Advanced Graphs

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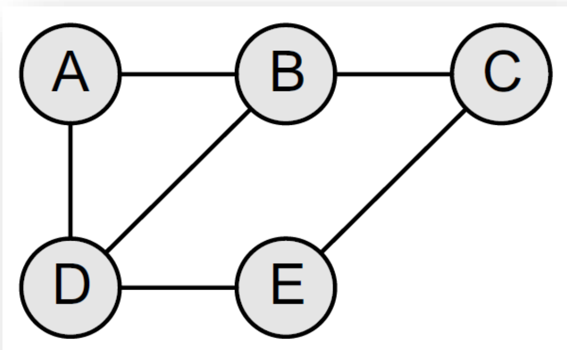
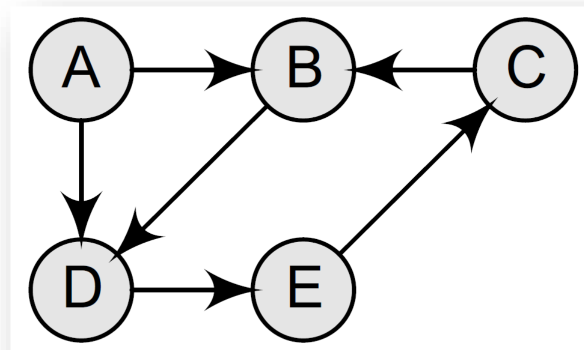
Review

- A graph G is defined as aset (V, E), where $V(G)$ represents the set of vertices and $E(G)$ represents the edges
  - For a given undirected graph with $V(G) = \{A, B, C, D, E\}$ and $E(G) = \{(A, B), (B, C), (A, D), (B, D), (D, E), (C, E)\}$
    - Five vertices or nodes and six edges in the graph





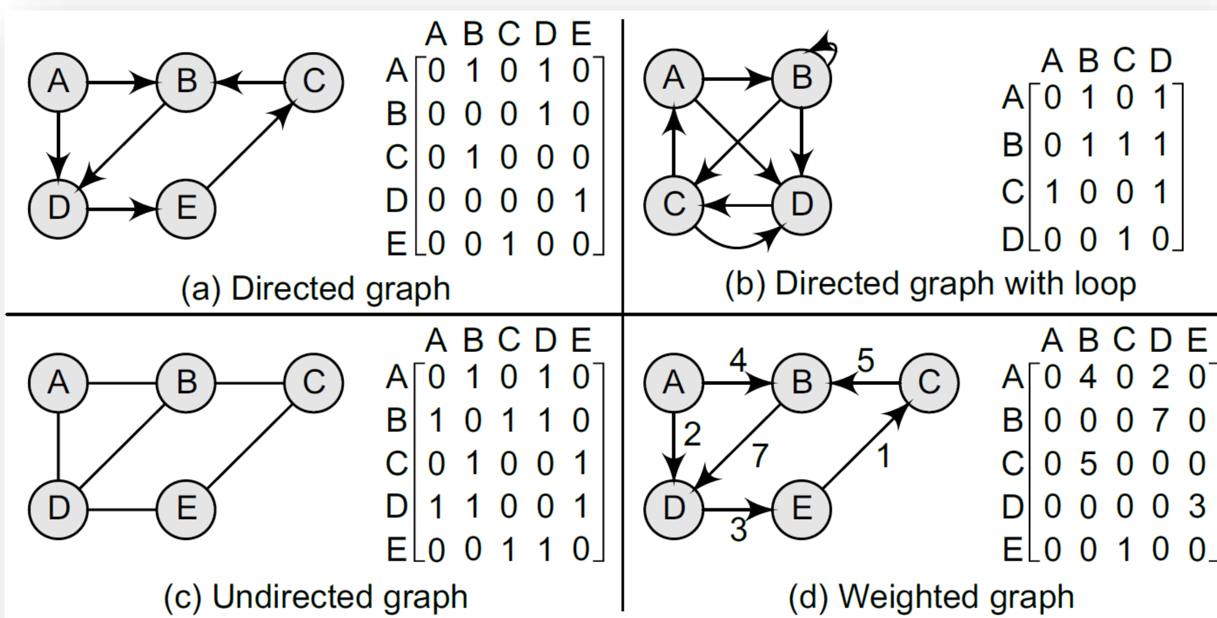
  - For a given directed graph, the edge (A, B) is said to initiate from node A (also known as initial node) and terminate at node B (terminal node)

Representation of Graphs

- There are three common ways of storing graphs in the computer's memory
  - **Sequential representation** by using an adjacency matrix
  - **Linked representation** by using an adjacency list that stores the neighbors of a node using a linked list
  - **Adjacency multi-list** which is an extension of linked representation

Sequential Representation.

