Procedimiento para hallar el MVE y las funciones de transferencia

El modelo linealizado tiene la siguiente forma [1, 2]:

$$\delta\dot{\vec{\chi}} = \begin{bmatrix} \frac{\partial \dot{G}}{\partial G} & \frac{\partial \dot{G}}{\partial X} & \frac{\partial \dot{G}}{\partial Y} \\ \frac{\partial \dot{X}}{\partial G} & \frac{\partial \dot{X}}{\partial X} & \frac{\partial \dot{X}}{\partial Y} \\ \frac{\partial \dot{Y}}{\partial G} & \frac{\partial \dot{Y}}{\partial X} & \frac{\partial \dot{Y}}{\partial Y} \end{bmatrix} \delta\vec{\chi} + \begin{bmatrix} \frac{\partial \dot{G}}{\partial h} & \frac{\partial \dot{G}}{\partial i} \\ \frac{\partial \dot{X}}{\partial h} & \frac{\partial \dot{X}}{\partial i} \\ \frac{\partial \dot{Y}}{\partial h} & \frac{\partial \dot{Y}}{\partial i} \end{bmatrix} \delta\vec{\upsilon}, \tag{1a}$$

$$\delta\vec{\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \delta\vec{\chi}. \tag{1b}$$

Ahora lo que se debe realizar es hallar las derivadas parciales de las tres ecuaciones diferenciales respecto a las tres variables de estados y a las dos entradas [1]. Por lo tanto, para $G(t)$ las derivadas parciales son:

$$\begin{aligned} \frac{\partial \dot{G}}{\partial G} &= \frac{\partial}{\partial G}\left[-p_1 G(t) - \left[G(t) + G_B\right] X(t) + h(t)\right] \\ &= \frac{\partial}{\partial G}\left[-p_1 G(t)\right] - \frac{\partial}{\partial G}\left[G(t) + G_B\right] X(t) + \frac{\partial}{\partial G} h(t) \\ &= -p_1 + \frac{\partial}{\partial G}\left[G(t) X(t)\right] \\ &= -p_1 + X(t), \end{aligned} \tag{2a}$$

$$\begin{aligned} \frac{\partial \dot{G}}{\partial X} &= \frac{\partial}{\partial X}\left[-p_1 G(t) - \left[G(t) + G_B\right] X(t) + h(t)\right] \\ &= \frac{\partial}{\partial X}\left[-p_1 G(t)\right] - \frac{\partial}{\partial X}\left[G(t) + G_B\right] X(t) + \frac{\partial}{\partial X} h(t) \\ &= -\left(G(t) + G_B\right), \end{aligned} \tag{2b}$$

$$\begin{aligned} \frac{\partial \dot{G}}{\partial Y} &= \frac{\partial}{\partial Y}\left[-p_1 G(t) - \left[G(t) + G_B\right] X(t) + h(t)\right] \\ &= \frac{\partial}{\partial Y}\left[-p_1 G(t)\right] - \frac{\partial}{\partial Y}\left[G(t) + G_B\right] X(t) + \frac{\partial}{\partial Y} h(t) \\ &= 0, \end{aligned} \tag{2c}$$

$$\begin{aligned}\frac{\partial \dot{G}}{\partial h} &= \frac{\partial}{\partial h}\left[-p_1 G(t) - \left[G(t) + G_B\right] X(t) + h(t)\right] \\ &= \frac{\partial}{\partial h}\left[-p_1 G(t)\right] - \frac{\partial}{\partial h}\left[G(t) + G_B\right] X(t) + \frac{\partial}{\partial h} h(t) \\ &= 1, \end{aligned} \tag{2d}$$

$$\begin{aligned}\frac{\partial \dot{G}}{\partial i} &= \frac{\partial}{\partial i}\left[-p_1 G(t) - \left[G(t) + G_B\right] X(t) + h(t)\right] \\ &= \frac{\partial}{\partial i}\left[-p_1 G(t)\right] - \frac{\partial}{\partial i}\left[G(t) + G_B\right] X(t) + \frac{\partial}{\partial i} h(t) \\ &= 0. \end{aligned} \tag{2e}$$

