Advanced Counting Techniques

Chapter 8

Recurrence Relations

Definition: A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, a_0, a_1, …, a_{n-1}, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.
- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Applications of Recurrence Relations

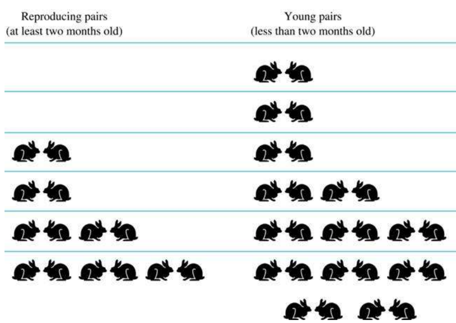
Section 8.1

Rabbits and the Fibonacci Numbers [1]

Example: A young pair of rabbits (one of each gender) is placed on an island. A pair of rabbits does not breed until they are 2 months old. After they are 2 months old, each pair of rabbits produces another pair each month. Find a recurrence relation for the number of pairs of rabbits on the island after n months, assuming that rabbits never die.

This is the original problem considered by Leonardo Pisano (Fibonacci) in the thirteenth century.

Rabbits and the Fibonacci Numbers₂



Month	Reproducing pairs	Young pairs	Total pairs
1	0	1	1
2	0	1	1
3	1	1	2
4	1	2	3
5	2	3	5
6	3	5	8

Modeling the Population Growth of Rabbits on an Island

Jump to long description

Rabbits and the Fibonacci Numbers₃

Solution: Let f_n be the number of pairs of rabbits after n months.

- There are is $f_1 = 1$ pairs of rabbits on the island at the end of the first month.
- We also have $f_2 = 1$ because the pair does not breed during the first month.
- To find the number of pairs on the island after n months, add the number on the island after the previous month, f_{n-1}, and the number of newborn pairs, which equals f_{n-2}, because each newborn pair comes from a pair at least two months old.

Consequently the sequence $\{f_n\}$ satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$ with the initial conditions $f_1 = 1$ and $f_2 = 1$.

The number of pairs of rabbits on the island after n months is given by the nth Fibonacci number.

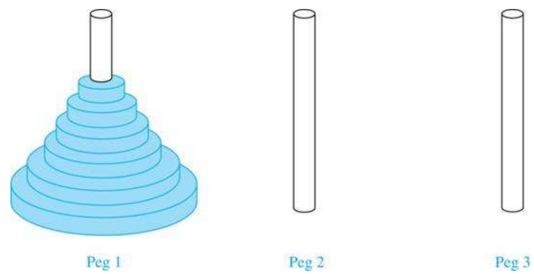
The Tower of Hanoi₁

In the late 19th century, the French mathematician Édouard Lucas invented a puzzle consisting of three pegs on a board with disks of different sizes. Initially all of the disks are on the first peg in order of size, with the largest on the bottom.

Rules: You are allowed to move the disks one at a time from one peg to another as long as a larger disk is never placed on a smaller.

Goal: Using allowable moves, end up with all the disks on the second peg in order of size with largest on the bottom.

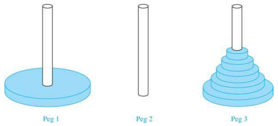
The Tower of Hanoi₂



The Initial Position in the Tower of Hanoi Puzzle

The Tower of Hanoi 3

Solution: Let $\{H_n\}$ denote the number of moves needed to solve the Tower of Hanoi Puzzle with n disks. Set up a recurrence relation for the sequence $\{H_n\}$. Begin with n disks on peg 1. We can transfer the top $n-1$ disks, following the rules of the puzzle, to peg 3 using H_{n-1} moves.



First, we use 1 move to transfer the largest disk to the second peg. Then we transfer the $n-1$ disks from peg 3 to peg 2 using H_{n-1} additional moves. This can not be done in fewer steps. Hence,

$$H_n = 2H_{n-1} + 1.$$

The initial condition is $H_1 = 1$ since a single disk can be transferred from peg 1 to peg 2 in one move.

