Graph Algorithms - Lecture 11

December 20, 2024

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Theorem 1

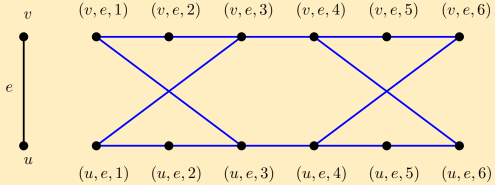
(Karp, 1972) SM $\leqslant_P$ CH.

Proof. Let $G = (V, E)$ and $j \in \mathbb{N}$ an instance for the problem SM. We will build in polynomial time (w.r.t. $n = |V|$) a graph H such that there is a stable set S in G with $|S| \geqslant j$ if and only if H is Hamiltonian. Let $k = n - j$. Suppose that $k > 0$ (to avoid trivial cases).

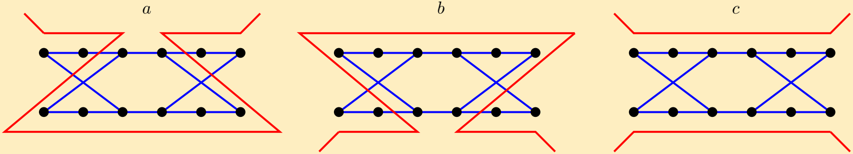
(i) Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k distinct vertices.

(ii) For each edge $e = uv \in E(G)$, consider the graph $G'_e = (V'_e, E'_e)$ with $V'_e = \{(w, e, i) : w \in \{u, v\}, i = \overline{1,6}\}$ and $E'_e = \{(w, e, i)(w, e, i+1) : w \in \{u, v\}, i = \overline{1,5}\} \cup \{(u, e, 1)(v, e, 3), (u, e, 3)(v, e, 1), (u, e, 4)(v, e, 6), (u, e, 6)(v, e, 4)\}$.

Proof cont'd.



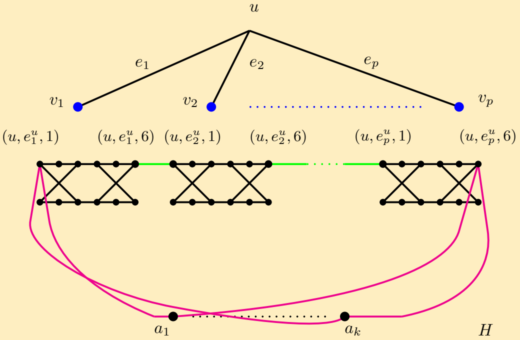
The graph has the property that, if it is an induced subgraph of a Hamiltonian graph H, and no one of the vertices (w,e,i) with $w \in \{u,v\}$ and $i = \overline{2,5}$ has another neighbor in H, then the only possibilities of traversing G'_e by a Hamiltonian cycle are:



Proof cont'd. Hence, if the Hamiltonian cycle "enters" in G'_e by a vertex corresponding to u ($(u, e, 1)$ or $(u, e, 6)$) then it "leaves" the graph G'_e also by a vertex corresponding to u.

(iii) For each vertex $u \in V$, consider (in an arbitrary order) the edges incident in G with u: $e^u_1 = uv_1, e^u_2 = uv_2, \ldots, e^u_p = uv_p$ ($p = d_G(u)$). Let $E''_u = \{(u, e^u_i, 6)(u, e^u_{i+1}, 1) \ : \ i = \overline{1, p-1}\}$ and $E'''_u = \{a_i(u, e^u_1, 1), a_i(u, e^u_p, 6) \ : \ i = \overline{1, k}\}$

Proof cont'd. The graph H has $V(H) = A \cup \left(\bigcup_{e \in E} V'_e \right)$ and $E(H) = \left(\bigcup_{e \in E} E'_e \right) \cup \left(\bigcup_{u \in V} (E''_u \cup E'''_u) \right)$. Clearly, it can be constructed from G in polynomial (in n) time complexity.



Proof cont'd. Now we show that there exists a stable set in G with at least j vertices if and only if H is Hamiltonian.

"$\Leftarrow$" If H is Hamiltonian, then there exists a Hamiltonian cycle, C, in H. Since A is a stable set in H, A decomposes the cycle C in exactly k internally disjoint paths: $D_{a_{i_1} a_{i_2}}, D_{a_{i_2} a_{i_3}}, \ldots, D_{a_{i_k} a_{i_1}}$.