Ahora, las derivadas parciales para $X(t)$ son:

$$\begin{aligned}\frac{\partial \dot{X}}{\partial G} &= \frac{\partial}{\partial G}\left[-p_2 X(t) + p_3 Y(t)\right] \\ &= -\frac{\partial}{\partial G}\left[p_2 X(t)\right] + \frac{\partial}{\partial G}\left[p_3 Y(t)\right] \\ &= 0, \end{aligned} \tag{3a}$$

$$\begin{aligned}\frac{\partial \dot{X}}{\partial X} &= \frac{\partial}{\partial X}\left[-p_2 X(t) + p_3 Y(t)\right] \\ &= -\frac{\partial}{\partial X}\left[p_2 X(t)\right] + \frac{\partial}{\partial X}\left[p_3 Y(t)\right] \\ &= -p_2, \end{aligned} \tag{3b}$$

$$\begin{aligned}\frac{\partial \dot{X}}{\partial Y} &= \frac{\partial}{\partial Y}\left[-p_2 X(t) + p_3 Y(t)\right] \\ &= -\frac{\partial}{\partial Y}\left[p_2 X(t)\right] + \frac{\partial}{\partial Y}\left[p_3 Y(t)\right] \\ &= p_3, \end{aligned} \tag{3c}$$

$$\begin{aligned}\frac{\partial \dot{X}}{\partial h} &= \frac{\partial}{\partial h}\left[-p_2 X(t) + p_3 Y(t)\right] \\ &= -\frac{\partial}{\partial h}\left[p_2 X(t)\right] + \frac{\partial}{\partial h}\left[p_3 Y(t)\right] \\ &= 0, \end{aligned} \tag{3d}$$

$$
\begin{aligned}
\frac{\partial \dot{X}}{\partial i} &= \frac{\partial}{\partial i}\left[-p_2 X(t) + p_3 Y(t)\right] \\
&= -\frac{\partial}{\partial i}\left[p_2 X(t)\right] + \frac{\partial}{\partial i}\left[p_3 Y(t)\right] \\
&= 0.
\end{aligned} \tag{3e}
$$

Por último, las derivadas parciales para $Y(t)$ son:

$$
\begin{aligned}
\frac{\partial \dot{Y}}{\partial G} &= \frac{\partial}{\partial G}\left[-p_4\left(Y(t) + Y_B\right) + \frac{i(t)}{V_L}\right] \\
&= \frac{\partial}{\partial G}\left[-p_4\left(Y(t) + Y_B\right)\right] + \frac{\partial}{\partial G}\left[\frac{i(t)}{V_L}\right] \\
&= 0,
\end{aligned} \tag{4a}
$$

$$
\begin{aligned}
\frac{\partial \dot{Y}}{\partial X} &= \frac{\partial}{\partial X}\left[-p_4\left(Y(t) + Y_B\right) + \frac{i(t)}{V_L}\right] \\
&= \frac{\partial}{\partial X}\left[-p_4\left(Y(t) + Y_B\right)\right] + \frac{\partial}{\partial X}\left[\frac{i(t)}{V_L}\right] \\
&= 0,
\end{aligned} \tag{4b}
$$

$$
\begin{aligned}
\frac{\partial \dot{Y}}{\partial Y} &= \frac{\partial}{\partial Y}\left[-p_4\left(Y(t) + Y_B\right) + \frac{i(t)}{V_L}\right] \\
&= \frac{\partial}{\partial Y}\left[-p_4\left(Y(t) + Y_B\right)\right] + \frac{\partial}{\partial Y}\left[\frac{i(t)}{V_L}\right] \\
&= -p_4,
\end{aligned} \tag{4c}
$$