The Tower of Hanoi 4

We can use an iterative approach to solve this recurrence relation by repeatedly expressing H_n in terms of the previous terms of the sequence.

$$\begin{aligned}
H_n &= 2H_{n-1} + 1 \\
&= 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1 \\
&= 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1 \\
&\ \ \vdots \\
&= 2^{n-1}H_1 + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \\
&= 2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 2 + 1 \qquad \textit{because } H_1 = 1 \\
&= 2^n - 1 \quad \textit{using the formula for the sum of the terms of a geometric series}
\end{aligned}$$

- There was a myth created with the puzzle. Monks in a tower in Hanoi are transferring 64 gold disks from one peg to another following the rules of the puzzle. They move one disk each day. When the puzzle is finished, the world will end.
- Using this formula for the 64 gold disks of the myth,
  $2^{64} - 1$ = 18,446, 744,073, 709,551,615
  days are needed to solve the puzzle, which is more than 500 billion years.
- Reve's puzzle (proposed in 1907 by Henry Dudeney) is similar but has 4 pegs. There is a well-known unsettled conjecture for the minimum number of moves needed to solve this puzzle.

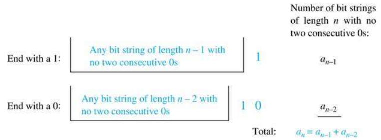
Counting Bit Strings 1

Example 3: Find a recurrence relation and give initial conditions for the number of bit strings of length n without two consecutive 0s. How many such bit strings are there of length five?

Solution: Let a_n denote the number of bit strings of length n without two consecutive 0s. To obtain a recurrence relation for $\{a_n\}$ note that the number of bit strings of length n that do not have two consecutive 0s is the number of bit strings ending with a 0 plus the number of such bit strings ending with a 1.

Now assume that $n \geq 3$.

- The bit strings of length n ending with 1 without two consecutive 0s are the bit strings of length $n-1$ with no two consecutive 0s with a 1 at the end. Hence, there are a_{n-1} such bit strings.
- The bit strings of length n ending with 0 without two consecutive 0s are the bit strings of length $n-2$ with no two consecutive 0s with 10 at the end. Hence, there are a_{n-2} such bit strings.



We conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$.

Bit Strings 2

The initial conditions are:

- $a_1 = 2$, since both the bit strings 0 and 1 do not have consecutive 0s.
- $a_2 = 3$, since the bit strings 01, 10, and 11 do not have consecutive 0s, while 00 does.

To obtain a_5, we use the recurrence relation three times to find that:

- $a_3 = a_2 + a_1 = 3 + 2 = 5$
- $a_4 = a_3 + a_2 = 5 + 3 = 8$
- $a_5 = a_4 + a_3 = 8 + 5 = 13$

Note that $\{a_n\}$ satisfies the same recurrence relation as the Fibonacci sequence. Since $a_1 = f_3$ and $a_2 = f_4$, we conclude that $a_n = f_{n+2}$.

Counting the Ways to Parenthesize a Product

Example: Find a recurrence relation for C_n, the number of ways to parenthesize the product of $n + 1$ numbers, $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, to specify the order of multiplication. For example, $C_3 = 5$, since all the possible ways to parenthesize 4 numbers are

$$((x_0 \cdot x_1) \cdot x_2) \cdot x_3, (x_0 \cdot (x_1 \cdot x_2)) \cdot x_3, (x_0 \cdot x_1) \cdot (x_2 \cdot x_3), x_0 \cdot ((x_1 \cdot x_2) \cdot x_3), x_0 \cdot (x_1 \cdot (x_2 \cdot x_3))$$

Solution: Note that however parentheses are inserted in $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_n$, one "$\cdot$" operator remains outside all parentheses. This final operator appears between two of the $n + 1$ numbers, say x_k and x_{k+1}. Since there are C_k ways to insert parentheses in the product $x_0 \cdot x_1 \cdot x_2 \cdot \cdots \cdot x_k$ and C_{n-k-1} ways to insert parentheses in the product $x_{k+1} \cdot x_{k+2} \cdot \cdots \cdot x_n$, we have

$$\begin{aligned} C_n &= C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-2} C_1 + C_{n-1} C_0 \\ &= \sum_{k=0}^{n-1} C_k C_{n-k-1} \end{aligned}$$

The initial conditions are $C_0 = 1$ and $C_1 = 1$.

