Quantum Computation and Quantum Information
Assignment 2

Gabriel Balarezo

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Exercise 1.2

Explain how a device which, upon input of one of two non-orthogonal quantum states $|\psi\rangle$ and $|\varphi\rangle$ correctly identified the state, could be used to build a device which cloned the states $|\psi\rangle$ and $|\varphi\rangle$, in violation of the non-cloning theorem. Conversely, explain how a device for cloning could be used to distinguish non-orthogonal quantum states.

Given a device that can distinguish between non-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$ (without measurement), we can then design a quantum circuit that clones the states $|\psi\rangle$ and $|\varphi\rangle$ as we like.
First, let's define the non-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$ as follows:

$$|\psi\rangle \in \{|\psi_+\rangle, |\psi_-\rangle\} \tag{1}$$
$$|\varphi\rangle \in \{|\varphi_+\rangle, |\varphi_-\rangle\} \tag{2}$$

then, there exists a single-qubit operator, i.e. a rotation such that:

$$\begin{aligned} U(\phi,\theta)\,|\psi_+\rangle &= |0\rangle \\ U(\phi,\theta)\,|\psi_-\rangle &= |1\rangle \\ U(\phi,\theta)\,|\varphi_+\rangle &= |0\rangle \\ U(\varphi,\theta)\,|\varphi_-\rangle &= |1\rangle \end{aligned}$$

and

$$\begin{aligned} U^{-1}(\phi,\theta)\,|0\rangle &= |\psi_+\rangle \\ U^{-1}(\phi,\theta)\,|1\rangle &= |\psi_-\rangle \\ U^{-1}(\phi,\theta)\,|0\rangle &= |\varphi_+\rangle \\ U^{-1}(\varphi,\theta)\,|1\rangle &= |\varphi_-\rangle \end{aligned}$$

Now, we can define a rotation from

$$|0\rangle \quad \text{to} \quad \cos\left(\frac{\phi}{2}\right)|0\rangle + \sin\left(\frac{\phi}{2}\right)e^{i\theta}|1\rangle = |\psi_+\rangle$$

and from

$$|1\rangle \quad \text{to} \quad \sin\left(\frac{\phi}{2}\right)|0\rangle + \cos\left(\frac{\phi}{2}\right)e^{i\theta}|1\rangle = |\psi_-\rangle$$

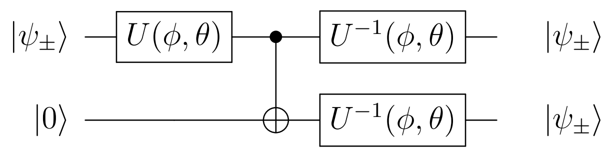
as follows:

$$U(\phi,\theta) = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) & \sin\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) & -\cos\left(\frac{\phi}{2}\right) \end{pmatrix} \tag{3}$$

and the inverse transformation is given by

$$U^{-1}(\phi,\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} \cos\left(\frac{\phi}{2}\right) & \sin\left(\frac{\phi}{2}\right) \\ \sin\left(\frac{\phi}{2}\right) & -\cos\left(\frac{\phi}{2}\right) \end{pmatrix} \tag{4}$$

Now, we can define a quantum circuit that clones the states $|\psi\rangle$ and $|\varphi\rangle$ as follows:



The same follows for the state $|\varphi\rangle$.

So far we have shown that a device that can distinguish between non-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$ can be used to build a device that clones the states $|\psi\rangle$ and $|\varphi\rangle$. But for this we need to know if our qubit is in either $|\psi\rangle$ or $|\varphi\rangle$, so we can plug the correct phase and angle into the rotation operator, which is in contradiction with the non-cloning theorem.

Conversely, given the aforementioned cloning circuit, we can use it to distinguish between non-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$ by

1. Making n copies of our unknown state $|\chi\rangle$ which is either $|\psi\rangle$ or $|\varphi\rangle$.
2. Measuring the $|\chi\rangle$ state in the computational basis $|\psi_\pm\rangle$, i.e $\langle\psi|\chi\rangle$.

if $|\chi\rangle = |\psi\rangle$, then the outcome is 1, and if $|\chi\rangle = |\varphi\rangle$, then the outcome is $\langle\psi|\varphi\rangle$.

$$\text{Probability} \quad |\langle\psi|\psi\rangle|^2 = 1 \quad \text{and probability} \quad |\langle\psi|\varphi\rangle|^2 < 1$$

Let $|\langle\chi|\varphi\rangle|^2 = p$, then the probability of measuring an overlap for n measurements of $\langle\psi|\phi\rangle$ is

$$p^n = |\langle\chi|\varphi\rangle|^{2n}$$

for example, $\left\{|0\rangle, \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}\right\}$

$$p = \left|\langle 0|\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)\right|^2 = \frac{1}{2}$$

$$p^n = \left(\frac{1}{2}\right)^n, \quad \text{and for four measurements} \quad p^4 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

So we can see that as long as we can make n copies of the state $|\chi\rangle$, we can distinguish between non-orthogonal states $|\psi\rangle$ and $|\varphi\rangle$.

Even if $|\psi\rangle \sim |\varphi\rangle$, we can still distinguish between them by making enough n copies of the state $|\chi\rangle$ and measuring them in the appropriate basis.