**Closed form solution to LQR Optimal Control applied to One-dimensional non-linear systems**

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We have the dynamical system:

$$\dot{x} = f(x) + g(x)u$$

With $x \in \mathbb{R}$ and $u \in \mathbb{R}$. Let's minimize the functional:

$$J = \frac{1}{2}\int_0^\infty x^T Q x + u^T R u \, dt$$

Using Variational Calculus optimization formula.

We'll optimize the following function:

$$A(x, \dot{x}, \lambda, u) = \frac{1}{2}(x^T Q x + u^T R u) + \lambda(f(x) + g(x)u - \dot{x})$$

$$A(x, \dot{x}, \lambda, u) = H(x, \lambda, u) - \lambda\dot{x}$$

$$\dot{x} = \frac{\partial H}{\partial \lambda}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

$$0 = \frac{\partial H}{\partial u}$$

$$0 = Ru + \lambda g(x) \rightarrow u = -R^{-1}g(x)\lambda$$

$$\dot{x} = f(x) + g(x)u$$

$$\dot{\lambda} = -Qx - \lambda(f'(x) + g'(x)u)$$

Let's see that the values of $H$ remains constant in the values of the optimum solutions:

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \lambda}\dot{\lambda} + \frac{\partial H}{\partial u}\dot{u} = \frac{\partial H}{\partial x}\left(\frac{\partial H}{\partial \lambda}\right) - \frac{\partial H}{\partial \lambda}\frac{\partial H}{\partial x} = 0$$

We also know:

$$H = 0$$

$$0 = (Qx^2 + R^{-1}g(x)^2\lambda^2) + 2\lambda(f(x) - R^{-1}g(x)^2\lambda)$$

$$0 = R^{-1}g(x)^2\lambda^2 - 2f(x)\lambda - Qx^2$$

$$\lambda(x) = \frac{f(x) + c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2 Q x^2}}{R^{-1}g(x)^2}$$

$$c(x) = \pm 1$$

So the optimal policy for a one dimensional system is:

$$u(x) = -\frac{f(x) + c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2 Q x^2}}{g(x)}$$

Substituting in the dynamical system:

$$\dot{x} = f(x) + g(x)u$$

$$\dot{x} = -c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2 Q x^2}$$

To prove optimality, it's enough to satisfy the HJB Equation:

Choosing $J^*$ as:

$$J^*(x) = \int_0^x \frac{f(x) + c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2 Q x^2}}{R^{-1}g(x)^2} dx$$

$$\frac{\partial J^*}{\partial x} = \lambda(x)$$

Choosing $J^*$ as

$$0 = \min_{\mathrm{u}} \left[\frac{1}{2}\left(x^T Q x + u^T R u\right) + \frac{\partial J^*}{\partial x}\left(f(x) + g(x)u\right)\right]$$

The following condition must hold $\forall x$

$$0 = \min_{\mathrm{u}} \left[\frac{1}{2}\left(x^T Q x + u^T R u\right) + \lambda(x)\left(f(x) + g(x)u\right)\right] = \min_{\mathrm{u}}[H(x, \lambda(x), u)]$$

$$0 = \frac{\partial H}{\partial u} \rightarrow\rightarrow u^*(x) = -R^{-1}g(x)\lambda(x)$$

Replacing back this equation, we get:

$$H(x, \lambda(x), u^*(x)) = 0$$

$$c(x) = \mathrm{sign}(x)$$

$$\lambda(x) = \frac{Qx^2}{\left(-f(x) + c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2Qx^2}\right)}$$

$$u(x) = -\frac{f(x) + c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2Qx^2}}{g(x)}$$

$$u(x) = -\frac{R^{-1}g(x)Qx^2}{\left(-f(x) + c(x)\sqrt{f(x)^2 + R^{-1}g(x)^2Qx^2}\right)}$$