Graph Algorithms - Lecture 2

October 11, 2024

Table of contents

Variations in the definition of a graph

Multigraph: $G = (V, E)$, where V is a non-empty set (of vertices), and E is the **multiset** (of edges) on V, i. e., there exists a map $m : \binom{V}{2} \to \mathbb{N}$.

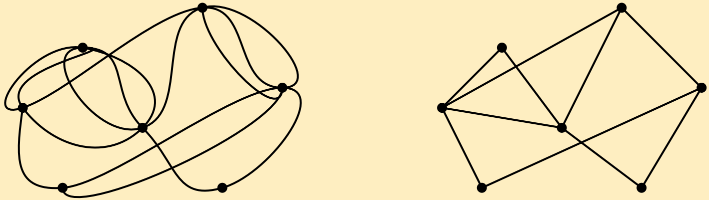
$e \in \binom{V}{2}$, with $m(e) > 0$ is an edge of the multigraph G; if $m(e) = 1$, then e is a **simple edge**, otherwise is a **multiple edge** of **multiplicity** $m(e)$.

The **support graph** of a multigraph, G, is the graph obtained from G by replacing each multiple edge by a simple one.

Variations in the definition of a graph

Example

A multigraph and its support graph:



Pseudograph (general graph): $G = (V, E)$, where V is a non-empty set (of vertices), and E is the multiset (of edges) on $V \cup \binom{V}{2}$, i. e., there exists a map $m : V \cup \begin{pmatrix} V \\ 2 \end{pmatrix} \to \mathbb{N}$.

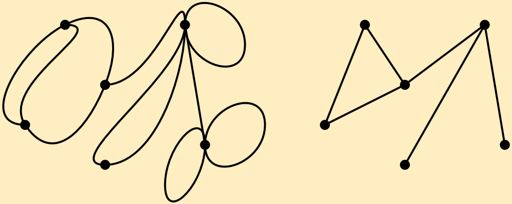
$e \in E \cap V$ (i. e., $|e| = 1$) is called a loop.

The support graph of a pseudograph G is the graph obtained from G by replacing each multiple edge by a simple one and by removing the loops.

Variations in the definition of a graph

Example

A pseudograph and its support graph:



Digraph (directed graph): $D = (V(D), E(D))$, where $V(D)$ is a non-empty set (of vertices), and $E(D) \subseteq V(D) \times V(D)$ is the set of arcs (or directed edges).

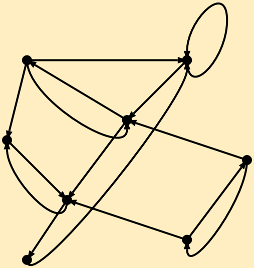
If $e \in E$ then $e = (u, v)$ (or simply $e = uv$) is an arc directed from u to v, and we say:

- u is the initial extremity (tail) of e, v is the final extremity (head) of e;
- u and v are adjacent;
- e is incident from u and into v;
- v is a succesor of u, and u is a predecessor of v etc.

Variations in the definition of a graph

Example

A digraph:

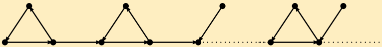


Variations in the definition of a graph

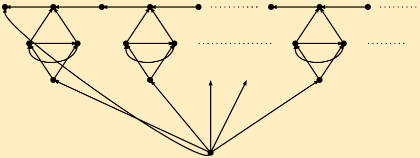
- Symmetric pair of arcs: (uv, vu). uv is called the converse of vu.
- The converse of a digraph D: replace each arc in D with its converse.
- The support graph of a digraph D: $M(D)$ replace each arc with the corresponding set of two vertices. $M(D)$ is a multigraph.
- When $M(D)$ is a (simple) graph, then D is called an oriented graph.
- Complete symmetric digraph: every two (distinct) vertices are joined by a symmetric pair of arcs.
- Tournament: an oriented complete graph (every two (distinct) vertices are joined by exactly one arc).

Infinite (di)graphs: the set of vertices and/or the set of edges (arcs) is countable infinite.

An infinite graph is locally finite if $N(v)$ is a finite set, for any vertex v.