$$
\begin{aligned}
\frac{\partial \dot{Y}}{\partial h} &= \frac{\partial}{\partial h}\left[-p_4\left(Y(t) + Y_B\right) + \frac{i(t)}{V_L}\right] \\
&= \frac{\partial}{\partial h}\left[-p_4\left(Y(t) + Y_B\right)\right] + \frac{\partial}{\partial h}\left[\frac{i(t)}{V_L}\right] \\
&= 0,
\end{aligned} \tag{4d}
$$

$$\begin{aligned}
\frac{\partial \dot{Y}}{\partial i} &= \frac{\partial}{\partial i}\left[-p_4\left(Y(t)+Y_B\right)+\frac{i(t)}{V_L}\right] \\
&= \frac{\partial}{\partial i}\left[-p_4\left(Y(t)+Y_B\right)\right]+\frac{\partial}{\partial i}\left[\frac{i(t)}{V_L}\right] \\
&= \frac{1}{V_L}. \qquad (4e)
\end{aligned}$$

Esto quiere decir que el modelo linealizado es [1]:

$$\delta\dot{\vec{\chi}} = \begin{bmatrix} -p_1+X(t) & -\left(G(t)+G_B\right) & 0 \\ 0 & -p_2 & p_3 \\ 0 & 0 & -p_4 \end{bmatrix}\delta\vec{\chi} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{V_L} \end{bmatrix}\delta\vec{v},$$

$$\delta\vec{\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\delta\vec{\chi}.$$

Utilizando el punto de operación $X(t) = 0$ y $G(t) = 0$ [2], el modelo se simplifica a:

$$\delta\dot{\vec{\chi}} = \begin{bmatrix} -p_1 & -G_B & 0 \\ 0 & -p_2 & p_3 \\ 0 & 0 & -p_4 \end{bmatrix}\delta\vec{\chi} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{V_L} \end{bmatrix}\delta\vec{v},$$

$$\delta\vec{\gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\delta\vec{\chi}.$$

Ahora, tal como se estudió en el curso, es posible pasar de un modelo en variables de estado a una o varias funciones de transferencia por medio de la siguiente ecuación [1]:

$$\mathbf{H}(s) = \left(\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}\right) \qquad (5a)$$

donde las matrices **A**, **B**, **C** y **D** son [1, 2]:

$$\mathbf{A} = \begin{bmatrix} -p_1 & -G_B & 0 \\ 0 & -p_2 & p_3 \\ 0 & 0 & -p_4 \end{bmatrix}, \qquad (5b)$$

$$\mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{V_L} \end{bmatrix}, \qquad (5c)$$

$$\mathbf{C} = \mathbf{I}, \qquad (5d)$$

$$\mathbf{D} = \mathbf{0}. \qquad (5e)$$

Por lo tanto, calculando $s\mathbf{I}-\mathbf{A}$:

$$\begin{aligned}
s\mathbf{I}-\mathbf{A} &= s\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-p_1 & -G_B & 0\\ 0 & -p_2 & p_3\\ 0 & 0 & -p_4\end{bmatrix}\\
&= \begin{bmatrix}s & 0 & 0\\ 0 & s & 0\\ 0 & 0 & s\end{bmatrix} - \begin{bmatrix}-p_1 & -G_B & 0\\ 0 & -p_2 & p_3\\ 0 & 0 & -p_4\end{bmatrix}\\
&= \begin{bmatrix}s+p_1 & G_B & 0\\ 0 & s+p_2 & -p_3\\ 0 & 0 & s+p_4\end{bmatrix}\\
&= \begin{bmatrix}s+28\cdot 10^{-3} & 110 & 0\\ 0 & s+25\cdot 10^{-3} & -130\cdot 10^{-6}\\ 0 & 0 & s+5/54\end{bmatrix}.
\end{aligned}$$