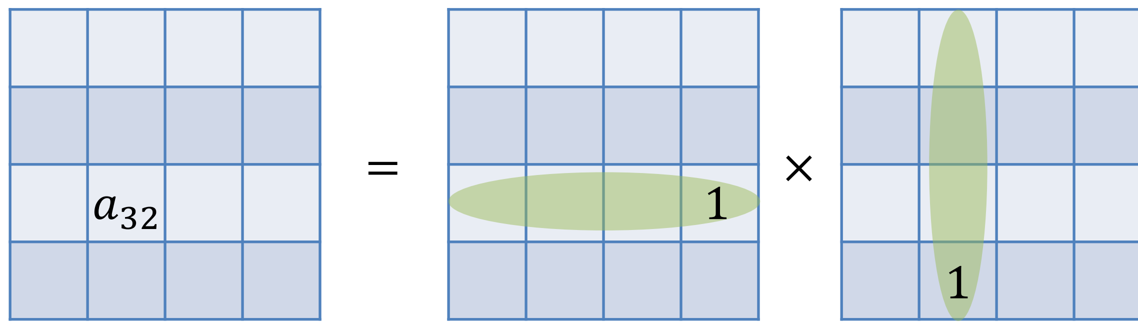
- For any graph G having n nodes, the **adjacency matrix** will have the dimension of $n \times n$
  - The rows and columns are labelled by graph vertices
  - An entry a_{ij} in the adjacency matrix will contain 1, if vertices v_i and v_j are adjacent to each other; otherwise, a_{ij} will set to 0
    - Since an adjacency matrix contains only 0s and 1s, it is called a ***bit matrix*** or a ***Boolean matrix***



(a) Directed graph
(b) Directed graph with loop
(c) Undirected graph
(d) Weighted graph

Sequential Representation..

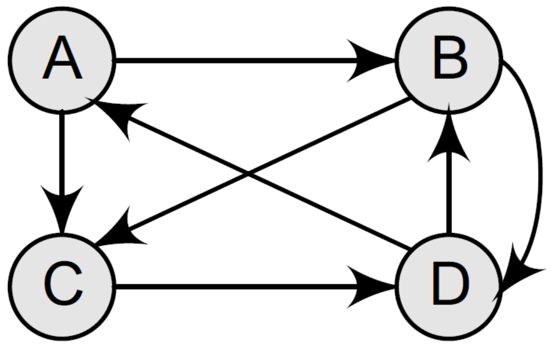
- From the original adjacency matrix, denoted by A^1
  - An entry 1 in the i^{th} row and j^{th} column means that there exists a path of length 1 from v_i to v_j



- Let's consider A^2
  - $A^2 = A^1 \times A^1$
  - $a_{ij}^2 = \sum a_{ik} a_{kj}$
  - If $a_{ij}^2 \geq 1$, $\exists k$ such that $a_{ik} = 1 \ \wedge a_{kj} = 1$
  - There are edges (v_i, v_k) and (v_k, v_j) in the graph, and they form a path from v_i to v_j of length 2

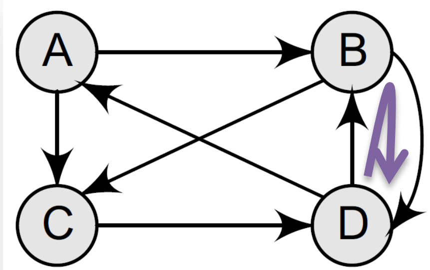
- Similarly, every entry in the i^{th} row and j^{th} column of A^n gives the number of paths of length n from node v_i to v_j

Sequential Representation…

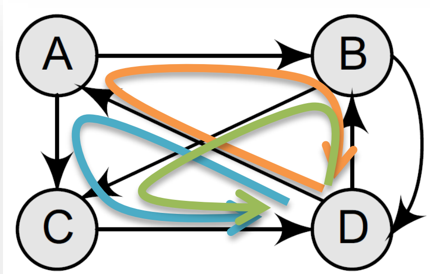


$$\begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

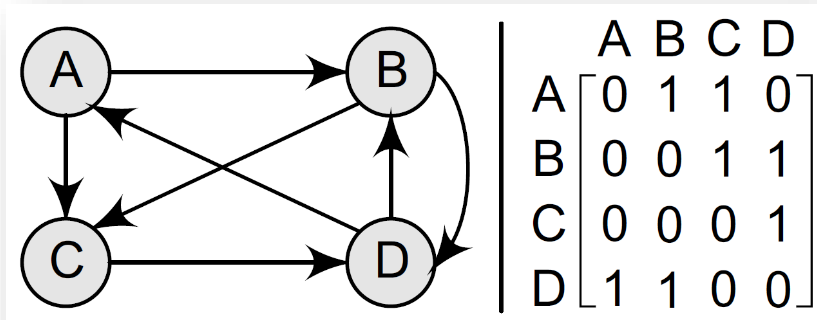
– $A^2 = A^1 \times A^1 = \begin{bmatrix} 0012 \\ 1101 \\ 1100 \\ 0121 \end{bmatrix}$



– $A^3 = A^2 \times A^1 = \begin{bmatrix} 2201 \\ 1221 \\ 0121 \\ 1113 \end{bmatrix}$



Sequential Representation....