The sequence $\{C_n\}$ is the sequence of **Catalan Numbers**.

Linear Homogeneous Recurrence Relations

Definition: A *linear homogeneous recurrence relation of degree* k *with constant coefficients* is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$, where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$

- *linear*: the right-hand side is a sum of the previous terms of the sequence each multiplied by a function of n.
- *homogeneous*: no terms occur that are not multiples of the a_js. Each coefficient is a constant.
- *degree k*: a_n is expressed in terms of the previous k terms of the sequence.

By strong induction, a sequence satisfying such a recurrence relation is uniquely determined by the recurrence relation and the k initial conditions $a_0 = C_1, a_0 = C_1, \ldots, a_{k-1} = C_{k-1}$.

Solving Linear Recurrence Relations

Section 8.2

Examples of Linear Homogeneous Recurrence Relations

$P_n = (1.11) P_{n-1}$	linear homogeneous recurrence relation of degree one
$f_n = f_{n-1} + f_{n-2}$	linear homogeneous recurrence relation of degree two
$a_n = a_{n-1} + a^2_{n-2}$	not linear
$H_n = 2H_{n-1} + 1$	not homogeneous
$B_n = nB_{n-1}$	coefficients are not constants

Solving Linear Homogeneous Recurrence Relations

The basic approach is to look for solutions of the form $a_n = r^n$, where r is a constant.

Note that $a_n = r^n$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$ if and only if

$r^n = c_1r^{n-1} + c_2r^{n-2} + \cdots + c_kr^{n-k}$.

Algebraic manipulation yields the *characteristic equation*:

$$r^k - c_1r^{k-1} - c_2r^{k-2} - \cdots - c_{k-1}r - c_k = 0$$

The sequence $\{a_n\}$ with $a_n = r^n$ is a solution if and only if r is a solution to the characteristic equation.

The solutions to the characteristic equation are called the *characteristic roots* of the recurrence relation. The roots are used to give an explicit formula for all the solutions of the recurrence relation.

Solving Linear Homogeneous Recurrence Relations of Degree Two

Theorem 1: Let c_1 and c_2 be real numbers. Suppose that $r^2 - c_1r - c_2 = 0$ has two distinct roots r_1 and r_2. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha_1r_1^n + \alpha_2r_2^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Using Theorem 1

Example: What is the solution to the recurrence relation

$a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution: The characteristic equation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and

only if $a_n = \alpha_1 2^n + \alpha_2(-1)^n$, for some constants α_1 and α_2.

To find the constants α_1 and α_2, note that

$a_0 = 2 = \alpha_1 + \alpha_2$ and $a_1 = 7 = \alpha_1 2 + \alpha_2(-1)$.

Solving these equations, we find that $\alpha_1 = 3$ and $\alpha_2 = -1$.

Hence, the solution is the sequence $\{a_n\}$ with $a_n = 3\cdot 2^n - (-1)^n$.

An Explicit Formula for the Fibonacci Numbers[1]

We can use Theorem 1 to find an explicit formula for the Fibonacci numbers. The sequence of Fibonacci numbers satisfies the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with the initial conditions: $f_0 = 0$ and $f_1 = 1$.

Solution: The roots of the characteristic equation $r^2 - r - 1 = 0$ are

$$r_1 = \frac{1+\sqrt{5}}{2}$$

$$r_2 = \frac{1-\sqrt{5}}{2}$$

Fibonacci Numbers[2]

Therefore by Theorem 1

$$f_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some constants α_1 and α_2.

Using the initial conditions $f_0 = 0$ and $f_1 = 1$, we have

$$f_0 = \alpha_1 + \alpha_2 = 0$$

$$f_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.$$

Solving, we obtain $\alpha_1 = \frac{1}{\sqrt{5}}, \quad \alpha_2 - \frac{1}{\sqrt{5}}.$

Hence, $f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$

Using Theorem 2

Example: What is the solution to the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution: The characteristic equation is $r^2 - 6r + 9 = 0$.

The only root is $r = 3$. Therefore, $\{a_n\}$ is a solution to the recurrence relation if and only if

$$a_n = \alpha_1 3^n + \alpha_2 n(3)^n$$

where α_1 and α_2 are constants.

