Pascal's Triangle

n=0 ⟶ 1 k=0

n=1 ⟶ 1 1 k=1

n=2 ⟶ 1 2 1 k=2

n=3 ⟶ 1 3 3 1 k=3

n=4 ⟶ 1 4 6 4 1 k=4

: 1 5 10 10 5 1

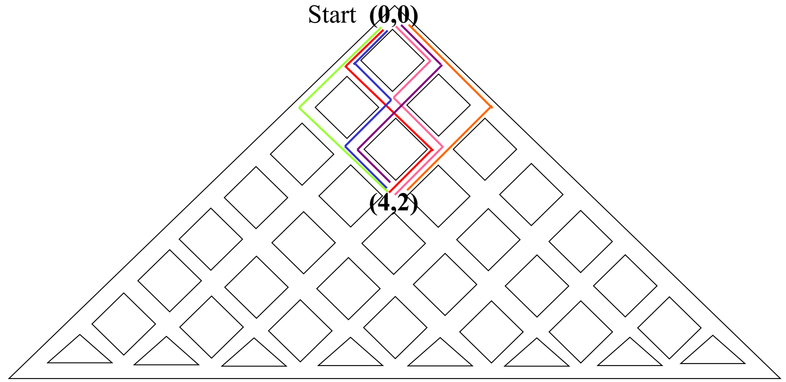
n=n

- Each number in the table except the first and last numbers in a row, is the sum of the two neighboring numbers in the preceding row. The *n*'s represent how far down you are, and the *k*'s are how many rights you have taken.

Block-Walking using Pascal's Triangle

- Begin at point (0,0), move down the page.
- Label each street corner in the network with a pair of numbers (*n,k*), where *n* indicates the number of blocks traversed from (0,0) and *k* the number of times the person chose the right branch at intersections.
- Write the route taken by a sequences of R's (right branches) and L's (left branches).
- *s(n,k)* = number of possible routes from (0,0) to corner (*n,k*)
- *s(n,k)=C(n,k)*

Block-walking



$$\binom{4}{2} = \frac{4!}{2!(4-2)!} = \frac{\cancel{4}*3*\cancel{2}*\cancel{1}}{\cancel{2}*\cancel{1}*\cancel{2}*1} = 6$$

Binomial Identities

(6) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$

(7) $\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\ldots+\binom{n+r}{r}=\binom{n+r+1}{r}$

(8) $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots+\binom{n}{r}=\binom{n+1}{r+1}$

(9) $\binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$

Proof (Identity 6)

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\ldots+\binom{n}{n}=2^n$$

- Committee argument:
  – 2 ways to count all subsets of any size of n people
    - Summing number of subsets of size 0, 1, 2… (left side of equation)
    - Counting all subsets by whether or not the first person is in the subset, whether the second person was in the subset, and so on, which gives $2*2**\ldots*2\ (n \text{ times}) = 2^n$

Example 2

Verify identity (8) by block-walking and committee-selection arguments.

$$(8)\quad \binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots+\binom{n}{r}=\binom{n+1}{r+1}$$

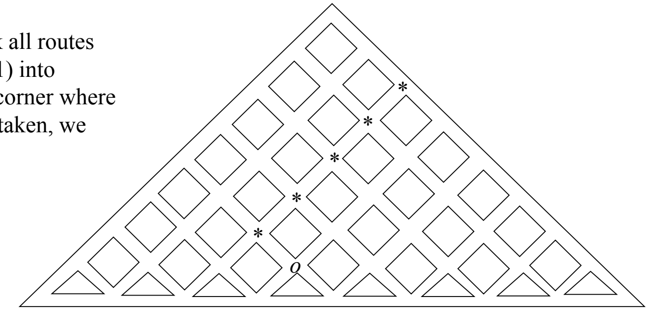
- Consider the case where $r = 2$ and $n = 6$.