The Danaids barrel

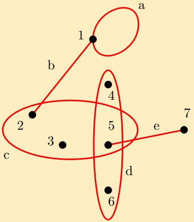


The sisyphus graph

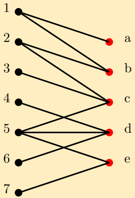
Hypergraphs (Finite Set Systems)

- Edges, now called hyperedges, are not restricted to be 2-subsets of the vertex set. A hyperedge is a non-empty subset of the vertex set.
- k-uniform hypergraph: every edge has cardinality k.

Every hypergraph can be represented as a bipartite graph:



Hypergraph H



Bipartite graph associated with H

Let $G = (V, E)$ be a graph and $v \in V$.

- Degree of vertex v: $d_G(v)$ = number of edges incident to v.
- v is an isolated vertex if $d_G(v) = 0$ and pendant (or leaf) if $d_G(v) = 1$.

$$\sum_{v \in V} d_G(v) = 2|E|.$$

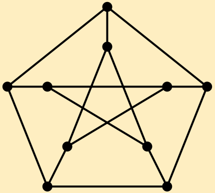
- Maximum degree $\Delta(G)$ and minimum degree $\delta(G)$:

$$\Delta(G) = \max_{v \in V} d_G(v), \;\; \delta(G) = \min_{v \in V} d_G(v).$$

- If $\Delta(G) = \delta(G) = k$, then G is k-regular.
- Null graph: a 0-regular graph.

Degrees

A 3-regular (cubic) graph: Petersen's graph



Degrees

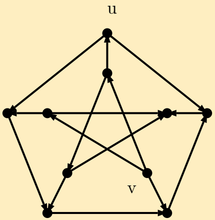
Let $G = (V, E)$ be a digraph and $v \in V$.

- Indegree of vertex v: $d^-_G(v) =$ number of arcs incident into v.
- Outdegree of vertex v: $d^+_G(v) =$ number of arcs incident from v.

$$\sum_{v \in V} d^+_G(v) = \sum_{v \in V} d^-_G(v) = |E|.$$

Example

$d^+_G(u) = 2$, $d^-_G(u) = 1$; $d^+_G(v) = 3$, $d^-_G(v) = 0$

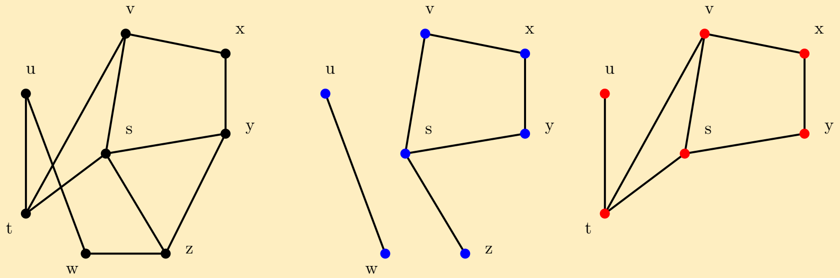


Let $G = (V(G), E(G))$ be a graph.

- Subgraph of G: a graph $H = (V(H), E(H))$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.
- Spanning subgraph of G: a subgraph H of G such that $V(H) = V(G)$.
- Subgraph spanned by $B \subseteq E(G)$ in G: a subgraph $H = (V(H), E(H))$ such that $E(H) = B$ and $V(H) = \cup_{uv \in B}\{u, v\}$; denoted by $\langle B \rangle_G$.
- Induced subgraph: a subgraph H of G such that $E(H) = \binom{V(H)}{2} \cap E(G)$. If $A \subseteq V(G)$; the subgraph induced by A in G is denoted by $[A]_G$ or $G[A]$.

Example

A graph G, a subgraph H' of G, and an induced subgraph of G: $H'' = G[\{u, v, x, y, s, t\}]$.



G H' H''

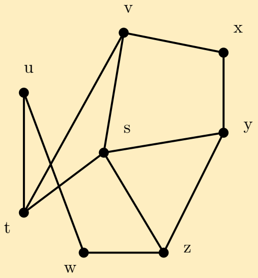
Let $G = (V(G), E(G))$ be a graph.

- If $A \subseteq V(G)$, then the subgraph $[V(G) \setminus A]_G$, denoted by $G - A$, is the subgraph obtained from G by deleting the vertices from A. $G - \{u\}$ is called a deleted subgraph and is denoted by $G - u$.
- If $B \subseteq E(G)$, then the subgraph $\langle E(G) \setminus B \rangle_G$, denoted by $G - B$, is the subgraph obtained from G by deleting the edges from B. $G - \{e\}$ is denoted by $G - e$.
- Similar definitions and notations for digraphs, multigraphs etc.

