt{k_n^2-\omega^2}} & : \omega<k_n<\infty \end{cases}. \end{aligned} \tag{5.4}$$

This is precisely the result we found by solving the stationary case!!!! As the cherry on top, in this case we did not even have to *impose* Sommerfeld's radiation condition, as it has *arisen naturally* from our model.

6 Bounding the error in the asymptotic regime

We now want to see *how fast* the solution tends to the stationary case. For this we will study the asymptotic behaviour of the error,

$$E := \int_t^\infty \sin\omega(t-\tau)\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}}\,d\tau, \tag{6.1}$$

in the regime $z \ll t$. We thus Taylor expand

$$\frac{J_1\left(k_n\sqrt{\tau^2-z^2}\right)}{\sqrt{\tau^2-z^2}} = \frac{J_1\left(k_n\tau[1+\mathcal{O}(z^2/\tau^2)]\right)}{\tau[1+\mathcal{O}(z^2/\tau^2)]} = \frac{J_1\left(k_n\tau\right)}{\tau}\left[1+\mathcal{O}(z^2/\tau^2)+\mathcal{O}(k_n z^2/\tau)\right].$$

Thus,

$$\begin{aligned} E &= \int_t^\infty \sin\omega(t-\tau)\frac{J_1\left(k_n\tau\right)}{\tau}\left[1+\mathcal{O}(z^2/\tau^2)+\mathcal{O}(k_n z^2/\tau)\right]\,d\tau, \\ &\le \int_t^\infty \sin\omega(t-\tau)\frac{J_1\left(k_n\tau\right)}{\tau}\,d\tau\left[1+\mathcal{O}(z^2/t^2)+\mathcal{O}(k_n z^2/t)\right], \\ &\le I\left[1+\mathcal{O}(z^2/t^2)+\mathcal{O}(k_n z^2/t)\right]. \end{aligned} \tag{6.2}$$

We now use the asymptotic behaviour of the Bessel function,

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}}\left(\cos\left(x-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)-\frac{\sin\left(x-\frac{\alpha\pi}{2}-\frac{\pi}{4}\right)}{x}+\mathcal{O}\left(x^{-2}\right)\right), \quad \text{for } x \gg \alpha^2-1/4, \tag{6.3}$$

where the so,

$$\begin{aligned} I &= \int_t^\infty \sin\omega(t-\tau)\frac{J_1\left(k_n\tau\right)}{\tau}\,d\tau, \\ &= -\sqrt{\frac{2}{\pi k_n}}\int_t^\infty \frac{\sin\omega(\tau-t)}{\tau^{3/2}}\left[\cos\left(k_n\tau-\frac{3\pi}{4}\right)-\frac{\sin\left(k_n\tau-\frac{3\pi}{4}\right)}{k_n\tau}+\mathcal{O}\left((k_n t)^{-2}\right)\right]d\tau, \\ &= -\sqrt{\frac{1}{2\pi k_n}}\int_t^\infty \frac{1}{\tau^{3/2}}\left[\sin\left((\omega-k_n)\tau-\omega t+\frac{3\pi}{4}\right)+\sin\left((\omega+k_n)\tau-\omega t+\frac{3\pi}{4}\right)+\right. \\ &\left.+\frac{\cos\left((\omega-k_n)\tau-\omega t+\frac{3\pi}{4}\right)-\cos\left((\omega+k_n)\tau-\omega t+\frac{3\pi}{4}\right)}{k_n\tau}+\mathcal{O}\left((k_n t)^{-2}\right)\right]d\tau. \end{aligned} \tag{6.4}$$