– We can further define a matrix $B^n = A^1 + \cdots + A^n$

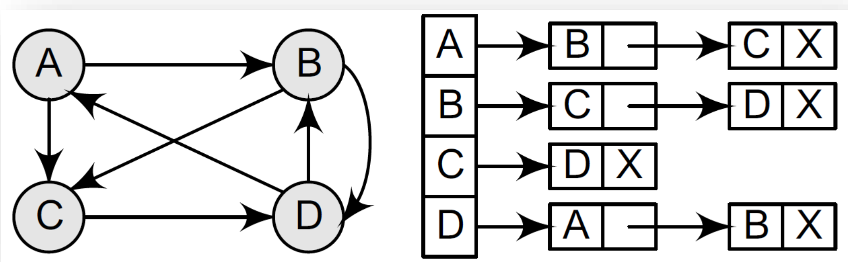
- $B^3 = A^1 + A^2 + A^3 = \begin{bmatrix} 0110 \\ 0011 \\ 0001 \\ 1100 \end{bmatrix} + \begin{bmatrix} 0012 \\ 1101 \\ 1100 \\ 0121 \end{bmatrix} + \begin{bmatrix} 2201 \\ 1221 \\ 0121 \\ 1113 \end{bmatrix} = \begin{bmatrix} 2323 \\ 2333 \\ 1222 \\ 2334 \end{bmatrix}$

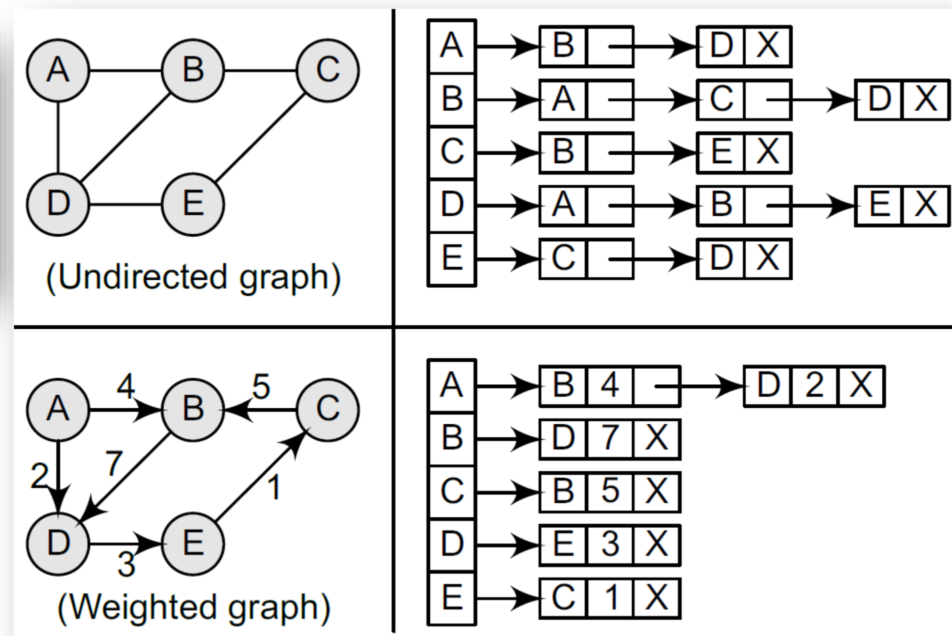
– A path matrix P can be obtained by setting an entry $p_{ij} = 1$ if b_{ij} is non-zero

- $P = \begin{bmatrix} 1111 \\ 1111 \\ 1111 \\ 1111 \end{bmatrix}$

Linked Representation

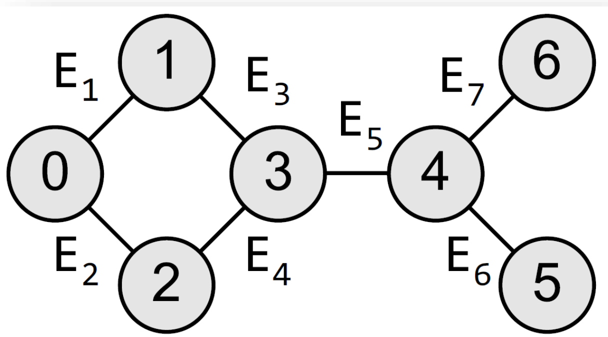
- An **adjacency list** is another way in which graphs can be represented in the computer's memory
  - It is often used for storing graphs that have a **small-to-moderate** number of edges
    - An adjacency list is preferred for representing **sparse graphs** in the computer's memory
      - ✓ Otherwise, an adjacency matrix is a good choice





Adjacency Multi-list.

- Graphs can also be represented using **multi-lists** which can be said to be modified version of adjacency lists
  - Adjacency multi-list is an **edge-based** rather than a **vertex-based** representation of graphs



Edge 1		0	1	Edge 2	Edge 3
Edge 2		0	2	NULL	Edge 4
Edge 3		1	3	NULL	Edge 4
Edge 4		2	3	NULL	Edge 5
Edge 5		3	4	NULL	Edge 6
Edge 6		4	5	Edge 7	NULL
Edge 7		4	6	NULL	NULL

Adjacency Multi-list..

