



Laboratorio #3

Jeremy Cáceres y Gabriel Lemus

Ejercicio #1

Utilizando la definición de suma (\oplus) para los números naturales unarios, llevar a cabo la suma entre tres $[s(s(s(0)))]$ y cuatro $[s(s(s(s(0))))]$. Debe elaborar todos los pasos de forma explícita. Como referencia, se presenta nuevamente la definición de suma para números naturales unarios:

$$n \oplus m := \begin{cases} m & \text{si } n = o \\ n & \text{si } m = o \\ \sigma(i \oplus m) & \text{si } n = \sigma(i) \end{cases}$$

Demostración:

$$\sigma(\sigma(\sigma(0))) \oplus \sigma(\sigma(\sigma(\sigma(0)))) \quad n = \sigma(\sigma(\sigma(0))), \quad i = \sigma(\sigma(0)) \quad (1)$$

$$\sigma[\sigma(\sigma(0)) \oplus \sigma(\sigma(\sigma(0))))] \quad n = \sigma(\sigma(0)), \quad i = \sigma(0) \quad (2)$$

$$\sigma[\sigma[\sigma(0) \oplus \sigma(\sigma(\sigma(0)))] \quad n = \sigma(0), \quad i = 0 \quad (3)$$

$$\sigma[\sigma[\sigma[0 \oplus \sigma(\sigma(\sigma(0)))] \quad n = 0 \quad \therefore 0 \oplus m = m \quad (4)$$

$$\sigma(\sigma(\sigma(\sigma(\sigma(\sigma(0))))) \quad 3 + 4 = 7 \quad (5)$$

Ejercicio #2

Definir inductivamente una función para multiplicar (\otimes) números naturales unarios.

Consejo: Puede apoyarse de la definición de suma estudiada durante la clase.

Definición:

$$a \otimes b := \begin{cases} 0 & \text{si } a = 0 \vee b = 0 \\ a & \text{si } b = \sigma(0) \\ b & \text{si } a = \sigma(0) \\ (a \otimes \sigma(i)) \oplus a & \text{si } b = \sigma(i) \end{cases}$$

Ejercicio #3

Verifique que su definición de multiplicación es correcta multiplicando los siguientes valores:

1. $\sigma(\sigma(\sigma(0))) \otimes 0$

$$\sigma(\sigma(\sigma(0))) \otimes 0 \quad a = \sigma(\sigma(\sigma(0))), b = 0 \quad (1)$$

$$0 \quad \therefore \sigma(\sigma(\sigma(0))) \otimes 0 = 0 \quad (2)$$

2. $\sigma(\sigma(\sigma(0))) \otimes \sigma(0)$

$$\sigma(\sigma(\sigma(0))) \otimes \sigma(0) \quad a = \sigma(\sigma(\sigma(0))), b = \sigma(0) \quad (1)$$

$$\sigma(\sigma(\sigma(0))) \quad \therefore \sigma(\sigma(\sigma(0))) \otimes \sigma(0) = \sigma(\sigma(\sigma(0))) \quad (2)$$

3. $\sigma(\sigma(\sigma(0))) \otimes \sigma(\sigma(0))$

$$\sigma(\sigma(\sigma(0))) \otimes \sigma(\sigma(0)) \quad a = \sigma(\sigma(\sigma(0))), b = \sigma(\sigma(0)) \quad (1)$$

$$(\sigma(\sigma(\sigma(0))) \otimes \sigma(0)) \oplus \sigma(\sigma(\sigma(0))) \quad a \otimes \sigma(0) = a \quad (2)$$

$$\sigma(\sigma(\sigma(0))) \oplus \sigma(\sigma(\sigma(0))) \quad \text{Definición de la suma.} \quad (3)$$

$$\sigma(\sigma(\sigma(\sigma(\sigma(0))))) \quad \therefore 3 \times 2 = 6 \quad (4)$$

Ejercicio #4

Demostrar utilizando inducción:

1. $a \oplus \sigma(\sigma(0)) = \sigma(\sigma(a))$

Caso Base:

- $a = 0$

Demostración.

$$0 \oplus \sigma(\sigma(0)) = \sigma(\sigma(0))$$

$$\sigma(\sigma(0)) = \sigma(\sigma(0))$$

□

Caso Inductivo:

- $a = \sigma(a)$
- **Hipótesis Inductiva:** $a \oplus \sigma(\sigma(0)) = \sigma(\sigma(a))$
- *Demostrar que:* $\sigma(a) \oplus \sigma(\sigma(0)) = \sigma(\sigma(\sigma(a)))$

Demostración.