We thus need to estimate integrals of the form $\int_t^\infty \frac{\sin(ax+b)}{x^c}\,dx$ and $\int_t^\infty \frac{\cos(ax+b)}{x^c}\,dx$. Using integration by parts we find that

$$\begin{aligned} \int_t^\infty \frac{\sin(ax+b)}{x^c}\,dx &= -\left.\frac{\cos(ax+b)}{ax^c}\right|_t^\infty - c\int_t^\infty \frac{\cos(ax+b)}{ax^{c+1}}\,dx \\ &= \frac{\cos(at+b)}{at^c} - c\int_t^\infty \frac{\cos(ax+b)}{ax^{c+1}}\,dx \\ &= \frac{\cos(at+b)}{at^c}\left[1+\mathcal{O}(1/t)\right], \end{aligned}$$

and similarly for $\int_t^\infty \frac{\cos(ax+b)}{x^c}\,dx = -\frac{\sin(at+b)}{at^c}\left[1+\mathcal{O}(1/t)\right]$. Thus,

$$\begin{aligned} I &= -\sqrt{\frac{1}{2\pi k_n t}}\left[\frac{\cos\bigl((\omega-k_n)t-\omega t+\frac{3\pi}{4}\bigr)}{(\omega-k_n)t}+\frac{\cos\bigl((\omega+k_n)t-\omega t+\frac{3\pi}{4}\bigr)}{(\omega+k_n)t}\right]\left[1+\mathcal{O}\left(1/k_n t\right)\right], \\ &= -\sqrt{\frac{1}{2\pi k_n t^3}}\left[\frac{\cos\bigl(k_n t-\frac{3\pi}{4}\bigr)}{(\omega-k_n)}+\frac{\cos\bigl(k_n t+\frac{3\pi}{4}\bigr)}{(\omega+k_n)}\right]\left[1+\mathcal{O}\left(1/k_n t\right)]\right], \\ &= \sqrt{\frac{1}{4\pi k_n t^3}}\left[\frac{\cos(k_n t)-\sin(k_n t)}{(\omega-k_n)}+\frac{\cos(k_n t)+\sin(k_n t)}{(\omega+k_n)}\right]\left[1+\mathcal{O}\left(1/k_n t\right)\right], \\ &= \sqrt{\frac{1}{\pi k_n t^3}}\frac{\omega\cos(k_n t)-k_n\sin(k_n t)}{\omega^2-k_n^2}\left[1+\mathcal{O}\left(1/k_n t\right)+\right]. \end{aligned} \tag{6.5}$$

We thus find that

$$\begin{aligned} E &\le \sqrt{\frac{1}{\pi k_n t^3}}\frac{\omega\cos(k_n t)-k_n\sin(k_n t)}{\omega^2-k_n^2}\left[1+\mathcal{O}\left(1/k_n t\right)+\mathcal{O}(z^2/t^2)+\mathcal{O}(k_n z^2/t)\right] \\ &\lesssim \sqrt{\frac{1}{\pi k_n t^3}}\frac{\omega\cos(k_n t)-k_n\sin(k_n t)}{\omega^2-k_n^2}. \end{aligned} \tag{6.6}$$

Note: This is problematic for $k_n^2 \approx \omega^2$ as there is a resonance. We need to take care of this in another way.

References

[1] Bateman Manuscript Project, H. Bateman, A. Erdélyi, and United States. Office of Naval Research. *Tables of Integral Transforms: Based, in Part, on Notes Left by Harry Bateman.* Number v. 2 in California Institute of technology. Bateman Manuscript project. A. Erdélyi, editor. W. Magnus, F. Oberhettinger, F.G. Tricomi, research associates. McGraw-Hill, 1954.

A Computing the inverse Laplace transform of $e^{-z\sqrt{s^2+k^2}}$

In equation (4.4) we stated that

$$\mathcal{L}_t^{-1}\{G(s)\}(s) = \mathcal{L}_t^{-1}\left\{e^{-z\sqrt{k^2+s^2}}\right\}(s) = \delta(t-z) - \theta(t-z)k_n z \frac{J_1\left(k_n\sqrt{t^2-z^2}\right)}{\sqrt{t^2-z^2}}. \tag{A.1}$$