To find the constants α_1 and α_2, note that

$$a_0 = 1 = \alpha_1 \quad \text{and} \quad a_1 = 6 = \alpha_1 \cdot 3 + \alpha_2 \cdot 3.$$

Solving, we find that $\alpha_1 = 1$ and $\alpha_2 = 1$.

Hence,

$$a_n = 3^n + n3^n .$$

The Solution when there is a Repeated Root

Theorem 2: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1r - c_2 = 0$ has one repeated root r_0. Then the sequence $\{a_n\}$ is a solution to the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if

$$a_n = \alpha_1 r_0^n + \alpha_2 n r_0^n = (\alpha_1 + \alpha_2 n) r_0^n$$

for $n = 0,1,2,\ldots$, where α_1 and α_2 are constants.

Solving Linear Homogeneous Recurrence Relations of Arbitrary Degree

This theorem can be used to solve linear homogeneous recurrence relations with constant coefficients of any degree when the characteristic equation has distinct roots.

Theorem 3: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$r^k - c_1r^{k-1} - \cdots - c_k = 0$

has k distinct roots $r_1, r_2, \ldots, r_k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots.. + c_k a_{n-k}$

if and only if

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^m + \cdots + \alpha_k r_k^n$$

for $n = 0, 1, 2, \ldots$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are constants.

The General Case with Repeated Roots Allowed

Theorem 4: Let $c_1, c_2, \ldots, c_k$ be real numbers. Suppose that the characteristic equation

$$r^k - c_1 r^{k-1} - \cdots - c_k = 0$$

has t distinct roots $r_1, r_2, \ldots, r_t$ with multiplicities $m_1, m_2, \ldots, m_t$, respectively so that $m_i \geq 1$ for $i = 0, 1, 2, \ldots, t$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots .. + c_k a_{n-k}$$

if and only if

$$\begin{aligned} a_n = &\left(\alpha_{1,0} + \alpha_{1,1} n + \cdots + \alpha_{1,m_1-1} n^{m_1-1}\right) r_1^n \\ &+ \left(\alpha_{2,0} + \alpha_{2,1} n + \cdots + \alpha_{2,m_2-1} n^{m_2-1}\right) r_2^n \\ &+ \cdots \\ &+ \left(\alpha_{t,0} + \alpha_{t,1} n + \cdots + \alpha_{t,m_t-1} n^{m_t-1}\right) r_t^n \end{aligned}$$

for $n = 0, 1, 2, \ldots$, where $\alpha_{i,j}$ are constants for $1 \leq i \leq t$ and $0 \leq j \leq m_{i-1}$.

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 2

The following are linear nonhomogeneous recurrence relations with constant coefficients:

$$a_n = a_{n-1} + 2^n$$
$$a_n = a_{n-1} + a_{n-2} + n^2 + n + 1$$
$$a_n = 3a_{n-1} + n3^n$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + n!$$

where the following are the associated linear homogeneous recurrence relations, respectively:

$$a_n = a_{n-1}$$
$$a_n = a_{n-1} + a_{n-2}$$
$$a_n = 3a_{n-1}$$
$$a_n = a_{n-1} + a_{n-2} + a_{n-3}$$

Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 1

Definition: A *linear nonhomogeneous recurrence relation with constant coefficients* is a recurrence relation of the form:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots .. + c_k a_{n-k} + F(n),$$

where $c_1, c_2, \ldots, c_k$ are real numbers, and $F(n)$ is a function not identically zero depending only on n.

The recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots .. + c_k a_{n-k},$$

is called the associated homogeneous recurrence relation.

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 1

Theorem 5: If $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous linear recurrence relation with constant coefficients

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + F(n),$$

then every solution is of the form $\{a_n^{(p)} + a_n^{(h)}\}$, where $\{a_n^{(h)}\}$ is a solution of the associated homogeneous recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

Solving Linear Nonhomogeneous Recurrence Relations with Constant Coefficients 2

Example: Find all solutions of the recurrence relation $a_n = 3a_{n-1} + 2n$. What is the solution with $a_1 = 3$?

Solution: The associated linear homogeneous equation is $a_n = 3a_{n-1}$. Its solutions are $a_n^{(h)} = \alpha 3^n$, where α is a constant.