- Using the adjacency multi-list, the inverse information for vertices can be derived

Edge 1		0	1	Edge 2	Edge 3
Edge 2		0	2	NULL	Edge 4
Edge 3		1	3	NULL	Edge 4
Edge 4		2	3	NULL	Edge 5
Edge 5		3	4	NULL	Edge 6
Edge 6		4	5	Edge 7	NULL
Edge 7		4	6	NULL	NULL

VERTEX	LIST OF EDGES
0	Edge 1, Edge 2
1	Edge 1, Edge 3
2	Edge 2, Edge 4
3	Edge 3, Edge 4, Edge 5
4	Edge 5, Edge 6, Edge 7
5	Edge 6
6	Edge 7

Search Algorithms

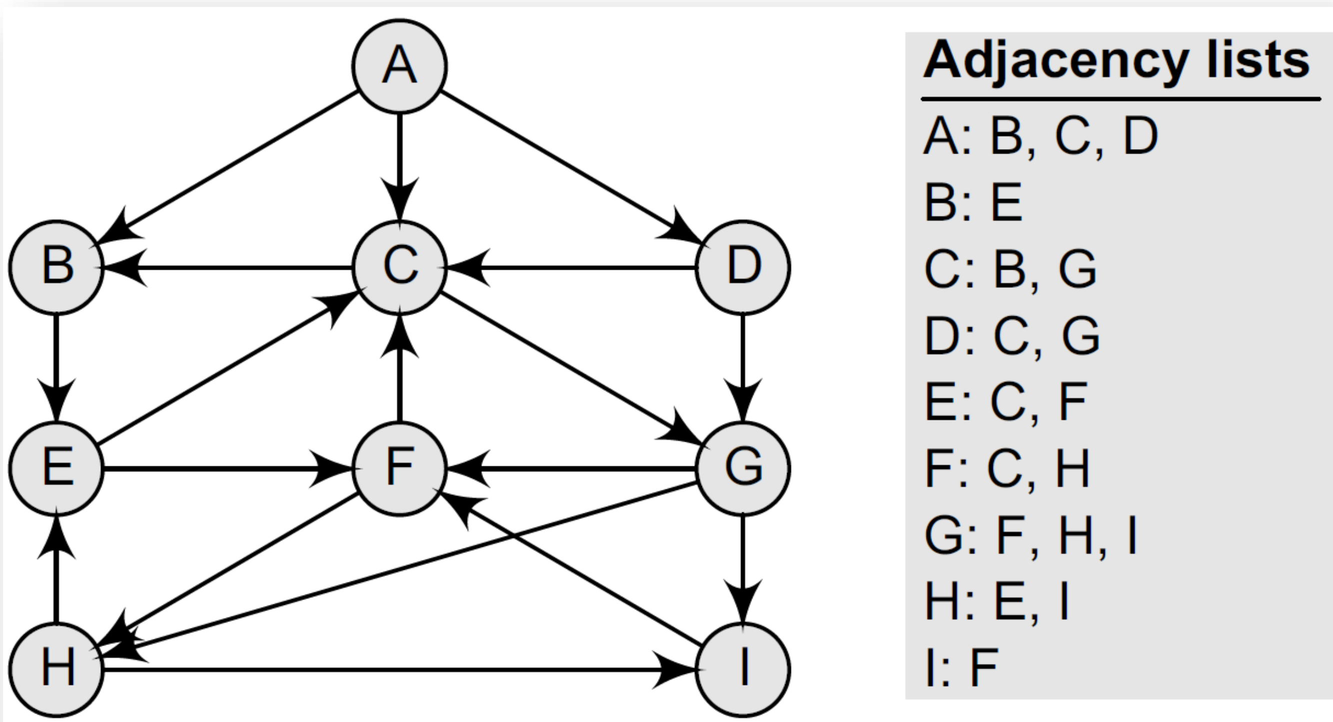
- In a graph structure, an important issue is to find a (minimal) path from a node to another node
  - Breadth-first search
    - BFS uses a **queue** as an auxiliary data structure to store nodes for further processing

      Queue: first-in-first-out
  - Depth-first search
    - DFS uses a **stack** to store nodes for further processing

      Stack: last-in-first-out

Breadth-first Search.

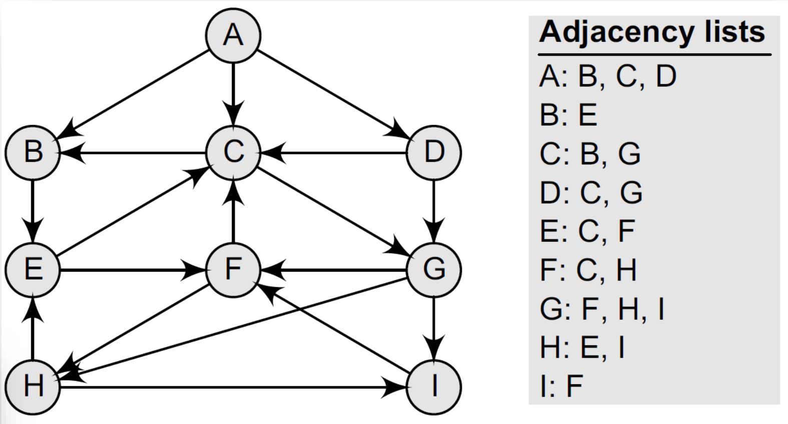
- Breadth-first search (BFS) is a graph search algorithm that begins at the predefined node and explores all the neighboring nodes until the target node is reached
  - Given a directed graph, please find a path from *A* to *I* by using BFS



Breadth-first Search..

