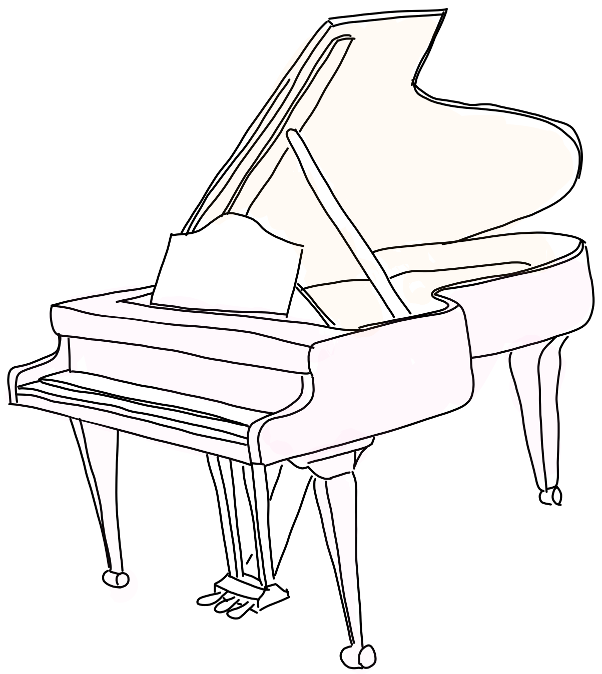

Peano Arithmetic

$$0 \in \mathbb{N}$$

$$\forall x \in \mathbb{N}.\ Sx \in \mathbb{N}$$

$$\mathbb{N} = 0 \mid S\,\mathbb{N}$$

$$\forall x : \mathbb{N}.\quad x = x \qquad \text{(refl)}$$

$$\forall x, y.\quad x = y \Rightarrow y = x \qquad \text{(sym)}$$

$$\forall x, y, z.\quad x = y \wedge y = z \Rightarrow x = z \qquad \text{(trans)}$$

$$\forall x, y.\quad x = y \Leftrightarrow S(x) = S(y) \qquad \text{(inj)}$$

$P(0)$ if a property P holds for 0

$\wedge$

$(\forall x.\ P(x) \Rightarrow P(S(x)))$

and assuming it holds for some x, we can show it also holds for $S(x)$

$\Rightarrow \forall n.\ P(n)$

then P holds for all numbers

$$0 + x = x \qquad (+_1)$$

$$S(x) + y = S(x + y) \qquad (+_2)$$

$\forall x\, y.\ x + y = y + x$

$P(x) = \forall y.\ x + y = y + x$

<u>theorem</u> $\forall n.\ P(n)$

<u>proof</u>

- prove $P(0) = \forall y.\ 0 + y = y + 0$

$0 + y =$ by $(+_1)$

$y =$???

$y + 0$

$Q(y) = y = y + 0$

lemma $\forall n.\, Q(n)$ (Q)

proof

- prove $Q(0) = 0 = 0 + 0$ by $(+_1)$
- prove $\forall y.\, Q(y) \Rightarrow Q(S(y))$

  assume $Q(y)$ i.e. $y = y + 0$ (ih)

  now show $S(y) + 0 =$ by $(+_2)$

  $S(y + 0) =$ by (ih) + (inj)

  $S(y)$ $\square$

$\forall x\, y.\ x + y = y + x$

$P(x) = \forall y.\ x + y = y + x$

<u>theorem</u> $\forall n.\ P(n)$

<u>proof</u>

- prove $P(0) = \forall y.\ 0 + y = y + 0$

$$
\begin{aligned}
0 + y &= \text{by } (+_1)\\
y &= \text{by } (Q)\\
y + 0
\end{aligned}
$$

- prove $\forall x.\ P(x) \Longrightarrow P(S(x))$

assume $P(x)$ i.e. $\forall y.\ x + y = y + x$ (ih)

now show $\forall y.\ S(x) + y = y + S(x)$

$$
\begin{aligned}
S(x) + y &= \text{by } (+_2)\\
S(x + y) &= \text{by (ih)} + \text{(inj)}\\
S(y + x) &= ???\\
y + S(x)
\end{aligned}
$$

$$R(a) = \forall b.\ a + S(b) = S(a+b)$$

<u>lemma</u> $\forall n\ R(n)$

<u>proof</u>

- prove $R(0) =$

$$\forall b.\ 0 + S(b) = S(0+b)$$

$0 + S(b) =$ by $(+_1)$

$S(b) =$ by $(+_1) + (\text{inj})$

$S(0+b)$

- prove $\forall a.\ R(a) \Longrightarrow R(S(a))$

assume $R(a)$ i.e.

$$\forall b.\ a + S(b) = S(a+b) \quad (\text{ih})$$

show $\forall b.\ S(a) + S(b) = S(S(a)+b)$

$S(a) + S(b) =$ by $(+_2)$

$S(a + S(b)) =$ by $(\text{ih}) + (\text{inj})$

$S(S(a+b)) =$ by $(+_2) + (\text{inj})$

$S(S(a)+b)$ $\square$

$$\forall a.\ 0 \le a$$

$$\forall a b k.\ a \le b \Rightarrow a + k \le b + k$$

$$\forall a b.\ a \le b \Rightarrow S(a) \le S(b)$$