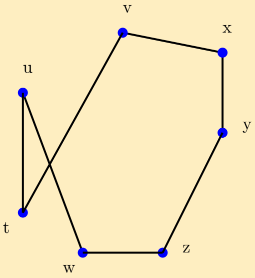
Subgraphs

Example

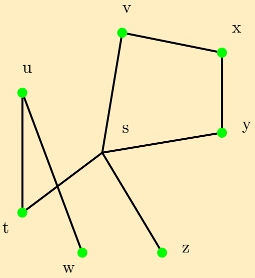
A graph G, $G - s$, and $G - \{vt, wz, zy\}$.



G



$G - s$

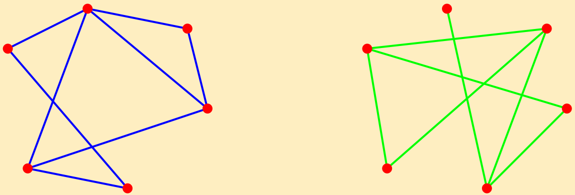


$G - \{vt, wz, zy\}$

Graph operations

Unary: $G = (V(G), E(G))$

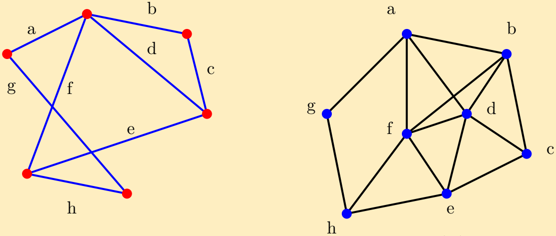
- The **complement** of G: the graph $\overline{G}$, with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \binom{V(G)}{2} \setminus E(G)$.



G $\qquad$ $\overline{G}$

Unary: $G = (V(G), E(G))$

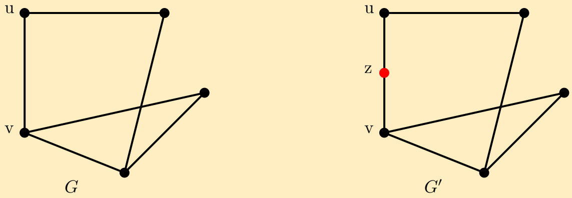
- The line graph of G: the graph $L(G)$, with $V(L(G)) = E(G)$ and $E(L(G)) = \{ef \ : \ e, f \in E(G), e \text{ and } f \text{ are adjacent in } G\}$.



G $L(G)$

Unary: $G = (V(G), E(G))$

- The graph obtained from G by insertion of a new vertex (z) on an edge $(e = uv)$: the graph G', with $V(G') = V(G) \cup \{z\}$ and $E(G') = E(G) \setminus \{uv\} \cup \{uz, zv\}$.

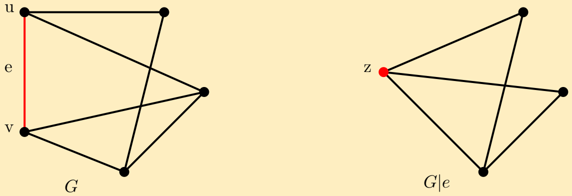


Unary: $G = (V(G), E(G))$

- The graph obtained from G by contracting the edge $e = uv \in E(G)$: the graph $G|e$ with

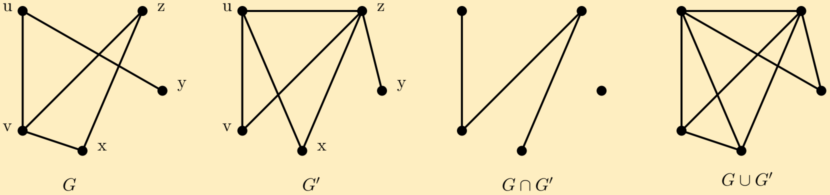
$$V(G|e) = V(G) \setminus \{u, v\} \cup \{z\},$$

$$E(G|e) = E([V(G) \setminus \{u, v\}]_G) \cup \{yz \ : \ yu \text{ or } yv \in E(G)\}.$$



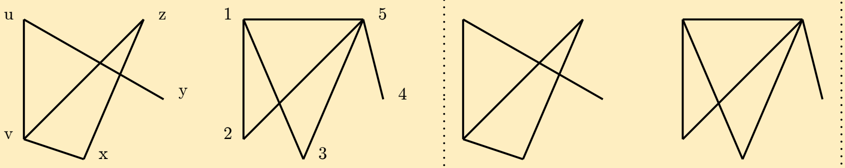
Binary: G, G' with $V(G) = V(G')$

- **Intersection** $G \cap G' = (V(G), E(G) \cap E(G'))$.
- **Union** $G \cup G' = (V(G), E(G) \cup E(G'))$.