Let $D_{a_{i_j} a_{i_{j+1}}}$ be such a path ($j+1 = 1 + (j \ (mod\ k))$). By the construction of H, it follows that the first vertex after a_{i_j} on this path will be $(v_{i_j}, e_1^{v_{i_j}}, 1)$ or $(v_{i_j}, e_p^{v_{i_j}}, 6)$, where $p = d_G(v_{i_j})$, $v_{i_j} \in V$.

After that, $D_{a_{i_j} a_{i_{j+1}}}$ will enter the component G'_{e_1} or G'_{e_p} that will be leaved also by a vertex corresponding to v_{i_j}. If the next vertex is not $a_{i_{j+1}}$, it will enter into the component corresponding to the next edge incident with v_{i_j}, that will be leaved also by a vertex corresponding to v_{i_j}.

Proof cont'd. It follows that each path $D_{a_{i_j} a_{i_{j+1}}}$ corresponds to an unique vertex $v_{i_j} \in V$, such that the first and the last edge of $D_{a_{i_j} a_{i_{j+1}}}$ are $a_{i_j}(v_{i_j}, e_t^{v_{i_j}}, x)$, $a_{i_{j+1}}(v_{i_j}, e_{t'}^{v_{i_j}}, x')$, with $t = 1$ and $t' = d_G(v_{i_j})$, $x = 1$, $x' = 6$ or $t = d_G(v_{i_j})$, $t' = 1$, $x = 6$, $x' = 1$.

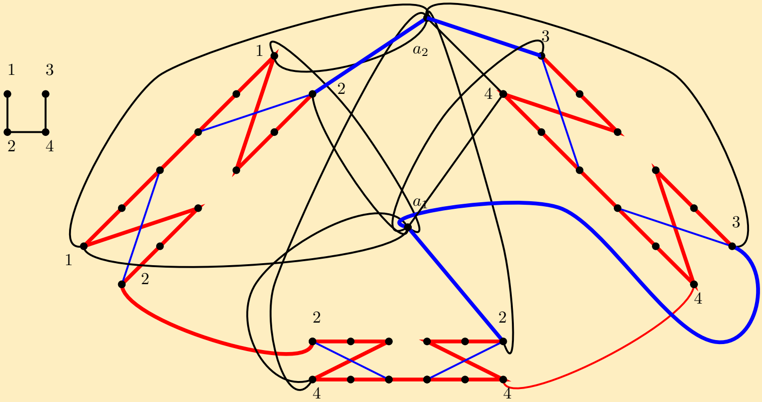
It follows that the vertices v_{i_j} are distinct.

Let $V^* = \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$. Since C is a Hamiltonian cycle in H, it follows that, $\forall e \in E$, there is a path $D_{a_{i_j} a_{i_{j+1}}}$ traversing G'_e, hence there exists a vertex $v \in V^*$ adjacent with e.

Therefore $S = V \setminus V^*$ is a stable set in G and $|S| = j$.

We proved that if H is Hamiltonian, then in G there is a stable set with j vertices, i. e., the answer to SM for the instance (G, j) is yes.

Proof cont'd.



Proof cont'd. " $\Rightarrow$ " Suppose that the answer to the SM's question for the instance (G, j) is affirmative, hence there exists S_0 a stable set in G with $|S_0| \geqslant j$. There exists $S \subseteq S_0$ with $|S| = j$. Let $V^* = V \setminus S = \{v_1, v_2, \ldots, v_k\}$.

Consider in H for each $e = uv \in E$:

- the two paths from case (c) in G'_e (depicted on one of the above slides) if $u, v \in V^*$.
- the path from case (b) in G'_e (depicted above) if $u \in V^*$ and $v \notin V^*$.
- the path from case (a) in G'_e (depicted above) if $u \notin V^*$ and $v \in V^*$.

To the union of all these paths we add the edges $a_i(v_i, e_1^{v_i}, 1)$, $(v_i, e_1^{v_i}, 6)(v_i, e_2^{v_i}, 1)$, $\ldots$, $(v_i, e_{p-1}^{v_i}, 6)(v_i, e_p^{v_i}, 1)$, $(v_i, e_p^{v_i}, 6)a_{i+1}$, (with $p = d_G(v_i)$), for $i = \overline{1, k}$.

It's easy to check that we obtain, in this way, a Hamiltonian cycle in H.