Example 2

•the corners (k, 2), k = 2,3,4,5,6 are marked with a * and corner (7,3) is marked with an o

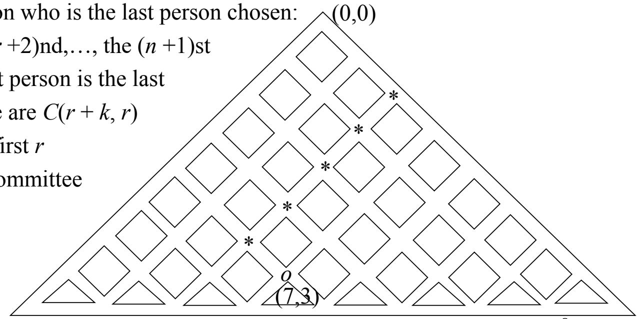
•Observe that the right branches at each starred corner are the locations of last possible right branches on routes from (0,0) to (7,3)

•In general, if we break all routes from (0,0) to (n+1, r +1) into subcases based on the corner where the last right branch is taken, we obtain identity (8)



Example 2

- Restate the block-walking model as a committee selection: if the kth turn is right, this corresponds to selecting the kth person to be on the committee; if the kth turn is left, the kth person is not chosen
- Break the ways to pick r +1 members of a committee from n +1people into cases depending on who is the last person chosen: the (r +1)st, the (r +2)nd,…, the (n +1)st
- If the (r + k + 1)st person is the last chosen, then there are $C(r + k, r)$ ways to pick the first r members of the committee
- Identity (8) now follows.



Example 3

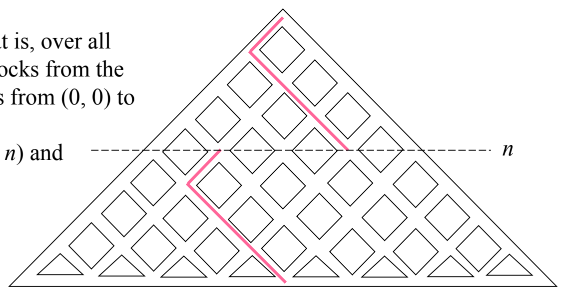
Verify identity (9) by a block-walking argument.

$$(9)\qquad \binom{n}{0}^2+\binom{n}{1}^2+\binom{n}{2}^2+\ldots+\binom{n}{n}^2=\binom{2n}{n}$$

Example 3

- The number of routes from (n, k) to $(2n, n)$ = number of routes from $(0, 0)$ to $(n, n-k)$
  [since both trips go a total of n blocks with $n-k$ to the right (and k to the left)]
- So number of ways to go from $(0, 0)$ to (n, k) and then on to $(2n, n)$ is $C(n, k)$ x $C(n, n-k)$
- By theorem (2), $C(n, n-k) = C(n, k)$
- Thus, the number of routes from $(0, 0)$ to $(2n, n)$ via (n, k) is $C(n,k)^2$

- Summing over all k – that is, over all intermediate corners n blocks from the start – we count all routes from $(0, 0)$ to $(2n, n)$
- So this sum equals $C(2n, n)$ and identity (9) follows.



Example 4

Evaluate the sum $1\times2\times3 \;+\; 2\times3\times4 \;+\ldots+\; (n-2)(n-1)n$

- The general term in this sum, $(k-2)(k-1)k$, is equal to

$$P(k,3)=\frac{k!}{(k-3)!}$$

- Recall that the numbers of r-permutations and of r-selections differ by a factor of $r!$

$$C(k,3)=\frac{k!}{(k-3)!3!}=P(k,3)=3!C(k,3)$$

Example 4

- So the given sum can be rewritten as

$$3!\binom{3}{3}+3!\binom{4}{3}+\ldots+3!\binom{n}{3}=3!\left(\binom{3}{3}+\binom{4}{3}+\ldots+\binom{n}{3}\right)$$

By identity (8) $\binom{r}{r}+\binom{r+1}{r}+\binom{r+2}{r}+\ldots+\binom{n}{r}=\binom{n+1}{r+1}$

this sum equals $3!\binom{n+1}{4}$