Ahora, para $(s\mathbf{I}-\mathbf{A})^{-1}$ se tiene que:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \lambda\cdot\begin{bmatrix}
\begin{vmatrix} s+25\cdot 10^{-3} & -130\cdot 10^{-6}\\ 0 & s+5/54\end{vmatrix} & -\begin{vmatrix} 110 & 0\\ 0 & s+5/54\end{vmatrix} & \begin{vmatrix} 110 & 0\\ s+25\cdot 10^{-3} & -130\cdot 10^{-6}\end{vmatrix}\\
-\begin{vmatrix} 0 & -130\cdot 10^{-6}\\ 0 & s+5/54\end{vmatrix} & \begin{vmatrix} s+28\cdot 10^{-3} & 0\\ 0 & s+5/54\end{vmatrix} & -\begin{vmatrix} s+28\cdot 10^{-3} & 0\\ 0 & -130\cdot 10^{-6}\end{vmatrix}\\
\begin{vmatrix} 0 & s+25\cdot 10^{-3}\\ 0 & 0\end{vmatrix} & -\begin{vmatrix} s+28\cdot 10^{-3} & 110\\ 0 & 0\end{vmatrix} & \begin{vmatrix} s+28\cdot 10^{-3} & 110\\ 0 & s+25\cdot 10^{-3}\end{vmatrix}
\end{bmatrix},$$

donde el valor de λ es:

$$\lambda = \frac{1}{(s+28\cdot 10^{-3})[(s+25\cdot 10^{-3})(s+5/54)]}.$$

Realizando los cálculos se obtiene que:

$$(s\mathbf{I}-\mathbf{A})^{-1} = \begin{bmatrix}\frac{250}{250s+7} & -\frac{1100000}{(250s+7)(40s+1)} & -\frac{7722}{(250s+7)(40s+1)(54s+5)}\\ 0 & \frac{40}{40s+1} & \frac{351}{1250(40s+1)(54s+5)}\\ 0 & 0 & \frac{54}{54s+5}\end{bmatrix}.$$

Ahora se tiene que $\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}$ es:

$$\begin{aligned}
\mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1} &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\frac{250}{250s+7} & -\frac{1100000}{(250s+7)(40s+1)} & -\frac{7722}{(250s+7)(40s+1)(54s+5)}\\ 0 & \frac{40}{40s+1} & \frac{351}{1250(40s+1)(54s+5)}\\ 0 & 0 & \frac{54}{54s+5}\end{bmatrix}\\
&= \begin{bmatrix}\frac{250}{250s+7} & -\frac{1100000}{(250s+7)(40s+1)} & -\frac{7722}{(250s+7)(40s+1)(54s+5)}\\ 0 & \frac{40}{40s+1} & \frac{351}{1250(40s+1)(54s+5)}\\ 0 & 0 & \frac{54}{54s+5}\end{bmatrix}.
\end{aligned}$$

Dado que la matriz D es nula, finalmente se calcula $\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$:

$$\begin{aligned}
\mathbf{H}(s) &= \begin{bmatrix} \frac{250}{250s+7} & -\frac{1100000}{(250s+7)(40s+1)} & -\frac{7722}{(250s+7)(40s+1)(54s+5)} \\ 0 & \frac{40}{40s+1} & \frac{351}{1250(40s+1)(54s+5)} \\ 0 & 0 & \frac{54}{54s+5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{V_L} \end{bmatrix} \\
&= \begin{bmatrix} \frac{250}{250s+7} & -\frac{1100000}{(250s+7)(40s+1)} & -\frac{7722}{(250s+7)(40s+1)(54s+5)} \\ 0 & \frac{40}{40s+1} & \frac{351}{1250(40s+1)(54s+5)} \\ 0 & 0 & \frac{54}{54s+5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & \frac{1}{120} \end{bmatrix} \\
&= \begin{bmatrix} \frac{250}{250s+7} & -\frac{1287}{20(54s+5)(40s+1)(250s+7)} \\ 0 & \frac{117}{50000(54s+5)(40s+1)} \\ 0 & \frac{9}{20(54s+5)} \end{bmatrix}
\end{aligned}$$