Because $F(n)= 2n$ is a polynomial in n of degree one, to find a particular solution we might try a linear function in n, say $p_n = cn + d$, where c and d are constants. Suppose that $p_n = cn + d$ is such a solution.

Then $a_n = 3a_{n-1} + 2n$ becomes $cn + d = 3(c(n-1) + d)+ 2n$.

Simplifying yields $(2 + 2c)n + (2d - 3c) = 0$. It follows that $cn + d$ is a solution if and only if

$2 + 2c = 0$ and $2d - 3c = 0$. Therefore, $cn + d$ is a solution if and only if $c = - 1$ and $d = - 3/2$.

Consequently, $a_n^{(p)} = -n - 3/2$ is a particular solution.

By Theorem 5, all solutions are of the form $a_n = a_n^{(p)} + a_n^{(h)} = -n - 3/2 + \alpha 3^n$, where α is a constant.

To find the solution with $a_1 = 3$, let $n = 1$ in the above formula for the general solution.

Then $3 = -1 - 3/2 + 3\alpha$, and $\alpha = 11/6$. Hence, the solution is $a_n = -n - 3/2 + (11/6)3^n$.

Counting Problems and Generating Functions 1

Example: Find the number of solutions of

$e_1 + e_2 + e_3 = 17$,

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions is the coefficient of x^{17} in the expansion of

$(x^2 + x^3 + x^4 + x^5)\ (x^3 + x^4 + x^5 + x^6)\ (x^4 + x^5 + x^6 + x^7)$.

This follows because a term equal to is obtained in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Generating Functions

Section 8.4

Generating Functions

Definition: The *generating function for the sequence* $a_0, a_1, \ldots, a_k, \ldots$ of real numbers is the infinite series

$$G(x) = a_0 + a_1 x + \cdots a_k x^k + \cdots = \sum_{k=0}^{\infty} a_k x^k$$

Examples:

- The sequence $\{a_k\}$ with $a_k = 3$ has the generating function $\sum_{k=0}^{\infty} 3x^k$.
- The sequence $\{a_k\}$ with $a_k = k + 1$ has the generating function has the generating function $\sum_{k=0}^{\infty} (k+1)x^k$.
- The sequence $\{a_k\}$ with $a_k = 2^k$ has the generating function has the generating function $\sum_{k=0}^{\infty} 2^k x^k$.

Generating Functions for Finite Sequences 1

Generating functions for finite sequences of real numbers can be defined by extending a finite sequence $a_0, a_1, \ldots, a_n$ into an infinite sequence by setting $a_{n+1} = 0$, $a_{n+2} = 0$, and so on.

The generating function $G(x)$ of this infinite sequence $\{a_n\}$ is a polynomial of degree n because no terms of the form $a_j x^j$ with $j > n$ occur, that is,

$$G(x) = a_0 + a_1 x + \cdots + a_n x^n.$$

Useful Generating Functions

TABLE 1 Useful Generating Functions.

$G(x)$	a_k
$(1+x)^n = \sum_{k=0}^{n} C(n,k)x^k = 1 + C(n,1)x + C(n,2)x^2 + \cdots + x^n$	$C(n,k)$
$(1+ax)^n = \sum_{k=0}^{n} C(n,k)a^k x^k = 1 + C(n,1)ax + C(n,2)a^2x^2 + \cdots + a^n x^n$	$C(n,k)a^k$
$(1+x^r)^n = \sum_{k=0}^{n} C(n,k)x^{rk} = 1 + C(n,1)x^r + C(n,2)x^{2r} + \cdots + x^{rn}$	$C(n,k/r)$ if $r \mid k$; 0 otherwise
$\frac{1-x^{n+1}}{1-x} = \sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n$	1 if $k \le n$; 0 otherwise
$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$	1
$\frac{1}{1-ax} = \sum_{k=0}^{\infty} a^k x^k = 1 + ax + a^2x^2 + \cdots$	a^k
$\frac{1}{1-x^r} = \sum_{k=0}^{\infty} x^{rk} = 1 + x^r + x^{2r} + \cdots$	1 if $r \mid k$; 0 otherwise
$\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1)x^k = 1 + 2x + 3x^2 + \cdots$	$k+1$
$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)x^k = 1 + C(n,1)x + C(n+1,2)x^2 + \cdots$	$C(n+k-1,k) = C(n+k-1,n-1)$
$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)(-1)^k x^k = 1 - C(n,1)x + C(n+1,2)x^2 - \cdots$	$(-1)^k C(n+k-1,k) = (-1)^k C(n+k-1,n-1)$
$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} C(n+k-1,k)a^k x^k = 1 + C(n,1)ax + C(n+1,2)a^2x^2 + \cdots$	$C(n+k-1,k)a^k = C(n+k-1,n-1)a^k$
$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$	$1/k!$
$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	$(-1)^{k+1}/k$

