Department of Aerospace Science and Technology
Launch Systems course
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Flipped Classroom - Staging

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Optimal Gross Lift-Off Mass (GLOM)

This paper aims to explore and identify the ideal structural parameters (ε) for each stage of a multistage launch vehicle. First, an extensive investigation of a three-stage launch design is carried out. The main objective is to use the Optimal Staging methodology to maximize the payload-to-total-mass ratio.

The method of Lagrange multipliers, which is well-suited for resolving restricted optimization problems including equality constraints, is used in the analysis to comprehensively tackle this issue. This mathematical approach ensures an optimal solution by enabling the systematic calculation of the objective function's boundaries while meeting the specified limitations.

Given a generic equality constraint $g(x, y) = 0$, the augmented function is constructed as:

$$J = f(x, y) + \lambda g(x, y), \tag{1}$$

where $f(x, y)$ is the objective function to be optimized, λ is the Lagrange multiplier, which accounts for the constraint.

Hence in this analysis, f and g are defined as:

$$f = \ln\left(\frac{m_\mathrm{p}}{m_0}\right) = \sum_{i=1}^{N} \ln\left(1 - \xi_{\mathrm{s},i}\right) \qquad\qquad g = \Delta V - \sum_{i=1}^{N} \Delta V_i \tag{2}$$

The purpose of this analysis is to optimize the structural parameters to minimize the initial total mass while satisfying the required target velocity ΔV computed as:

$$\Delta V = \sum_{i=1}^{3} c_i \ln\left(\frac{c_i\lambda - 1}{c_i\varepsilon_{s_i}\lambda}\right),$$

where c_i is the exhaust velocity for each stage. Solving this equation for λ ensures the target velocity is achieved with minimal mass. To simplify the formulation, finite approximations of drag and gravity losses are neglected in this study.

Once λ is determined, the mass ratio for each stage is computed as:

$$n_{v_i} = \frac{c_i\lambda - 1}{c_i\varepsilon_{s_i}\lambda}.$$

Iteratively, from the payload mass up to the first stage, the launch vehicle's total mass—including propellant and structural masses—is determined using these mass ratios. The calculated GLOM, given the parameters, is $m_0 = 220420kg$. Given the substantial impact that structural efficiency has on the vehicle's performance, this conclusion emphasizes the significance of optimizing the mass distribution.

b) Sensitivity Analysis of GLOM to ε_s Variations

To evaluate the robustness of the vehicle design, a sensitivity analysis was conducted by varying the structural mass indices (ε_s) of each stage by ±10%. For each perturbation, the GLOM was recalculated using the same Lagrange multiplier method. This analysis evaluates how much structural mass affects the overall performance of the launch vehicle. The results of the sensitivity analysis are visualized in two key graphs.

The GLOM as a function of the structural mass indices (ε_{s1}, ε_{s2}, and ε_{s3}) is shown in three dimensions in Figure 1. Since the first stage contributes the most to the total mass, the color

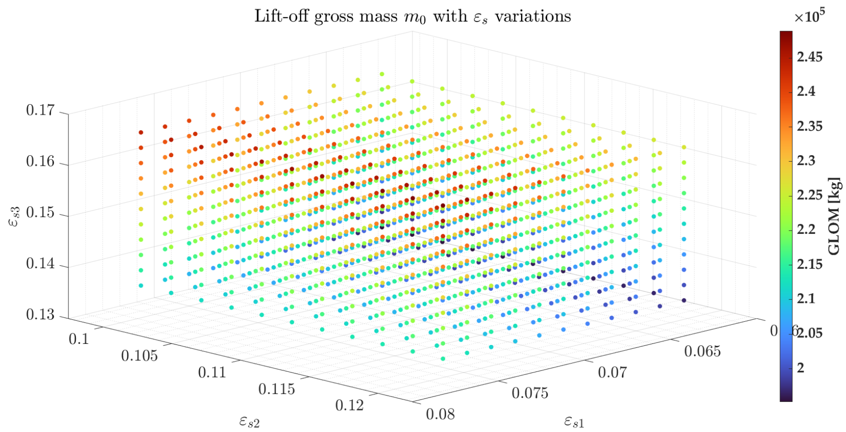

Figure 1: Gross Lift-Off Mass (GLOM) as a function of variations in structural mass indices (ε_{s1}, ε_{s2}, ε_{s3}).

distribution shows how changes in ε_{s1} cause the biggest changes in GLOM. Because of the third stage's lower mass percentage, the influence of ε_{s3} is less noticeable. The GLOM evolution over samples produced by altering the structural indices is shown in Figure 2. The GLOM evolution is displayed in the upper plot, and it increases dramatically when ε_{s1} increases. The percentage change in ε_s for each step is shown in the lower plot, and there is a clear relationship between the GLOM increments and the highest peak of ε_{s1} perturbations. These results, which were calculated iteratively, validate how important it is to optimize ε_{s1} in order to guarantee the robustness and effectiveness of the design.