$\square$

TSP Given $G = (V, E)$ a graph and $d : E \to \mathbb{R}_+$ a non-negative weight function on its edges, find a Hamiltonian cycle, H_o, s.t. the sum of the weights of the edges of H_o is minimum (over all Hamiltonian cycles of G).

Let the graph G be a network consisting of a set, V, of cities together with a set, E, of direct routes between cities and the weight function d giving, for each edge $uv \in E$, $d(uv) =$the distance on the direct route between cities u and v. Fixing a starting city v_0, the Hamiltonian cycle H_o represents the shortest way of visiting all the cities exactly once (except v_0) by a traveling salesman starting from v_0 and returning to v_0.

This is probably, the most studied NP-hard optimization problem!

We consider the following equivalent formulation of this problem.

TSP Given $n \in \mathbb{N}$ $(n \geqslant 3)$ and $d : E(K_n) \to \mathbb{R}_+$, find H_o, Hamiltonian cycle in K_n, with $d(H_o)$ minimum over all Hamiltonian cycles of K_n, where

$$d(H_o) = \sum_{e \in E(H_o)} d(e).$$

If the graph G, on which it is required to solve TSP, is not the complete graph K_n, then we can introduce the missing edges with a very large weight, $M \in \mathbb{R}_+$, where $M > |V| \cdot \max_{e \in E(G)} d(e)$.

Note that we are limited here only to the symmetric TSP, a similar (asymmetric) problem can be be considered for the case when G is a digraph.

In the study of the time complexity of this problem, we will take $d(e) \in \mathbb{N}$, for each edge e.

The associated decision problem is

DTSP

Instance: $n \in \mathbb{N}$ ($n \geqslant 3$), $d : E(K_n) \to \mathbb{N}$ and $B \in \mathbb{N}$.

Question: Is there H_o, Hamiltonian cycle in K_n, s. t. $d(H_o) \leqslant B$?

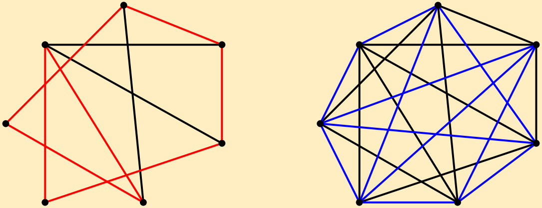
Theorem 2

CH $\leqslant_P$ DTSP.

Proof. Let $G = (V, E)$ ($|V| = n$) be an instance of the problem CH. We construct in polynomial time an instance of the problem DTSP, $d : E(K_n) \to \mathbb{N}$ and $B \in \mathbb{N}$, such that there exists a Hamiltonian cycle in K_n of total weight not greater than B if and only if G is a Hamiltonian graph.

Let
$$d(vw) = \begin{cases} 1, & \text{if } vw \in E(G) \\ 2, & \text{if } vw \in E(\overline{G}) \end{cases} \quad \text{and } B = n.$$

Then, there is in K_n a Hamiltonian cycle of weight $\leqslant n$ if and only if there is a Hamiltonian cycle C in K_n such that $\forall e \in E(C)$, $d(e) = 1$, that is, if and only if G has a Hamiltonian cycle:



G

weight 2
weight 1

It follows that TSP is NP-hard problem. □

A possible approach is to consider approximation algorithms $\mathcal{A}$, which build, in polynomial time, for each instance of TSP, a Hamiltonian cycle $H_{\mathcal{A}}$ of K_n, approximating the optimal solution H_o.

The quality of the approximation can be expressed using the following ratios:

$$R_{\mathcal{A}}(n) = \sup_{d:E(K_n)\to\mathbb{R}_+,\, d(H_o)\neq 0} \frac{d(H_{\mathcal{A}})}{d(H_o)}$$

$$R_{\mathcal{A}} = \sup_{n\geqslant 3} R_{\mathcal{A}}(n).$$

Obviously, the approximation algorithm A is useful only if $R_{\mathcal{A}}$ is finite. Unfortunately, if the weight function d is arbitrary, finding an algorithm $\mathcal{A}$ such that $R_{\mathcal{A}}$ is finite is as hard as solving exactly TSP. More precisely, we have the following result:

Theorem 3

If there is a polynomial time approximation algorithm $\mathcal{A}$ for TSP s.t. $R_{\mathcal{A}} < \infty$, then the problem CH can be solved in polynomial time.