Note: The series for the last two generating functions can be found in most calculus books when power series are discussed.

Generating Functions for Finite Sequences 2

Example: What is the generating function for the sequence 1,1,1,1,1,1?

Solution: The generating function of 1,1,1,1,1,1 is

$1 + x + x^2 + x^3 + x^4 + x^5$.

By Theorem 1 of Section 2.4, we have

$(x^6 - 1)/(x - 1) = 1 + x + x^2 + x^3 + x^4 + x^5$

when $x \neq 1$.

Consequently $G(x) = (x^6 - 1)/(x - 1)$ is the generating function of the sequence.

Counting Problems and Generating Functions 1

Example: Find the number of solutions of

$e_1 + e_2 + e_3 = 17$,

where e_1, e_2, and e_3 are nonnegative integers with $2 \le e_1 \le 5$, $3 \le e_2 \le 6$, and $4 \le e_3 \le 7$.

Solution: The number of solutions is the coefficient of x^{17} in the expansion of

$(x^2 + x^3 + x^4 + x^5)\,(x^3 + x^4 + x^5 + x^6)\,(x^4 + x^5 + x^6 + x^7)$.

This follows because a term equal to is obtained in the product by picking a term in the first sum x^{e_1}, a term in the second sum x^{e_2}, and a term in the third sum x^{e_3}, where $e_1 + e_2 + e_3 = 17$.

There are three solutions since the coefficient of x^{17} in the product is 3.

Counting Problems and Generating Functions[2]

Example: Use generating functions to find the number of k-combinations of a set with n elements, i.e., $C(n,k)$.

Solution: Each of the n elements in the set contributes the term $(1 + x)$ to the generating function

$$f(x) = \sum_{k=0}^{n} a^k x^k.$$

Hence $f(x) = (1 + x)^n$ where $f(x)$ is the generating function for $\{a^k\}$, where a^k represents the number of k-combinations of a set with n elements.

By the binomial theorem, we have
$$f(x) = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where
$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Hence,
$$C(n,k) = \frac{n!}{k!(n-k)!}$$

Example

- Find a generating function of $a_n = a_{n-1} + a_{n-2}$
  - $g(x) - a_0 - a_1 x = \sum_{n=2}^{\infty} a_n x^n$
$$= \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n$$
$$= x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2}$$
$$= x(g(x) - a_0) + x^2 g(x)$$
- If $a_0 = a_1 = 1$ is given, we have
$$g(x) = \frac{1}{1 - x - x^2}$$

Example

- If $a_n = a_{n-1} + n$, find a generating function of a_n
  - $g(x) - a_0 = \sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} (a_{n-1} + n) x^n$
$$= x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + \sum_{n=1}^{\infty} n x^n$$
    - $\sum_{n=1}^{\infty} a_{n-1} x^{n-1} = a_0 + a_1 x + a_2 x^2 + \cdots = g(x)$
    - $\sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = x \left(\frac{1}{1-x}\right)' = \frac{x}{(1-x)^2}$
  - $g(x) - a_0 = x g(x) + \frac{x}{(1-x)^2}$
- If $a_0 = 1$ is given, $g(x) - 1 = x g(x) + \frac{x}{(1-x)^2}$
  - $g(x) = \frac{x}{(1-x)^3} + \frac{1}{1-x} \rightarrow a_n = \binom{n+1}{2} + 1$