Binary: G, G' with $V(G) \cap V(G') = \varnothing$

- **Disjoint union** $G \dot{\cup} G' = (V(G) \cup V(G'), E(G) \cup E(G'))$.

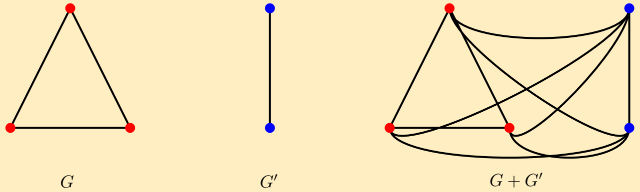


G $\quad$ G' $\quad$ $G \dot{\cup} G'$

Graph operations

Binary: G, G' with $V(G) \cap V(G') = \varnothing$

- Join (sum) $G + G' = \overline{\overline{G} \dot{\cup} \overline{G'}}$.



Graph operations

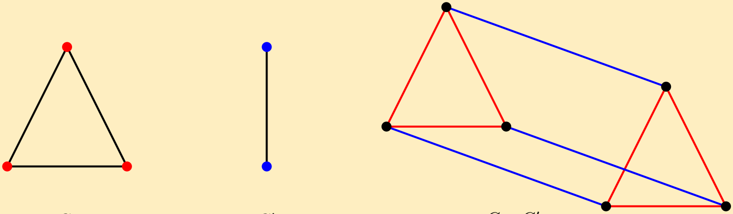
Binary: G, G' with $V(G) \cap V(G') = \varnothing$

- The cartesian product of graphs G and G': the graph $G \times G'$ with

$$V(G \times G') = V(G) \times V(G').$$

$$E(G \times G') = \{(u, u')(v, v') \,:\, u, v \in V(G), u', v' \in V(G'),$$

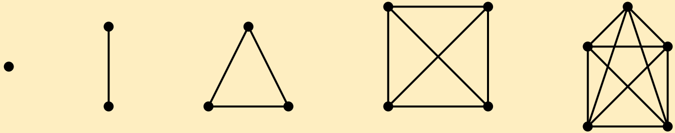
$$u = v \text{ and } u'v' \in E(G') \text{ or } u' = v' \text{ and } uv \in E(G)\}.$$



G $\quad$ G' $\quad$ $G \times G'$

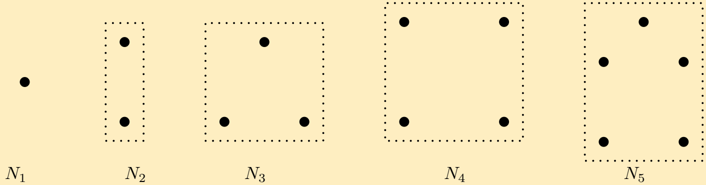
Graph classes - Complete graphs

The complete graph of order n, K_n: $|V(K_n)| = n$ and $E(K_n) = \binom{V(K_n)}{2}$.

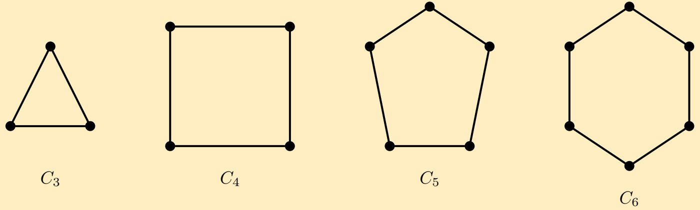


K_1 K_2 K_3 K_4 K_5

The null graph of order n, N_n: $|V(N_n)| = n$ and $E(N_n) = \varnothing$.