- **QUEUE** is used to hold the nodes that have to be processed, **ORIG** is used to keep track of the origin of each edge
  - Step 1:

QUEUE =	A
ORIG =	\0



  - Step 2:

QUEUE =	A	B	C	D
ORIG =	\0	A	A	A

  - Step 3:

QUEUE =	A	B	C	D	E
ORIG =	\0	A	A	A	B

  - Step 4:

QUEUE =	A	B	C	D	E	G
ORIG =	\0	A	A	A	B	C

Breadth-first Search...

– Step 5:

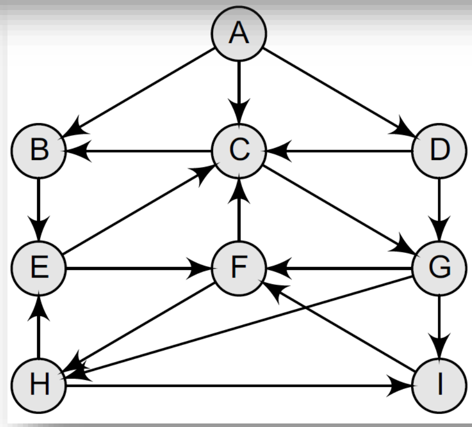
QUEUE =	A	B	C	D	E	G
ORIG =	\0	A	A	A	B	C

– Step 6:

QUEUE =	A	B	C	D	E	G	F
ORIG =	\0	A	A	A	B	C	E

– Step 7:

QUEUE =	A	B	C	D	E	G	F	H	I
ORIG =	\0	A	A	A	B	C	E	G	G



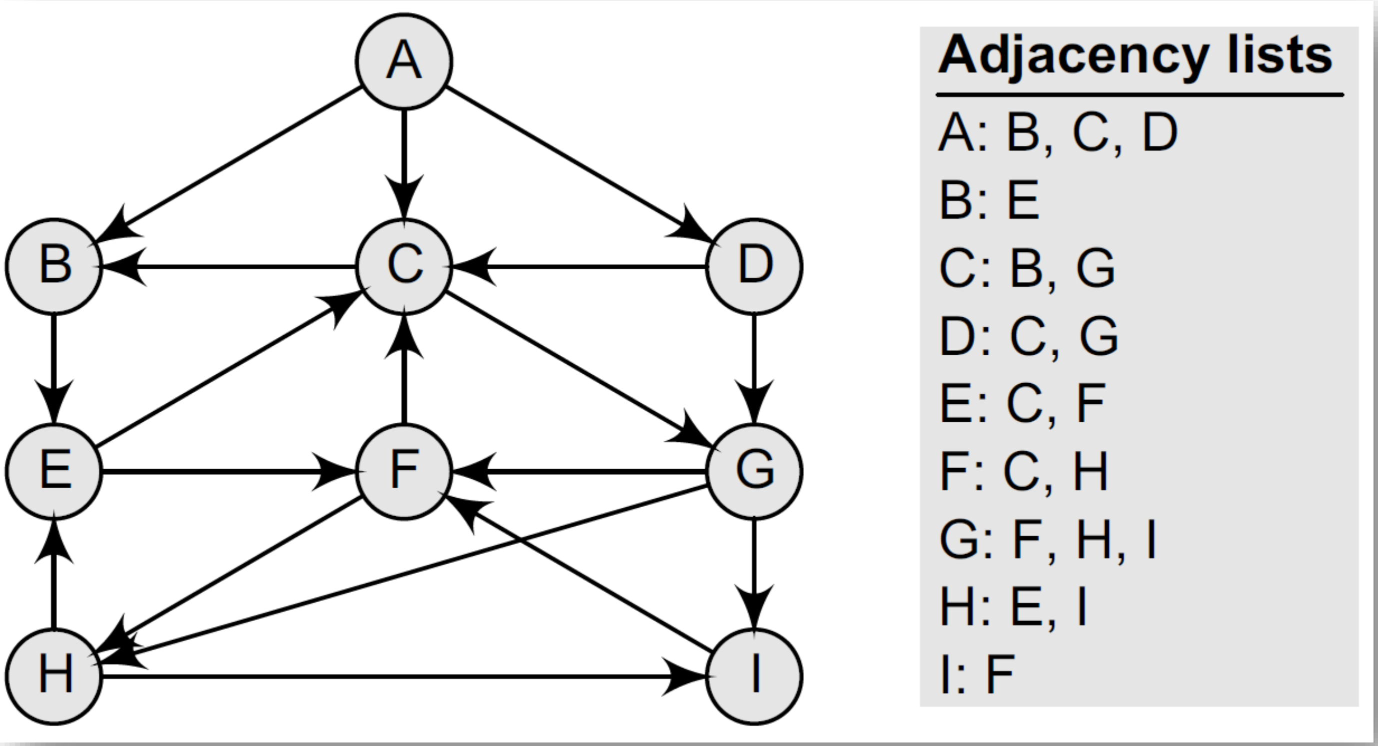
Adjacency lists

A: B, C, D
B: E
C: B, G
D: C, G
E: C, F
F: C, H
G: F, H, I
H: E, I
I: F

Breadth-first Search….