Example

- Count the number of n-digit decimal numbers that contain an even number of 0s.
  - $a_n = 9a_{n-1} + (10^{n-1} - a_{n-1}) = 8a_{n-1} + 10^{n-1}$
  - $a_0 = 1$
  - $g(x) - 1 = \sum_{n=1}^{\infty} (8a_{n-1} + 10^{n-1}) x^n$
$$= 8x \sum_{n=1}^{\infty} a_{n-1} x^{n-1} + x \sum_{n=1}^{\infty} 10^{n-1} x^{n-1}$$
$$= 8x g(x) + \frac{x}{1 - 10x}$$
  - $(1 - 8x) g(x) = \frac{1-9x}{1-10x} \rightarrow$

$$g(x) = \frac{1 - 9x}{(1 - 8x)(1 - 10x)} = \frac{1}{2} \frac{2 - 18x}{(1 - 8x)(1 - 10x)} = \frac{1}{2}\left(\frac{1}{1 - 8x} + \frac{1}{1 - 10x}\right)$$

  - $a_n = \frac{1}{2}(8^n + 10^n)$

Example

Find the coefficient of x^{16} in $\left(x^2 + x^3 + \cdots\right)^5$.

- Since $\left(x^2 + x^3 + \cdots\right)^5 = x^{10}(1 + x + \cdots)^5$, we need to find the coefficient of x^6 in $(1 + x + \cdots)^5 = 1/(1 - x)^5$. From expansion (5), the coefficient of x^6 is C(6+5-1,6)=C(10,6).
- More generally, the coefficient of x^r in $\left(x^2 + x^3 + \cdots\right)^5$ is C(r-10+5-1,r-10)

Example

- Count the number of ways to distribute 25 identical balls into 7 distinct boxes, where the first box can get ≤ 10 balls.
  - Generating function: $\left(1 + x + \cdots + x^{10}\right)(1 + x + \cdots)^6$
  - The goal is to get the coefficient of x^{25}
    - $(1 + x + \cdots + x^{10})(1 + x + \cdots)^6 = \left(\frac{1-x^{11}}{1-x}\right)\left(\frac{1}{1-x}\right)^6 = (1 - x^{11})\left(\frac{1}{1-x}\right)^7$
    - The coefficient is $C(25 + 7 - 1{,}25) \times 1 + C(14 + 7 - 1{,}14) \times (-1)$

Example

Find the number of ways to collect \$15 from 20 people if the first 19 people can give \$1 (or nothing) and the 20th person can give either \$1 or \$5 (or nothing).

- The first 19 people can be represented by $(1 + x)^{19}$
  - Each can give \$0 or \$1
- The 20th person can be represented by $\left(1 + x + x^5\right)$
  - He/She can give \$0, \$1, or \$5
- The resulted generating function: $(1 + x)^{19}\left(1 + x + x^5\right)$
- To get the coefficient of x^{15}, from the binomial theorem and Rule (6) we get $1 \times C(19{,}15) + 1 \times C(19{,}14) + 1 \times C(19{,}10)$.

Example

How many ways are there to select 25 toys from 7 types of toys with between 2 and 6 of each type?

- Generating function: $\left(x^2 + \cdots + x^6\right)^7$
- Need the coefficient of x^{25}
  - $\left(x^2 + \cdots + x^6\right)^7 = x^{14}(1 + \cdots + x^4)^7 = x^{14}\left(\frac{1-x^5}{1-x}\right)^7 = x^{14}\left(1 - x^5\right)^7\left(\frac{1}{1-x}\right)^7$
  - Now, find the coefficient of x^{11} in $\left(1 - x^5\right)^7\left(\frac{1}{1-x}\right)^7$
    - $\left(1 - x^5\right)^7 = 1 - C(7{,}1)x^5 + C(7{,}2)x^{10} - C(7{,}3)x^{15} + higher\ degrees$
    - So, the coefficient we need is $1 \times C(11 + 7 - 1{,}11) - C(7{,}1) \times C(6 + 7 - 1{,}6) + C(7{,}2) \times C(1 + 7 - 1{,}1)$

Example

Verify the binomial identity $\sum_{k=0}^{n} C(n,k)^2 = C(2n,n)$

- $\sum_{k=0}^{n} C(n,k)^2 = \sum_{k=0}^{n} C(n,k)C(n,n-k)$.
- It is the coefficient of x^n in $(1+x)^n(1+x)^n = (1+x)^{2n}$