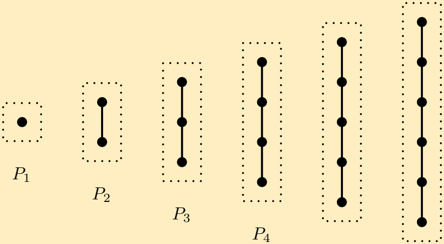


The cycle of order n, C_n: $V(C_n) = \{1, 2, \ldots, n\}$ and $E(C_n) = \{12, 23, \ldots, n-1n, n1\}$.



Graph classes - Paths P_n

The **path of order n**, P_n: $V(P_n) = \{1, 2, \ldots, n\}$ and $E(P_n) = \{12, 23, \ldots, n-1n\}$.



P_1 P_2 P_3 P_4 P_5 P_6

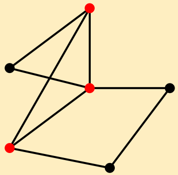
Graph classes - Cliques

A k-subset of vertices of a graph G that induces a complete graph is called a k-clique.

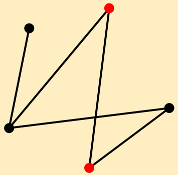
$$\text{clique number of G} : \omega(G) = \max_{Q \text{ clique in } G} |Q|.$$

Remark. $\omega(G) = \alpha(\overline{G})$.

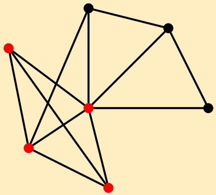
Example



$\omega(G) = 3$



$\omega(G) = 2$



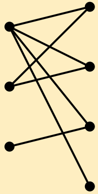
$\omega(G) = 4$

Bipartite graph: a graph G with the property that $V(G)$ can be partitioned in two stable sets.

If $V(G) = S \cup T$, $S \cap T = \varnothing$, $S, T \neq \varnothing$, S, T stable sets in G, then G is denoted $G = (S, T; E(G))$.

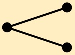
Complete bipartite graph: $G = (S, T; E(G))$, with $uv \in E(G)$, $\forall u \in S$ and $\forall v \in T$; denoted by $K_{s,t}$, where $s = |S|$, $t = |T|$.

Example

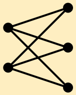


A bipartite graph

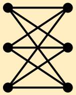
$K_{1,1}$



$K_{1,2}$



$K_{2,3}$



$K_{3,3}$

Planar graph: a graph that can be represented in a plane such that to each vertex corresponds a point of that plane and to each edge corresponds a simple curve joining the points corresponding to its extremities and these curves intersects only at their endpoints.
A graph which is not planar is a **non-planar graph**.

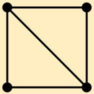
Planar graphs: **Decision problem**

PLAN	Instance:	G graph.
	Question:	Is G planar?

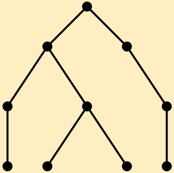
belongs to P (Hopcroft, Tarjan, 1972, $\mathcal{O}(n+m)$).

Example

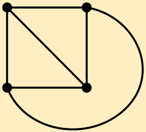
Planar and non-planar graphs.



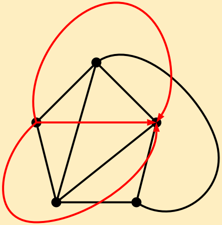
planar graph



planar graph



planar graph



K_5 non-planar graph

Graph classes - $\mathcal{F}$-free graphs

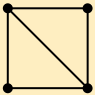
- The usual way to define a class of graphs by forbidding certain subgraphs.
- If $\mathcal{F}$ is a set of graphs then a graph G is said to be $\mathcal{F}$-free if G contains no induced subgraph isomorphic to a member of $\mathcal{F}$.
- If $\mathcal{F}$ is a singleton, $\mathcal{F} = \{H\}$, then we simply write H-free.

Example

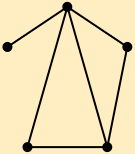
- the class of null graphs is exactly the class of K_2-free graphs.
- a P_3-free graph is a disjoint union of complete graphs.
- Triangulated (chordal) graphs: $(C_k)_{k \geqslant 4}$-free graphs.

Example

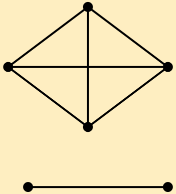
$\mathcal{F}$-free graphs.



a $2K_2$-free graph



a C_4-free graph



a P_3-free graph

Let $G = (V, E)$ be a graph.