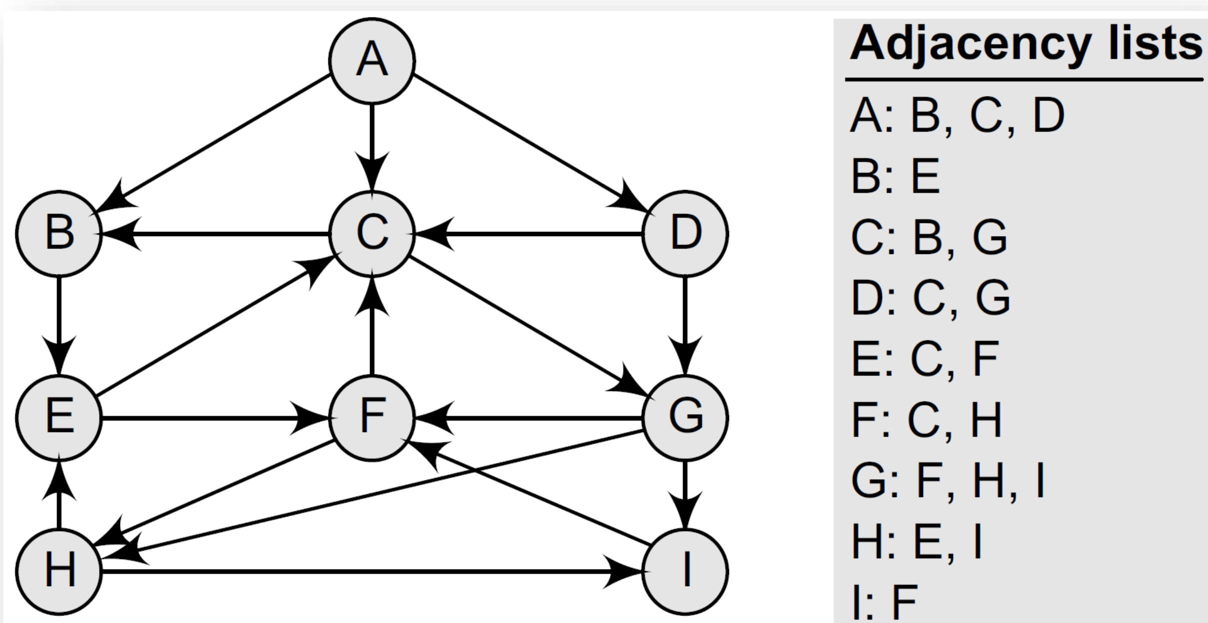
– Final, by referring to ORIG, the minimum path is $A \rightarrow C \rightarrow G \rightarrow I$

QUEUE =	A	B	C	D	E	G	F	H	I
ORIG =	\0	A	A	A	B	C	E	G	G



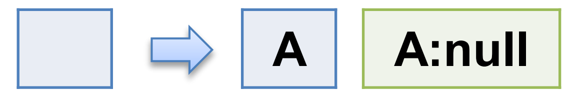
Depth-first Search.

- The depth-first search algorithm progresses by expanding the starting node of G and then going deeper and deeper until the goal node is found, or until a node that has no children is encountered
  - Given a graph G and its adjacency list, please find a path from A to I by using DFS

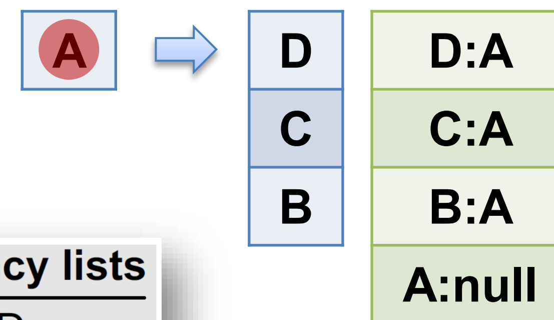


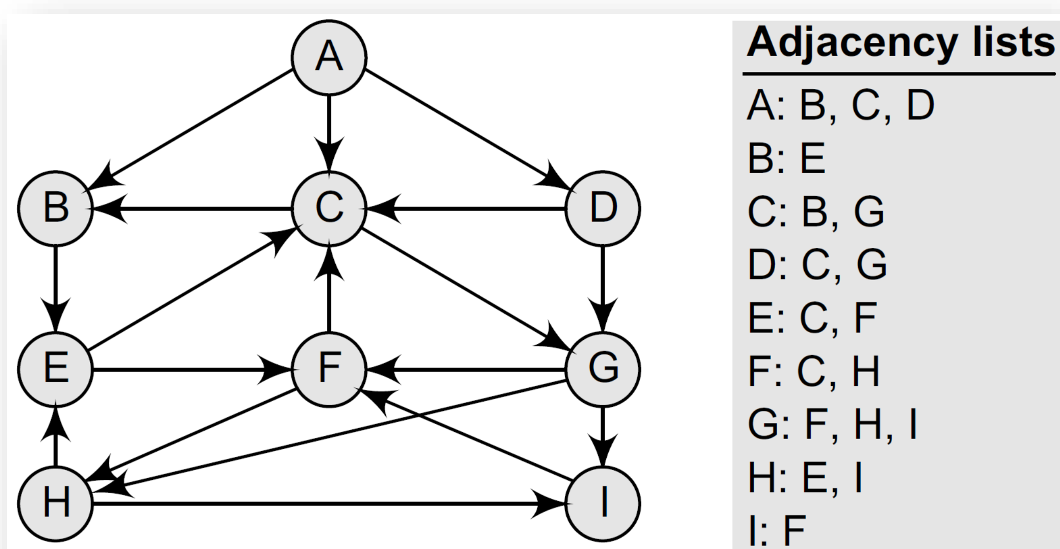
Depth-first Search..