Sea ahora $\mathbf{H}(s) = \begin{bmatrix} H_{G,h} & H_{G,i} \\ 0 & H_{X,i} \\ 0 & H_{Y,i} \end{bmatrix}$. Por ende, se puede observar claramente que:

$$H_{X,i} = \frac{117}{50000(54s+5)(40s+1)} \tag{6a}$$

$$H_{Y,i} = \frac{9}{20(54s+5)} \tag{6b}$$

$$H_{G,h} = \frac{250}{250s+7} \tag{6c}$$

$$H_{G,i} = -\frac{1287}{20(54s+5)(40s+1)(250s+7)} \tag{6d}$$

Para el análisis en frecuencia, se reacomodaron las funciones de transferencia en la forma de constantes de tiempo, procedimiento que se muestra a continuación [1]:

- Utilización de insulina en el compartimento remoto $X(t)$

$$H_{X,i} = \frac{117}{50000(54s+5)(40s+1)} \tag{7a}$$

  Esta expresión ya está factorizada, por lo que se muestra en términos de las constantes de tiempo:

$$H_{X,i} = \frac{117}{50000 \cdot 5(\frac{54}{5}s+1)(40s+1)} \tag{7b}$$

  Finalmente:

$$H_{X,i} = \frac{4.678 \cdot 10^{-4}}{(\frac{s}{0.025}+1)(\frac{s}{0.0926}+1)} \tag{7c}$$

- Desviación de la insulina plasmática $Y(t)$

$$H_{Y,i} = \frac{9}{20(54s+5)} \tag{8a}$$

Esta expresión ya está factorizada, por lo que se muestra en términos de las constantes de tiempo:

$$H_{Y,i} = \frac{9}{20 \cdot 5(\frac{54}{5}s + 1)} \tag{8b}$$

Finalmente:

$$H_{Y,i} = \frac{0.09}{(\frac{s}{0.09259} + 1)} \tag{8c}$$

- Desviación de la glucosa plasmática $G(t)$. En este caso se trabaja con dos funciones de transferencia, por lo que se debe realizar el mismo procedimiento para ambas:

$$H_{G,h} = \frac{250}{250s + 7} \tag{9a}$$

Esta expresión ya está factorizada, por lo que se muestra en términos de las constantes de tiempo:

$$H_{G,h} = \frac{\frac{250}{7}}{(\frac{1}{\frac{7}{250}s} + 1)} \tag{9b}$$

Finalmente:

$$H_{G,h} = \frac{35.71}{(\frac{s}{0.028} + 1)} \tag{9c}$$

Ahora:

$$H_{G,i} = \frac{-1287}{20\,(54s + 5)\,(40s + 1)\,(250s + 7)} \tag{10a}$$

Factorizando se obtiene que:

$$H_{G,i} = \frac{\frac{-1287}{20 \cdot 7 \cdot 5}}{\left(\frac{54}{5}s + 1\right)(40s + 1)\left(\frac{250}{7}s + 1\right)} \tag{10b}$$

Finalmente:

$$H_{G,i} = \frac{-1.838}{(\frac{s}{0.0926} + 1)(\frac{s}{0.0251} + 1)(\frac{s}{0.0279} + 1)} \tag{10c}$$

Referencias

[1] J. D. Rojas Fernández and M. Espinoza Bolaños, *Modelado y análisis de sistemas lineales*, Mar. 2021.

[2] L. Kovacs, B. Palancz, and Z. Benyo, "Design of luenberger observer for glucose-insulin control via mathematica," in *2007 29th Annual International Conference of the IEEE Engineering in Medicine and Biology Society*, 2007, pp. 624–627.