Quantum Computation and Quantum Information
Assignment 3

Gabriel Balarezo

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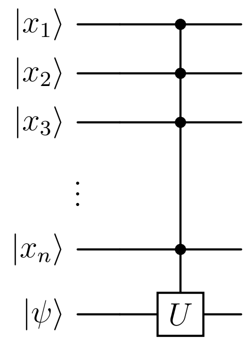
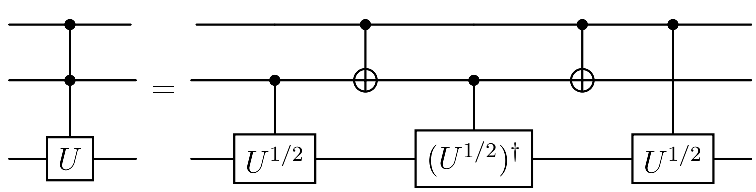
Exercise 4.30

Suppose U is a single-qubit unitary operation. Find a circuit containing $\mathcal{O}(n^2)$ Toffoli, CNOT and single qubit gates which implements a $C^n(U)$ gate (for $n > 3$), using no work qubits.

So, the circuit we want to implement is the following:

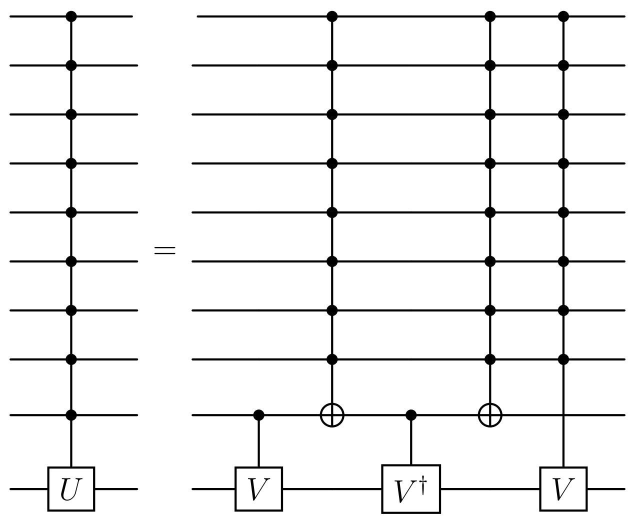


For this, we will Lemma 6.1 from Ref. [1], which states that for any unitary 2×2 matrix U, a $C^2(U)$ gate can be simulated by the following circuit:



Proof of this lemma can be found in the same reference.

A generalisation of Lemma 6.1 can be found in Lemma 7.5 of the same reference, which states that for any unitary 2×2 matrix U, a $C^{n-1}(U)$ gate can be simulated by the following circuit (illustrated for $n = 9$)



given V a unitary matrix such that $V^2 = U$. Another version of this implementation can be found in Ref. [2]

What we want to show is that the circuit above can be implemented using $\mathcal{O}(n^2)$ basic operations.

We can notice that this circuit is a recursive implementation of Lemma 7.5. Let C_{n-1} denote the cost of implementing a $C^{n-1}(U)$. We can also notice that the cost of applying $C^1(V)$ and $C^1(V^\dagger)$ gates is $\mathcal{O}(1)$. Corollary 7.4 from Ref.[1] states that for an n-bit circuit (for $n \geq 7$), a $C^{n-2}(X)$ gate can be simulated using $\mathcal{O}(n)$ basic operations. The cost of implementing $C^{n-2}(V)$ gate is C_{n-2} (by recursion). This implies that the cost of implementing the circuit above is $C_{n-1} = C_{n-2} + \mathcal{O}(n)$. This is a linear recurrence relation, and its solution is $C_{n-1} = \mathcal{O}(n^2)$.

Note: Illustrations were made using the package `quantikz2`.

References

[1] A. Barenco, C. H. Bennett, R. Cleve, D. P. DiVincenzo, N. Margolus, P. Shor, T. Sleator, J. A. Smolin, and H. Weinfurter, Physical Review A **52**, 3457–3467 (1995).

[2] Y. Liu, G. L. Long, and Y. Sun, Analytic constructions of general n-qubit controlled gates (2007), arXiv:0708.3274 [quant-ph] .