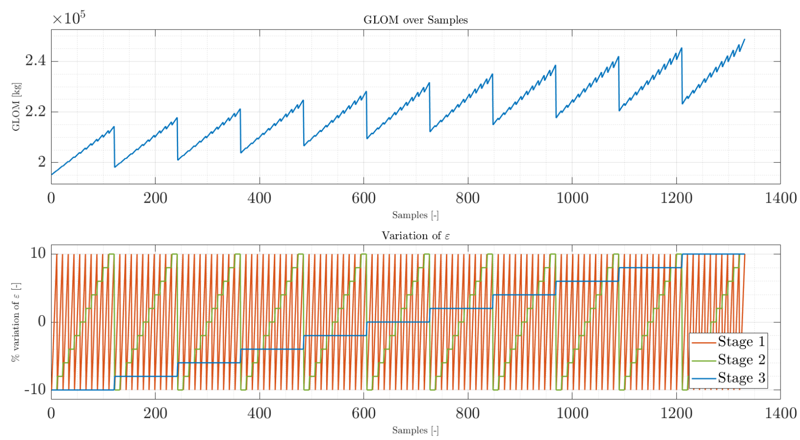

Figure 2: Top plot: Evolution of Gross Lift-Off Mass (GLOM) over sample variations in structural indices. Bottom plot: Percentage variations in structural indices (ε_s) for each stage.

c) Analysis of the TSTO Configuration

This section examines if using a Two-Stage-To-Orbit (TSTO) vehicle to complete the mission is evaluated. The structural mass fractions were assumed to be $\varepsilon_{s1} = 0.07$ for the first stage and $\varepsilon_{s2} = 0.09$ for the second stage. The specific impulses of the LOX-LCH$_4$ engines are $I_{sp1} = 330\,\mathrm{s}$ and $I_{sp2} = 380\,\mathrm{s}$ for the first and second stage, respectively.

As mentioned in the first paragraph, the goal of the mission was to reach a total ΔV of $10\,\mathrm{km/s}$, without assuming any losses. The Lagrange multiplier technique was used to determine the ideal

GLOM, and Table 2 displays the outcomes:

Stage	ε_s	I_{sp} [s]	Mass [kg]
1	0.07	330	139'106
2	0.09	380	26'433

Table 1: Stage Parameters and Masses

Total (GLOM) [kg]
170'539

Table 2: Optimal GLOM

The outcomes demonstrate a balance between the structural efficiency and propellant performance of both stages.

With the proper mass distribution and structural parameters, the results confirm that a TSTO design is feasible to comply with the mission requirements. The selected values for ε_s and I_{sp} are consistent with current rocket designs and estimates from the literature.

d) Optimization of ΔV Allocation for TSTO

In point 4, the objective was to optimize the distribution of the ΔV between the two stages of a Two-Stage-To-Orbit (TSTO) vehicle in order to minimize the Gross Lift-Off Mass (GLOM). The analysis accounted for total gravity and aerodynamic losses estimated at 2 km/s and focused on finding the optimal α, which represents the fraction of the ideal ΔV assigned to the first stage, while assuming that all losses were compensated by the first stage ($\beta = 1$). Losses are considered already integrated in the mission ΔV budget so the ideal ΔV_{orbit} to reach orbit is equal to the difference between the budget and the total losses.

The α parameter was varied between 0.1 and 0.9, and for each value, the ΔV assigned to each stage was calculated, followed by the mass ratios and the structural and propellant masses of both stages. The analysis yielded the following results in terms of optimal α and optimal GLOM:

Table 3: Optimization Results for TSTO

Optimal Parameter	Value
Optimal α	0.3250
Minimized GLOM [kg]	170'539

This result emphasizes the importance of an appropriate ΔV distribution to achieve an efficient design. The balanced allocation ensures a minimal initial mass and highlights the critical role of optimizing ΔV distribution in launcher design.

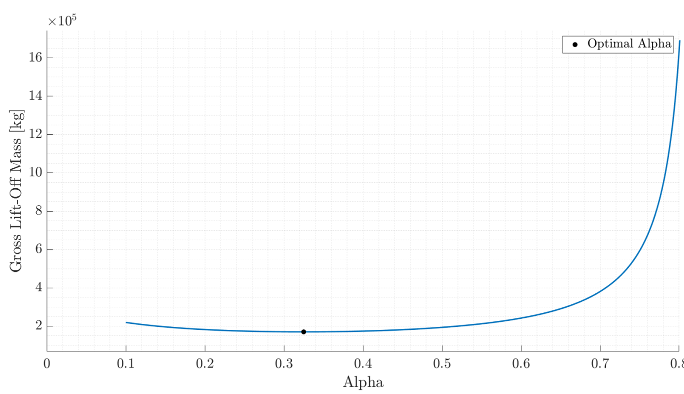

Figure 3: Gross Lift-Off Mass with α