Proof. Let $\mathcal{A}$ be a polynomial time approximation algorithm with $R_{\mathcal{A}} < \infty$. Hence there exists $k \in \mathbb{N}$ such that $R_{\mathcal{A}} \leqslant k$.

Let $G = (V, E)$ be an arbitrary graph, instance of CH. If $n = |V|$, then we consider $d : E(K_n) \to \mathbb{N}$ defined by

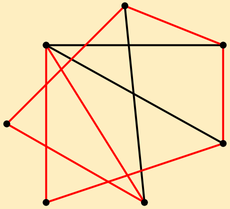
$$d(uv) = \begin{cases} 1, & \text{if } uv \in E(G) \\ kn, & \text{if } uv \notin E(G) \end{cases} .$$

Obviously, G is Hamiltonian if and only if H_o, the optimal solution of this instance of TSP, satisfies $d(H_o) = n$.

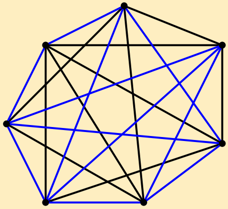
Apply $\mathcal{A}$ to solve approximately this instance of TSP.

- If $d(H_{\mathcal{A}}) \leqslant kn$, then $d(H_{\mathcal{A}}) = n$ and $H_{\mathcal{A}}$ is optimal.
- If $d(H_{\mathcal{A}}) > kn$, then $d(H_o) > n$. Indeed, assuming that $d(H_o) = n$, we have $\dfrac{d(H_{\mathcal{A}})}{d(H_o)} \leqslant k$, therefore $d(H_{\mathcal{A}}) \leqslant kd(H_o) = kn$, contradiction.

It follows that G is Hamiltonian if and only if $d(H_{\mathcal{A}}) \leqslant kn$, and, since A runs in polynomial time, it follows that CH can be solved in polynomial time. $\square$



G



weight $7k$

weight 1

Remark

The Theorem 3 can be formulated equivalently:

Theorem

If P $\neq$ NP, then there is no polynomial time approximation algorithm $\mathcal{A}$ for TSP with $R_{\mathcal{A}} < \infty$.

Theorem 4

If in TSP the weight function d satisfies

$$\forall u, v, w \in V(K_n) \text{ distinct, } d(uv) \leqslant d(uw) + d(wv),$$

then there is a polynomial time approximation algorithm $\mathcal{A}$ with $R_{\mathcal{A}} = 2$.

The idea is to compute a minimum cost spanning tree of K_n and then create a hamiltonian cycle based on this tree:

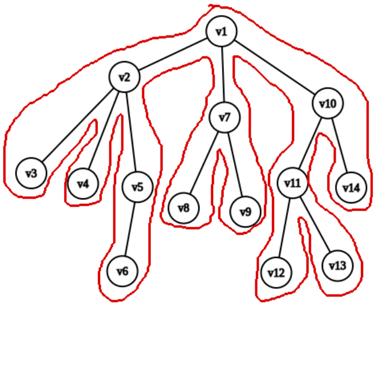
compute T^0 a MST of (K_n, d); // using Prim's algorithm.
$dfs(T^0, v_1)$ and let $v_1, v_2, \ldots, v_n$ be the dfs order; // v_1 could be any vertex;
return the cycle $\{v_1, v_2, \ldots, v_n, v_1\}$;

Proof.

- Let H_o be a minimum cost hamiltonian cycle in (K_n, d); by removing an edge e from H_o one gets a spanning tree $T = H_o - e$, hence $d(T) = d(H_o - e) \leqslant d(H_o)$.
- Consider the walk, P, of T^0 which traverse each edge exactly twice: $d(P) = 2d(T^0)$.
- By keeping only the first occurrence of each vertex, except v_1 for which we keep the last occurrence also, we get the hamiltonian cycle H.
- On the other hand,

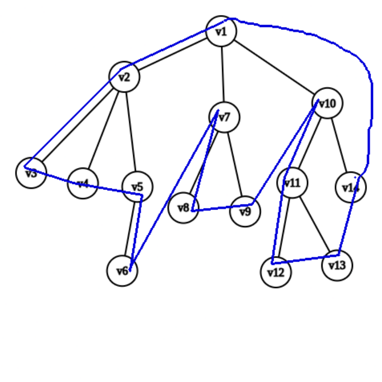
$$d(H) \overset{(*)}{\leqslant} d(P) = 2d(T^0) \leqslant 2d(T) \leqslant 2d(H_o).$$

- (*) comes from the triangle inequality.