Let us prove this. We start by introducing the related function

$$H(s) = \frac{d}{ds}[e^{zs}G(s)] = \frac{d}{ds}\left[e^{zs-z\sqrt{k^2+s^2}}\right] = z\frac{\sqrt{k^2+s^2}-s}{\sqrt{k^2+s^2}}e^{zs-z\sqrt{k^2+s^2}}, \tag{A.2}$$

which we Taylor expand

$$\begin{aligned} H(s) &= z\frac{\sqrt{k^2+s^2}-s}{\sqrt{k^2+s^2}}\sum_{n=0}\frac{\left(zs-z\sqrt{k^2+s^2}\right)^n}{n!} \\ &= kz\sum_{n=0}\frac{(-kz)^n}{n!}\frac{\left(\sqrt{k^2+s^2}-s\right)^{n+1}}{k^{n+1}\sqrt{k^2+s^2}}. \end{aligned}$$

We now take the inverse Laplace transform of this, using the identity

$$\mathcal{L}_t\left\{J_n(kt)\theta(t)\right\}(s) = \frac{\left(\sqrt{k^2+s^2}-s\right)^n}{k^n\sqrt{k^2+s^2}}, \quad n > -1,\ \mathrm{Re}(s) > 0. \tag{A.3}$$

So we find that

$$h(t) \equiv \mathcal{L}_s^{-1}\left\{H(s)\right\}(t) = kz\sum_{n=0}\frac{(-kz)^n}{n!}J_{n+1}(kt)\theta(t).$$

Now, we want to use the multiplication theorem for Bessel functions

$$\lambda^{-\nu}J_\nu(\lambda x) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\left(1-\lambda^2\right)x}{2}\right)^n J_{\nu+n}(x), \tag{A.4}$$

to simplify the expression. We thus rewrite

$$\begin{aligned} \sum_{n=0}\frac{(-kz)^n}{n!}J_{n+1}(kt)\theta(t) &= \sum_{n=0}\frac{(1-\lambda^2)^n(kt)^n}{n!}J_{n+1}(kt) \\ &= \lambda^{-1}J_1(\lambda kt), \end{aligned}$$

where $\lambda = \sqrt{2z/t+1}$. So we find

$$h(t) = \frac{kz}{\sqrt{2z/t+1}}J_1\left(\sqrt{2z/t+1}kt\right)\theta(t) = \frac{kz\sqrt{t}}{\sqrt{2z+t}}J_1\left(k\sqrt{t(2z+t)}\right)\theta(t). \tag{A.5}$$

Now, we have that

$$-tG(s) = \frac{d}{ds}G(s) = \frac{d}{ds}[e^{-zs}e^{zs}G(s)] = -ze^{-zs}[e^{zs}G(s)] + e^{-zs}\frac{d}{ds}[e^{zs}G(s)] = -zG(s) + e^{-zs}H(s),$$

so we find that

$$G(s) = \frac{e^{-zs}}{z-t}H(s), \tag{A.6}$$

thus

$$g(t)\theta(t) = \mathcal{L}_s^{-1}\left\{\frac{e^{-zs}}{z-t}H(s)\right\}(t) = \frac{1}{z-t}\mathcal{L}_s^{-1}\left\{e^{-zs}H(s)\right\}(t) = -\frac{h(t-z)}{t-z}\theta(t-z), \tag{A.7}$$

So we find that

$$g(t)\theta(t) = -\frac{kz}{\sqrt{t^2-z^2}}J_1\left(k\sqrt{t^2-z^2}\right)\theta(t-z).$$

But this is one $\delta(t-z)$ short! The reason why we are missing this δ is because most of the steps taken above were considering the $z > 0$ case. If $z = 0$, several of the steps break. But this can be fixed by using the initial value theorem, which states that

$$f(0^+) = \lim_{s\to\infty} sF(s), \tag{A.8}$$

which in this case implies that

$$g(t=0^+) = \lim_{s\to\infty} se^{-z\sqrt{k^2+s^2}} = \delta(z), \quad z \geq 0,$$

thus $g(t)$ must have a $\delta(t-z)$[1]. Thus

$$g(t) = \delta(t-z) - \frac{kz}{\sqrt{t^2-z^2}} J_1\left(k\sqrt{t^2-z^2}\right)\theta(t-z), \tag{A.9}$$

as claimed. This result can also be found in [1].

[1] A $\delta(t+z)$ is not possible because of the $\theta(t-z)$!