- Given a graph G and its adjacency list, please find a path from A to I by using DFS
  - Step1
    - Push A in stack



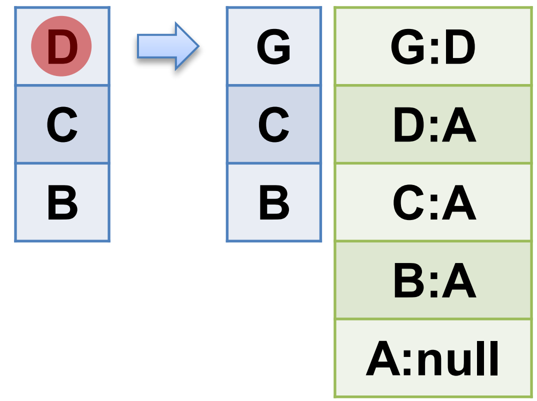
  - Step2
    - Pop the top element of the stack (i.e., A)
    - Push all the neighbors of A onto the stack



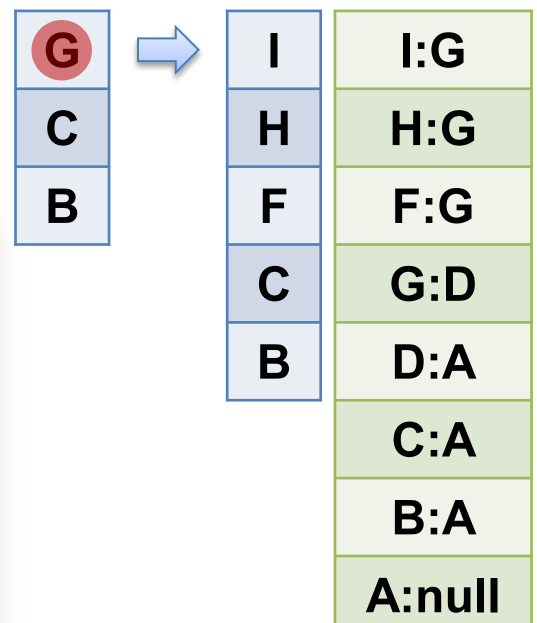


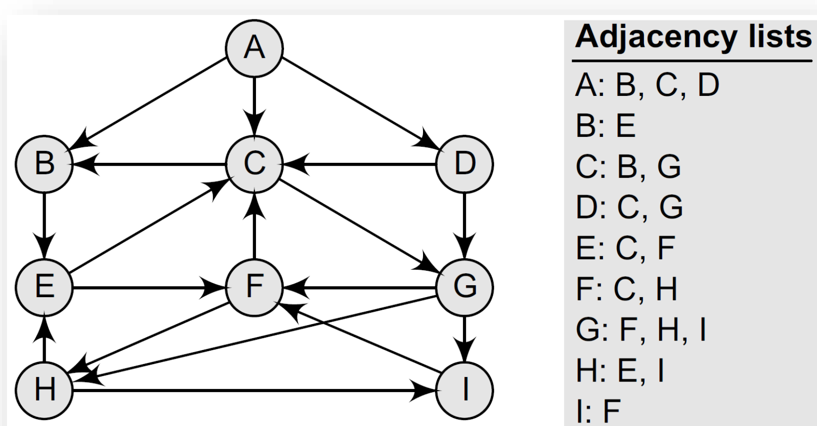
Depth-first Search…

- Given a graph G and its adjacency list, please find a path from A to I by using DFS
  - Step3
    - Pop the top element of the stack (i.e., D)
    - Push all the neighbors of D onto the stack



  - Step4
    - Pop the top element of the stack (i.e., G)
    - Push all the neighbors of G onto the stack





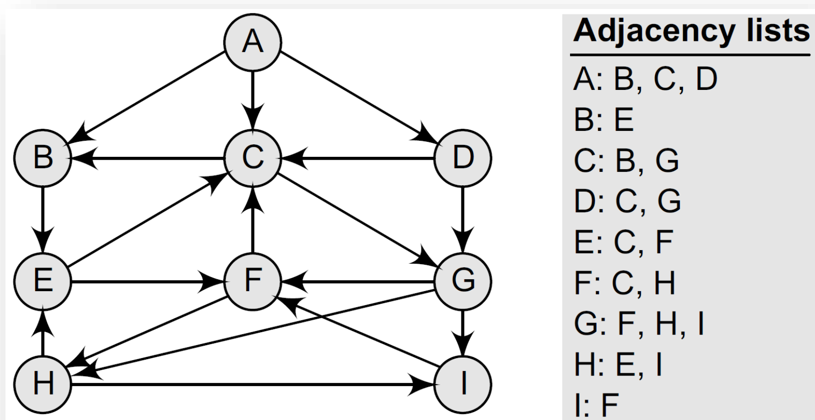
Depth-first Search....