$$\sigma(a) \oplus \sigma(\sigma(0)) = \sigma(\sigma(\sigma(a))) \quad \text{Definición de la suma.} \quad (1)$$

$$\sigma[a \oplus \sigma(\sigma(0))] = \sigma(\sigma(\sigma(a))) \quad \text{La suma es conmutativa.} \quad (2)$$

$$\sigma[\sigma(\sigma(0)) \oplus a] = \sigma(\sigma(\sigma(a))) \quad \text{Definición de la suma.} \quad (3)$$

$$\sigma[\sigma[\sigma(0) \oplus a]] = \sigma(\sigma(\sigma(a))) \quad \text{Definición de la suma.} \quad (4)$$

$$\sigma[\sigma[\sigma[0 \oplus a]]] = \sigma(\sigma(\sigma(a))) \quad 0 + a = a \quad (5)$$

$$\sigma(\sigma(\sigma(a))) = \sigma(\sigma(\sigma(a))) \quad (6)$$

□

$$2. \ a \otimes b = b \otimes a$$

Caso Base:

- $a = 0$

Demostración.

$$0 \otimes b = b \otimes 0$$

$$0 = 0$$

□

Caso Inductivo:

- $a = \sigma(a)$
- **Hipótesis Inductiva:** $a \otimes b = b \otimes a$
- *Demostrar que:* $\sigma(a) \otimes b = b \otimes \sigma(a)$

Demostración.

$$\sigma(a) \otimes b = b \otimes \sigma(a) \quad \text{Definición de la multiplicación.} \quad (1)$$

$$\sigma(a) \otimes b = (b \otimes a) \oplus b \quad \text{Por hipótesis inductiva.} \quad (2)$$

$$\sigma(a) \otimes b = (a \otimes b) \oplus b \quad \text{Definición de la multiplicación.} \quad (3)$$

$$(a \otimes b) \oplus b = (a \otimes b) \oplus b \quad (4)$$

□

3. $a \otimes (b \otimes c) = (a \otimes b) \otimes c$

Caso Base:

- $c = 0$

Demostración.

$$\begin{aligned} a \otimes (b \otimes 0) &= (a \otimes b) \otimes 0 & i &= a \otimes b \\ a \otimes 0 &= i \otimes 0 \\ 0 &= 0 \end{aligned}$$

□

Caso Inductivo:

- $c = \sigma(c)$
- **Hipótesis Inductiva:** $a \otimes (b \otimes c) = (a \otimes b) \otimes c$
- *Demostrar que:* $a \otimes (b \otimes \sigma(c)) = (a \otimes b) \otimes \sigma(c)$

Demostración.

$$\begin{aligned} a \otimes (b \otimes \sigma(c)) &= (a \otimes b) \otimes \sigma(c) && \text{Definición de la multiplicación.} && (1) \\ a \otimes (b \otimes \sigma(c)) &= ((a \otimes b) \otimes c) \oplus (a \otimes b) && \text{Por hipotesis inductiva.} && (2) \\ a \otimes (b \otimes \sigma(c)) &= (a \otimes (b \otimes c)) \oplus (a \otimes b) && \text{La suma es conmutativa.} && (3) \\ a \otimes (b \otimes \sigma(c)) &= (a \otimes b) \oplus (a \otimes (b \otimes c)) && \text{La multiplicación es distributiva.} && (4) \\ a \otimes (b \otimes \sigma(c)) &= a \otimes (b \oplus (b \otimes c)) && \text{La suma es conmutativa.} && (5) \\ a \otimes (b \otimes \sigma(c)) &= a \otimes ((b \otimes c) \oplus b) && \text{Definición de la multiplicación.} && (6) \\ a \otimes (b \otimes \sigma(c)) &= a \otimes (b \otimes \sigma(c)) && && (7) \end{aligned}$$

□

4. $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$

Caso Base:

- $c = 0$

Demostración.

$$\begin{aligned} (a \oplus b) \otimes 0 &= (a \otimes 0) \oplus (b \otimes 0) && i &= a \oplus b \\ i \otimes 0 &= 0 \oplus 0 \\ 0 &= 0 \end{aligned}$$

□

Caso Inductivo:

- $c = \sigma(c)$
- **Hipótesis Inductiva:** $(a \oplus b) \otimes c = (a \otimes c) \oplus (b \otimes c)$
- *Demostrar que:* $(a \oplus b) \otimes \sigma(c) = (a \otimes \sigma(c)) \oplus (b \otimes \sigma(c))$

Demostración.

- | | |
|--|--------------------------------------|
| 1) $(a \oplus b) \otimes \sigma(c) = (a \otimes \sigma(c)) \oplus (b \otimes \sigma(c))$ | Definición de la multiplicación. |
| 2) $(a \oplus b) \otimes c \oplus (a \oplus b) = (a \otimes \sigma(c)) \oplus (b \otimes \sigma(c))$ | Por hipótesis de inducción. |
| 3) $(a \otimes b) \oplus (b \otimes c) \oplus (a \oplus b) = (a \otimes \sigma(c)) \oplus (b \otimes \sigma(c))$ | La suma es asociativa y conmutativa. |
| 4) $((a \otimes c) \oplus a) \oplus ((b \otimes c) \oplus b) = (a \otimes \sigma(c)) \oplus (b \otimes \sigma(c))$ | Definición de la multiplicación. |
| 5) $(a \otimes \sigma(c)) \oplus (b \otimes \sigma(c)) = (a \otimes \sigma(c)) \oplus (b \otimes \sigma(c))$ | |

□