The walk P:

$v_1, v_2, v_3, v_2, v_4, v_2, v_5, v_6, v_5, v_2, v_1, v_7, v_8, v_7, v_9, v_7, v_1, v_{10}, v_{11}, v_{12}, v_{11}, v_{13}, v_{11}, v_{10}, v_{14}, v_{10}, v_1$



The hamiltonian cycle H:

$v_1, v_2, v_3, \cancel{v_2}, v_4, \cancel{v_2}, v_5, v_6, \cancel{v_5}, \cancel{v_2}, \cancel{v_1}, v_7, v_8, \cancel{v_7}, v_9, \cancel{v_7}, \cancel{v_1}, v_{10}, v_{11}, v_{12}, \cancel{v_{11}}, v_{13}, \cancel{v_{11}}, \cancel{v_{10}}, v_{14}, \cancel{v_{10}}, v_1$

Theorem 5

(Christofides,1976) If in TSP the weight function d satisfies

$$\forall u, v, w \in V(K_n) \text{ distinct, } d(uv) \leqslant d(uw) + d(wv),$$

then there is a polynomial time approximation algorithm $\mathcal{A}$ with $R_{\mathcal{A}} = 3/2$.

The proof is given in the Annex.

Let $G = (V, E)$ be a graph, $V = \{1, 2, \ldots, n\}$ and let π be a permutation of V. We construct a vertex-coloring $c : V \to \{1, \ldots, \chi(G, \pi)\}$.

```
c(π_1) ← 1; χ(G, π) ← 1; S_1 ← {π_1};
for (i = \overline{2, n}) do
    j ← 0;
    repeat
        j + +; v ← first vertex (in π), from S_j s. t. π_i v ∈ E(G);
        if (∃v) then
            first(π_i, j) ← v;
        else
            first(π_i, j) ← 0; c(π_i) ← j; S_j ← S_j ∪ {π_i};
        end if
    until (first(π_i, j) = 0 or j = χ(G, π))
    if (first(π_i, j) ≠ 0) then
        c(π_i) ← j + 1; S_{j+1} ← {π_i}; χ(G, π) ← j + 1;
    end if
end for
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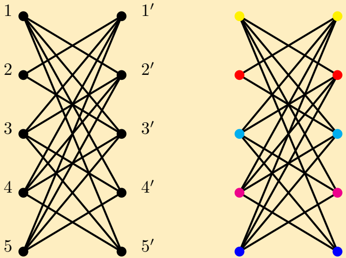
Approaching NP-hard graph problems - Greedy-color algorithm

Note that the number of colors used by the greedy algorithm is not greater than $1 + \Delta(G)$. It follows that $\chi(G) \leqslant 1 + \Delta(G)$.

$\chi(G, \pi)$ the number of colors returned by the Greedy-color algorithm, can be arbitrarily larger than $\chi(G)$. For example, let G be the graph obtained from the complete bipartite graph $K_{n,n}$, with vertex set $\{1, 2, \ldots, n\} \cup \{1', 2', \ldots, n'\}$, by removing the edges $11', 22', \ldots, nn'$.



If $\pi = (1, 1', 2, 2', \ldots, n, n')$, then the Greedy-color algorithm returns $c(1) = c(1')$, $c(2) = c(2') = 2$, $c(n) = c(n') = n$. Therefore, $\chi(G, \pi) = n$, while $\chi(G) = 2$.

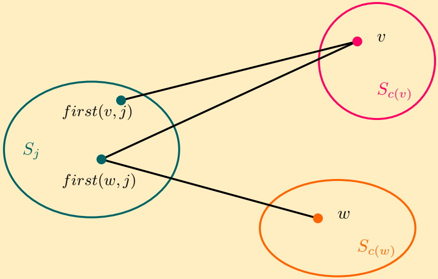
On the other hand, for any graph G, there is a permutation π of its vertices such that $\chi(G, \pi) = \chi(G)$.

Indeed, let $S_1, S_2, \ldots, S_{\chi(G)}$ be the coloring classes of an optimal coloring, such that each S_i is a maximal (w. r. t. inclusion) stable set in $G - \bigcup_{j=1}^{i-1} S_j$. Let π be a permutation which induces an ordering s.t. vertices occurs in the non-decreasing order of their colors. Then, $\chi(G, \pi) = \chi(G)$.

A sufficient condition for the correctness of the Greedy-color algorithm:

Theorem

If, $\forall vw \in E$ and $\forall j < \min\{c(v), c(w)\}$ such that $first(v, j) < first(w, j)$ in the ordering given by π, we have $first(w, j)v \in E$, then $\chi(G, \pi) = \chi(G)$.