- Walk of length r from u to v in G: any sequence of vertices and edges of the form

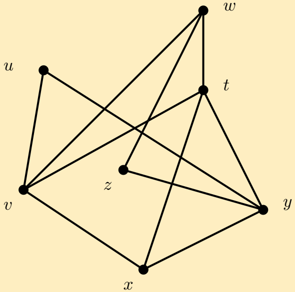
$$(u =)v_0, v_0v_1, v_1, \ldots, v_{r-1}, v_{r-1}v_r, v_r(= v).$$

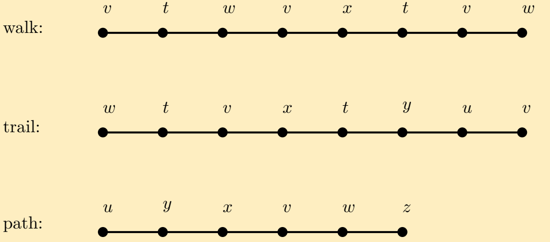
  u and v are the extremities of the walk.
- Trail: a walk with distinct edges.
- Path: a walk with distinct vertices.

  A vertex is a walk (trail, path) of length 0.

Example

Walks, trails, paths.



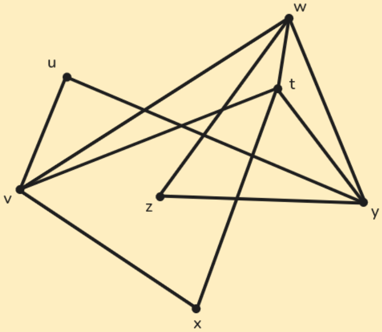


Let $G = (V, E)$ be a graph.

- Closed walk: a walk from u to u.
- Closed trail: a trail from u to u.
- Cycle or closed path: a walk with vertices that are distinct except the extremities which are equal.
- A cycle is even or odd depending on the parity of its length.
- The length of the shortest cycle (if any) is the girth, $g(G)$, of G.
- The maximum length of a cycle is the circumference, $c(G)$, of G.

Example

Closed walks, closed trails. cycles.



A closed walk: $u, uy, y, yt, t, tv, v, vw, w, wv, v, vu, u$.

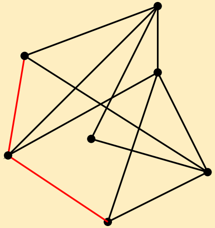
A closed trail: $v, vw, w, wz, z, zy, y, yw, w, wt, t, tx, x, xv, v,$

A cycle: $u, uv, v, vx, x, xt, t, tw, w, wy, y, yu, u$.

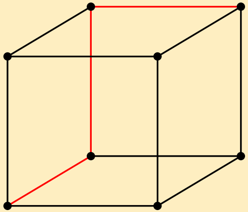
Let $G = (V, E)$ be a graph.

- The distance in G from u to v, $d_G(u, v)$, is the length of the shortest path in G from u to v (if any).
- The diameter of the graph G, $d(G)$, is:

$$d(G) = \max_{u,v \in V} d_G(u, v).$$



$d(G) = 2$



$d(G) = 3$

Let $D = (V, E)$ be a digraph.

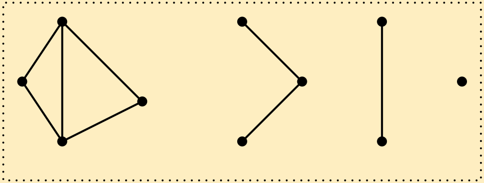
All the above definitions are preserved by considering arcs (directed edges) instead of edges.

Let $G = (V, E)$ be a graph.

- Connected graph: there is a path between any pair of vertices. Otherwise the graph is disconnected.
- Connected component of a graph G: a maximal connected subgraph, H, of G (i. e., there is no connected subgraph H' of G, $H' \neq H$, H being a subgraph of H').
- Every graph can be expressed as a disjoint union of its connected components.
- The following binary relation is an equivalence relation: $\rho \subseteq V \times V$, given by $u \rho v$ (i. e., $(u, v) \in \rho$) if there is a path in G between u and v.
- The connected components of G are the subgraphs induced by the equivalence classes of ρ.

Paths and Cycles - Connectivity

Example

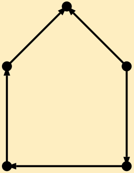


four connected components

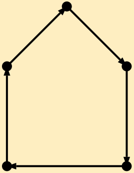
Let $D = (V, E)$ be a digraph.