- Given a graph G and its adjacency list, please find a path from A to I by using DFS
  - Step5
    - Pop the top element of the stack (i.e., I)
    - Since I is the target node, so there is a path from A to I

$ADGI$

I
H
F
C
B

I:G
H:G
F:G
G:D
D:A
C:A
B:A
A:null



Depth-first Search…..

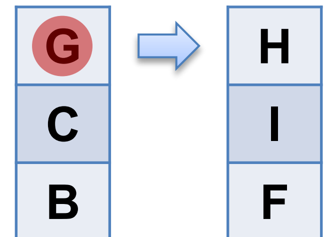
- If you are not lucky enough
  - Given a graph G and its adjacency list, please find a path from A to I by using DFS
    - Step4

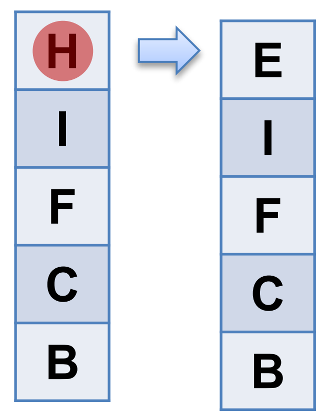
      Pop the top element of the stack (i.e., G)

      Push all the neighbors of G onto the stack
    - Step5

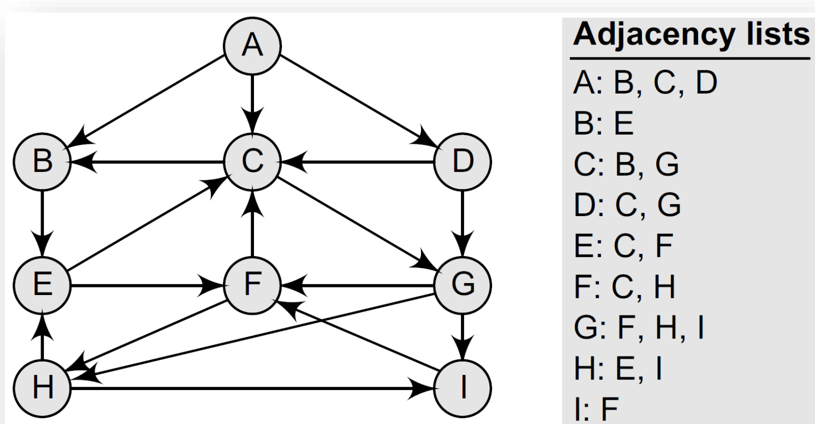
      Pop the top element of the stack (i.e., H)

      Push all the neighbors of H onto the stack





E:H
H:G
I:G
F:G
G:D
D:A
C:A
B:A
A:null

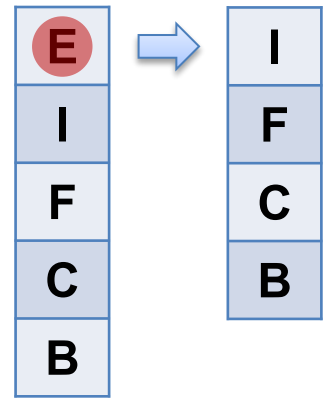


Depth-first Search……

- If you are not lucky enough
  - Given a graph G and its adjacency list, please find a path from A to I by using DFS
    - Step6

      Pop the top element of the stack (i.e., E)

      Push all the neighbors of E onto the stack

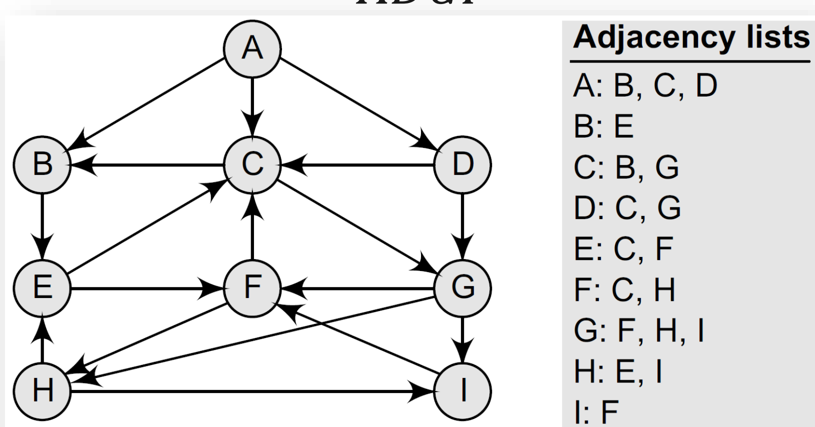


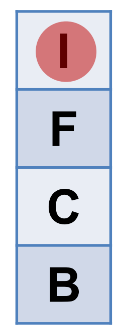
    - Step7

      Pop the top element of the stack (i.e., I)

      Since I is the target node, so there is a path from A to I

      $ADGI$





E:H
H:G
I:G
F:G
G:D
D:A
C:A
B:A
A:null

Questions?

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