Proof. Omitted (and left as an exercise).

The above condition can be tested in $\mathcal{O}(m)$ as a last step in the algorithm. If it is fulfilled, then we have a certificate for the optimality of the number of color found.

The above theorem follows by proving that if the condition stated holds, then $\chi(G,\pi) = \omega(G) \leqslant \chi(G)$. On the other hand, it is well known that there are graphs G for which $\chi(G) - \omega(G)$ is arbitrarily large; for such a graph G no permutation π satisfies the condition in the theorem.

Exercise 1. The regions (including the infinite region) formed by n circles in the plane can be colored with two colors such that any two regions that share a common boundary arc should be colored differently.

Exercise 2. Let $G = (V, E)$ be a graph having no disjoint odd cycles. Prove that G is 5-colorable.

Exercise 3. For a given graph H we define the average degree as $ad(H) = \frac{2|E(H)|}{|V(H)|}$. For a graph G, the maximum average degree is

$$mad(G) = \max\{ad(H) : H \text{ induced subgraph of } G\}$$

Let $k \geqslant 3$ an integer. Prove that, if a graph G has its cromatic number strictly greater than k and its maximum average degree at most k, then G contains an induced k-regular subgraph.

Exercise 4.

(a) Prove that any graph can be vertex-colored with $\Delta(G) + 1$ colors.

(b) Consider the following recursive algorithm for coloring the vertices of a given 3-colorable graph with n vertices:

color(G) {
if $(\Delta(G) \leqslant \sqrt{n})$ then
  color all the vertices of G with $\Delta(G) + 1$ new colors;
else
  let $x_0 \in V(G)$ s. t. $d_G(x_0) = \Delta(G)$;
  color x_0 with a new color;
  color the vertices of $[N_G(x_0)]_G$ with two new colors;
  $G \leftarrow G - (\{x_0\} \cup N_G(x_0))$;
  color(G);
end if }

Prove that the above algorithm uses $\mathcal{O}(\sqrt{n})$ colors.

Exercise 5. Consider the following problem:

Set Cover

Instance: A non empty set X, a family of subsets of X: $\mathcal{F} = \{X_1, \ldots, X_p\}$ and $k \in \mathbb{N}^*$.

Question: X can be covered with at most k subsets of $\mathcal{F}$?

Let us consider the following heuristic (a greedy type algorithm) for solving this problem

$\mathcal{F}' \leftarrow \varnothing$;
while ($\exists x \in X$ uncovered by $\mathcal{F}'$) do
  let X_i a subset in $\mathcal{F}$ with the largest number of uncovered elements;
  $\mathcal{F}' \leftarrow \mathcal{F}' \cup \{X_i\}$;
end while
return $\mathcal{F}'$;

Prove that if m is the minimum cardinality of an optimal subfamily of $\mathcal{F}$, then the above algorithm has gives a subfamily of at most $m \ln n$ subsets ($n = |X|$).

Exercise 6. Let $G = (V, E)$ be a graph with n vertices and m edges. An ordering $\{x_1, x_2, \ldots, x_n\}$ of the vertices of G is called k-bounded if in the digraph $\vec{G}$, obtained from G by replacing the edge $x_i x_j$ with an arc $x_{\min\{i,j\}}, x_{\max\{i,j\}}$, we have $d^+_{\vec{G}}(x) \leqslant k, \forall\, x \in V$.

