Multi-Robot Formation Control

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Problem is defined as follows: Suppose you have a set of differential wheeled robots. You want them to be located in a predefined formation, but you can send them only two variables to ALL robots: Angular and linear velocity. The robots all have a different distance from wheel to wheel. This is crucial to make the system of n robots controllable under only 2 variables. Problem is solved with a change of variable, a constant angular velocity for all robots, and a LQR control applied to the new system (which is linear).

Let's take the kynematic model of a car

$$\dot{x} = \frac{R}{2}(\omega_l + \omega_r) * \cos\varphi$$

$$\dot{y} = \frac{R}{2}(\omega_l + \omega_r) * \sin\varphi$$

$$\dot{\varphi} = \frac{l}{R}(-\omega_l + \omega_r)$$

Suppose we are allowed so send them only the rotation of the wheels, i.e ω_l and ω_r. Suppose also that all robots have the same radius of the wheel, but different length between wheels. Also we define de coordinates (x_i, y_i, φ_i) as the same coordinates as before, but for each robot i.

Defining also:

$v = \frac{R}{2}(\omega_l + \omega_r)$, $\omega = \frac{1}{R}(-\omega_l + \omega_r)$

We have $\forall i \in 1,2, \dots, n$:

$$\dot{x}_i = v * \cos\varphi_i$$

$$\dot{y}_i = v * \sin\varphi_i$$

$$\dot{\varphi}_i = l_i\omega$$

Let's note that (v, ω) is bijective to (ω_l, ω_r), so since now, we are going to assume that we can send to ALL robots directly (v, ω).

The key "trick" is make the following change of variable:

$$z_{1i} = -(x_i - x_{if})\sin\varphi_i + (y_i - y_{if})\cos\varphi_i$$

$$z_{2i} = (x_i - x_{if})\cos\varphi_i + (y_i - y_{if})\sin\varphi_i$$

$$\dot{z}_{1i} = -\omega l_i z_{2i}$$

$$\dot{z}_{2i} = v + \omega l_i z_{1i}$$

Defining a diagonal matrix $L = \{l_i \delta_{ij}\}_{\substack{i=1 \\ j=1}}^{n}$, which contains the value of l_i on its diagonal and zero everywhere else, and defining also the following state variables we have:

$$z_1 = \begin{bmatrix} z_{11} \\ z_{12} \\ \vdots \\ z_{1n} \end{bmatrix} \rightarrow \dot{z}_1 = -\omega L z_2$$

$$z_2 = \begin{bmatrix} z_{21} \\ z_{22} \\ \vdots \\ z_{2n} \end{bmatrix} \rightarrow \dot{z}_2 = \omega L z_1 + \mathbf{1}_{nx1} v$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \rightarrow \dot{z} = \omega \begin{bmatrix} \mathbf{0}_{nxn} & -L \\ L & \mathbf{0}_{nxn} \end{bmatrix} z + \begin{bmatrix} \mathbf{0}_{nx1} \\ \mathbf{1}_{nx1} \end{bmatrix} v$$

Taking the matrix:

$$A = \begin{bmatrix} \mathbf{0}_{nxn} & -L \\ L & \mathbf{0}_{nxn} \end{bmatrix}; B = \begin{bmatrix} \mathbf{0}_{nx1} \\ \mathbf{1}_{nx1} \end{bmatrix}; u_1 = \omega; u_2 = v$$

We have the dynamical system:

$$\dot{z} = u_1 A z + u_2 B$$

$$\dot{z} = u_1 g_1(z) + u_2 g_2(z)$$

With:

$$A = \begin{bmatrix} \mathbf{0}_{nxn} & -L \\ L & \mathbf{0}_{nxn} \end{bmatrix}; B = \begin{bmatrix} \mathbf{0}_{nx1} \\ \mathbf{1}_{nx1} \end{bmatrix}; u_1 = \omega; u_2 = v$$