- Weakly connected (or simply, connected) digraph: its support graph $M(D)$ is connected.
- Unilaterally connected digraph: there is a path from u to v or from v to u, for any two vertices $u, v \in V$.
- Strongly connected digraph: there is a path from u to v, for any two vertices $u, v \in V$.

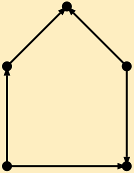
Example



unilaterally connected



strongly connected



weakly connected

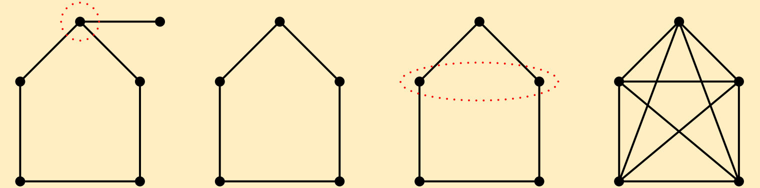
Paths and Cycles - Connectivity

Let $G = (V, E)$ be a connected graph.

- Cut vertex : a vertex $v \in V$ such that $G - v$ is not connected.
- Vertex cutset: a set of vertices $S \subseteq V$ such that $G - S$ is not connected.
- A tree is a connected graph without cycles.
- A graph whose connected components are trees is a forest.

More general: in a graph G that is not necessarily connected $v \in V(G)$ is a cut vertex if $G - v$ has more connected components than G.

Example



cut vertex | no cut vertex | vertex cutset | no vertex cutset

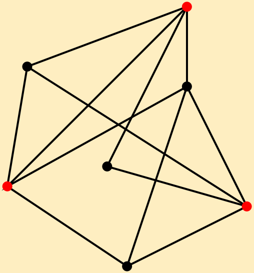
Let $G = (V, E)$ be a graph.

- For $p \in \mathbb{N}^*$, G is a p-connected graph if
  - $|V| = p$ and $G = K_p$ or
  - $|V| \geqslant p + 1$ and G has no vertex cutset of cardinality less than p.
- Obviuosly, G is 1-connected if and only if is connected.
- The vertex-connectivity number, $k(G)$, of the graph G is

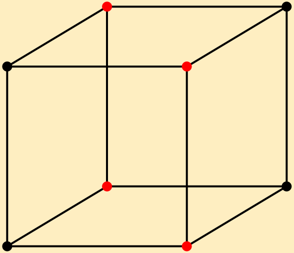
$$k(G) = \max\{p \in \mathbb{N}^* \ : \ G \text{ is } p - \text{connected}\}.$$

Paths and Cycles - Connectivity

Example



$k(G) = 2$



$k(G) = 3$

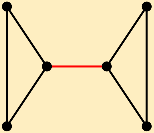
Let $G = (V, E)$ be a connected graph.

- Cut edge (or bridge): an edge $e \in E$ such that $G - e$ is not connected.
- Edge-cutset: A subset of edges $S \subseteq E$ such that $G - S$ is not connected.
- For $p \in \mathbb{N}^*$, G is a p-edge-connected graph if G has no edge-cutset of cardinality less than p.
- The edge-connectivity number, $\lambda(G)$, of the graph G is

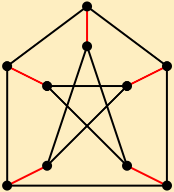
$$\lambda(G) = \max\{p \in \mathbb{N}^* \ : \ G \text{ is } p-\text{edge-connected}\}.$$

More general: in a graph G that is not necessarily connected $e \in V(G)$ is a cut edge if $G - e$ has more connected components than G.

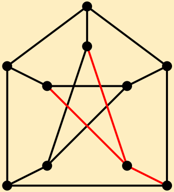
Example



a cut edge



an edge-cutset

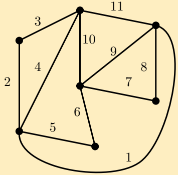


$\lambda(G) = 3$

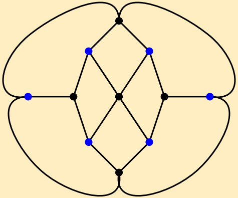
Let G be a (di)graph.

- G is Eulerian if there is a closed trail in G passing through each edge of G.
- G is Hamiltonian if there is a cycle in G passing through each vertex of G.