(a) Devise an algorithm to test in $\mathcal{O}(n+m)$ time complexity if G, has a *k-bounded* order, where $k \in \mathbb{N}$.

(b) Use the above algorithm in order to determine in $\mathcal{O}((n+m)\log n)$ time the number

$$o(G) = \min\{k \in \mathbb{N} \,:\, G \text{ has a } k\text{-bounded order.}\}$$

(c) Prove that any graph G has a vertex coloring which uses $o(G)+1$ colors.

Exercise 7. A greedy algorithm for vertex coloring of a graph $G = (V, E)$ is the following: first, choose an D-ordering of vertices $V = \{v_1, v_2, \ldots, v_n\}$, i. e., $d_G(v_1) \geqslant d_G(v_2) \geqslant \ldots \geqslant d_G(v_n)$, second v_i gets the smallest color unused by any of its already colored neighbors. Now consider the following decision problem

3GCOL

instance: $G = (V, E)$ a graph.

question: Has G a D-ordering s. t. the above heuristic uses at most 3 colors?

Prove that 3COL $\leqslant_P$ 3GCOL.

Exercise 8. Let $G = (V, E)$ be a graph with $V = \{1, 2, \ldots, n\}$ and $\omega(G) = 2$. We define a new graph $M(G)$ by considering the disjoint union of G and $K_{1,n}$ (whose bipartition is $(\{0\}, \{1', 2', \ldots, n'\})$) and adding all the edges $\{i'j, ij' : ij \in E(G)\}$.

(a) Prove that $\omega(M(G)) = 2$ and $\chi(M(G)) = \chi(G) + 1$.

(b) Show that, for every $p \in \mathbb{N}^*$, there exists a K_3-free graph having cromatic number p.

Exercise 9. Let $\mathcal{S}$ be a society formed with n individuals. Each person, $i \in \mathcal{S}$, knows a subset $c(i) \subseteq \mathcal{S} \setminus \{i\}$ of other persons. Two different persons $i, i' \in \mathcal{S}$ cannot be part of the same jury if one knows the other (we can have single member juries).

Prove that if each person knows at most k other persons ($|c(i)| \leqslant k$, $\forall i \in \mathcal{S}$), then there exists a family of at most $(2k + 1)$ disjoint juries that cover together the entire society.

Theorem 6

(Christofides,1976) If in TSP the weight function d satisfies

$$\forall u, v, w \in V(K_n) \text{ distinct, } d(uv) \leqslant d(uw) + d(wv),$$

then there is a polynomial time approximation algorithm $\mathcal{A}$ with $R_{\mathcal{A}} = 3/2$.

Proof. Let $\mathcal{A}$ be the following algorithm:

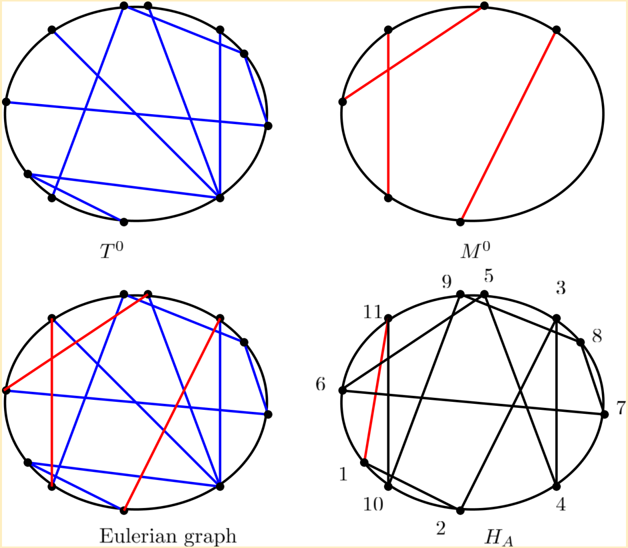
- Find T^0 the edge set of MST in K_n (the cost of each edge e is $d(e)$) (this takes polynomial time using any MST algorithms).
- Find M^0 a minimum weight perfect matching in the subgraph induced in K_n by the set of vertices of odd degrees of T^0 (this takes polynomial time using any maximum matching algorithm).

Proof cont'd.

- In the multi-graph obtained from $\langle T^0 \cup M^0 \rangle_{K_n}$, by duplicating the edges from $T^0 \cap M^0$ (it is connected and all vertices have even degree) find a closed Euler trail, $(v_{i_1}, v_{i_2}, \ldots, v_{i_n})$. Eliminate all the multiple occurrences of the internal vertices to obtain a Hamiltonian cycle $H_{\mathcal{A}}$ in K_n with the edge set $H_{\mathcal{A}} = \{v_{j_1}v_{j_2}, v_{j_2}v_{j_3}, \ldots, v_{j_n}v_{j_1}\}$ (both constructions need $\mathcal{O}(n^2)$ time, the closed Euler trail can be found with Hierholzer's algorithm).