Note that the system is driftless. Now, let's see the controllability of the system. For that, we take Lie Bracket to the direction vectors g_i, and we must be sure that they are linear independent of each other. In other words:

$$rank\{g_1, g_2, [g_1, g_2], [[g_1, g_2], g_1] \dots, \} = 2n$$

$$[g_1, g_2] = \frac{\partial g_2}{\partial z^T} g_1 - \frac{\partial g_1}{\partial z^T} g_2 = -AB$$

$$[[g_1, g_2], g_1] = [-AB, g_1] = -A^2 B$$

It can be prove by induction that the next terms are $-A^k B$. We need now:

$$rank\{Az, B, -AB, -A^2 B, \dots\} = 2n$$

The pair (A, B) is controllable in linear sense, because the matrix $\Gamma = [B\ AB\ A^2B \dots A^{2n-1}B]$ is full rank. This means, that the columns are linearly independent and automatically our last desired condition is hold (operating enough brackets until we get $-A^{2n-1}B$). Note also that we are relying on the fact that the diagonal matrix L has all its components different. Otherwise, the matrix Γ would not be full rank.

Note also that this controllability is in the sense of the z variables. Remember that z_1 and z_2 are rotations of an angle φ_i of the coordinates of (x, y) to the final point (x_f, y_f).

In the MATLAB program we define a constant angular velocity u_1. All robots will be spinning at that rate times l_i (Recall $\dot{\varphi}_i = l_i\omega$, $\omega = u_1$). So we have the new system:

$$\dot{z} = (u_{10}A)z + Bu_2 = A_0 z + Bu_2$$

This system has the classic form of the linear systems in modern control. So we can apply a simple LQR controller here. Simulation and graphics are showed in MATLAB script. There is also an additional commented code where we make the angular velocity variable piecewise according to the total distance from each robot to their final points.

Simple Proof of requirement: $l_i \neq l_j$

Suppose we have two robots with $l_i = l_j$. Recalling their dynamics

$$\frac{1}{2}\left(\left(x_i - x_{if}\right)^2 + \left(y_i - y_{if}\right)^2\right) = \frac{1}{2}\left(z_{1i}^2 + z_{2i}^2\right)$$

$$V_1 = \frac{1}{2}\left(z_{1i}^2 + z_{2i}^2 + z_{1j}^2 + z_{2j}^2\right)$$

$$\dot{V}_1 = \left(z_{2i} + z_{2j}\right)v = V_2 v$$

$$\dot{V}_2 = 2v + \omega l_i\left(z_{1i} + z_{1j}\right) = 2v + \omega l_i V_3$$

$$\dot{V}_3 = -\omega l_i\left(z_{2i} + z_{2j}\right) = -\omega l_i V_2$$

$$\dot{V} = \begin{bmatrix} \dot{V}_1 \\ \dot{V}_2 \\ \dot{V}_3 \end{bmatrix} = \begin{bmatrix} V_2 \\ 2 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ V_3 \\ -V_2 \end{bmatrix} \omega l_i = u_1 g_1(V) + u_2 g_2(V)$$

Recalling controllability condition:

$$[g_1, g_2] = \begin{bmatrix} V_3 \\ 0 \\ 2 \end{bmatrix}$$

$$\begin{vmatrix} V_2 & 0 & V_3 \\ 2 & V_3 & 0 \\ 0 & -V_2 & 2 \end{vmatrix} = 0 \rightarrow rank\{g_1, g_2, [g_1, g_2]\} \neq 3 \rightarrow Not\ controllable$$

But not controllability doesn't imply directly we can drive V_1 to 0. In fact, this is possible but only in a subspace of all the state space. In that case we must compute the Reachable set of the system, the values $(x_{if}, x_{jf}, y_{if}, y_{jf})$ for which it is possible to have $V_1 \rightarrow 0$.

But in general, it is not always possible to drive V_1 to 0. Since it is the sum of squares of distances to the final points from the actual points, we will not be able to reach our desired position from any initial position, just only a subset of all possible points.