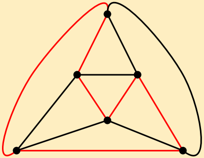
Polyinomial time recognition of Eulerian (di)graphs (Euler, 1736).



an Eulerian graph



a non Hamiltonian graph
(bipartite of odd order)



a Hamiltonian graph

Hamiltonian problems

HAM Instance: G a graph.
Question: Is G Hamiltonian?

NP-complete (Karp, 1972).

NH Instance: G a graph.
Question: Is G not Hamiltonian?

NH $\in$ NP?

Exercise 1. Let G_1 and G_2 be two graphs. Prove that if G_1 and G_2 are connected, then $G_1 \times G_2$ is connected.

Exercise 2. For $p \in \mathbb{N}^*$, set $G_p = K_2 \times K_2 \times \ldots \times K_2$ (p times).

(a) Find the order and the dimension of G_p.

(b) Show that G_p is bipartite and determine $\alpha(G_p)$.

Exercise 3. Let D be a tournament containing a cycle C of length $n \geqslant 4$. Show that for every vertex u of C one can find, in $\mathcal{O}(n)$ time, a cycle of length 3 containing u.

Exercise 4. Let G be a connected graph with $n \geqslant 2$ vertices and m edges. Prove that:

(a) If G has exactly one cycle, then $m = n$.

(b) If G has no leaves, then $m \geqslant n$.

(c) If G is a tree, then it has at least two leaves.

Exercise 5. Let G be a graph with $n \geqslant 2$ vertices. Prove that:

(a) If G is connected, then it contains at least one vertex that is not a cut vertex.

(b) If $n \geqslant 3$ and G is connected, then it contains two vertices that are not cut vertices.

(c) True or false: if G is connected and $x \in V(G)$ is not a cut vertex, then x is a leaf in a certain spanning tree of G?

Exercise 6. Let G be a connected graph that doesn't contain two pendant nodes (leaves) having a common neighbor. Prove that there exist two adjacent vertices those removal doesn't disconnect G.

Exercise 7. Let G be a graph and H its line-graph ($H = L(G)$). Prove that H is $K_{1,3}$-free.

Exercise 8. Let G be a graph. Prove that:

(a) If G has exactly two odd degree vertices, then these two vertices are linked with a path in G.

(b) If G is connected with all vertices of even degree, then G has an edge which is not a bridge (cut-edge).

(c) If G is connected with all vertices of even degree, then G contains no bridge (cut-edge).

(d) If G is connected with all vertices of even degree, then G is Eulerian.

Exercise 9. Let G be a graph. Prove that

(a) The number of odd degree vertices is even.

(b) If G is connected and has k odd degree vertices, then G is the union of $[k/2]$ edge-disjoint trails.

Exercise 10. Let G be a graph such that $N_G(u) \cup N_G(v) = V(G)$, $\forall u, v \in V(G)$, $u \neq v$. Prove that G is a complete graph.

Exercise 11. Let G be a graph having the property that $d_G(u)+d_G(v) \geqslant |G| - 1$, $\forall u, v \in V(G)$, $u \neq v$. Prove that the diameter of $d(G) \leqslant 2$.

Exercise 12. Let $G = (V, E)$ be a graph with $V = \{v_1, v_2, \ldots, v_n\}$ such that $d_G(v_1) \leqslant d_G(v_2) \leqslant \ldots \leqslant d_G(v_n)$. Prove that G is connected if $d_G(v_p) \geqslant p$, for every $p \leqslant n - d_G(v_n) - 1$.

Exercise 13. Let $G = (V, E)$ be a graph and S be a stable set of G. Prove that S is of maximum cardinality if and only if for every stable set of G, $S' \subseteq V \setminus S$, we have

$$|S'| \leqslant |N_G(S') \cap S|.$$

Exercise 14. Let $G = (V, E)$ be a digraph. A strongly connected component of G is a maximal (related to inclusion) sub-digraph H, of G which is strongly connected (i. e., H is strongly connected and there is no strongly connected sub-digraph H' of G, $H \subsetneq H'$).

Define the following binary relation on V, $\rho \subseteq V \times V$, given by: $u\rho v$ (i. e., $(u, v) \in \rho$), if there is a directed path in G from u to v and a directed path from v to u.

(a) Prove that ρ is an equivalence relation.

(b) Show that the strongly connected components of G are the sub-digraphs induced by the equivalence classes of ρ.