$H_{\mathcal{A}}$ is an approximative solution of TSP given by Christofides. Let $m = \lfloor n/2 \rfloor$ and H_o be the optimal solution. We prove (after Cornuejols & Nemhauser) that

$$\forall n \geqslant 3,\ d(H_{\mathcal{A}}) \leqslant \frac{3m-1}{2m} d(H_o).$$


T^0
M^0
Eulerian graph
9
5
3
11
8
6
7
1
10
4
2
H_A

Proof cont'd. Let $H_o = \{v_1v_2, v_2v_3, \ldots, v_nv_1\}$ (if necessary, we can rename the nodes).

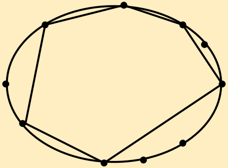
Let $W = \{v_{i_1}, v_{i_2}, \ldots, v_{i_{2k}}\}$ the set of odd degree vertices in $\langle T^0\rangle_{K_n}$, $i_1 < i_2 < \ldots < i_{2k}$. Let $H = \{v_{i_1}v_{i_2}, v_{i_2}v_{i_3}, \ldots, v_{i_{2k-1}}v_{i_{2k}}, v_{i_2k}v_{i_1}\}$ be the cycle generated by W in K_n. Applying repeatedly the triangle inequality, we obtain $d(H) \leqslant d(H_o)$, (the weight of each chord, $d(v_{i_j}v_{i_{j+1}})$ is upper bounded by the sum of the weights of the edges on H_o joining the extremities of the chord $v_{i_j}v_{i_{j+1}}$).

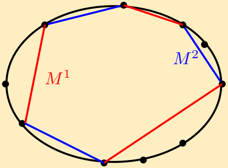
Since H is an even cycle, it is the union of two perfect matchings in $[W]_{K_n}$, $M^1 \cup M^2$. Suppose that $d(M^1) \leqslant d(M^2)$.

By the choosing of M^0, we have $d(M^0) \leqslant d(M^1) \leqslant (1/2)[d(M^1) + d(M^2)] = (1/2)d(H) \leqslant (1/2)d(H_o)$. Let $\alpha \in \mathbb{R}_+$ s. t. $d(M^0) = \alpha d(H_o)$. Obviously, $0 < \alpha \leqslant 1/2$.

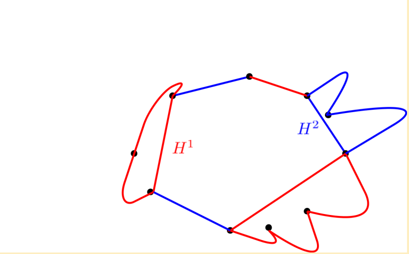
Approaching NP-hard graph problems - Christofides Algorithm

Proof cont'd.





Decompose H_o into $H^1 \cup H^2$ by taking into H^i the edges of H_o connecting the extremities of each chord in M^i: $(v_{i_j} v_{i_{j+1}} \in M^i \Rightarrow v_{i_j} v_{i_j+1}, \ldots v_{i_{j+1}-1} v_{i_{j+1}} \in H^i)$.



Proof cont'd. By the triangle inequality, $d(H^i) \geqslant d(M^i)$, $i = 1, 2$.
At least one of H^1 or H^2 has at most $m = \lfloor n/2 \rfloor$ edges. Suppose that H^1 has this property. Since $d(H^1) \geqslant d(M^1) \geqslant d(M^0) = \alpha d(H_o)$, it follows that there exists $e \in H^1$ such that $d(e) \geqslant (\alpha/m)d(H_o)$.
Let T be the spanning tree obtained from H_o by deleting an edge of maximum weight. We have $d(T) = d(H_o) - \max\limits_{e \in E(H_o)} d(e) \leqslant d(H_o) - (\alpha/m)d(H_o)$.
Since T^0 is MST in K_n it follows that $d(T^0) \leqslant d(H_o)(1 - \alpha/m)$.
Using the triangle inequality, we have

$$d(H_{\mathcal{A}}) \leqslant d(T^0) + d(M^0) \leqslant d(H_o)\left(1 - \frac{\alpha}{m}\right) + \alpha d(H_o) =$$

$$= \left(1 + \frac{\alpha(m-1)}{m}\right) d(H_o) \overset{\alpha \leqslant 1/2}{\leqslant} \frac{3m-1}{2m} d(H_o), \forall n \geqslant 3. \square$$