On a Variational Gradient Descent Technique for solving Optimal Control Problems

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Abstract — **This study presents a technique to approximate the optimal solutions of a functional, and in particular, to solve optimal control problems in dynamical systems where it can be performed a special case of Feedback Linearization, when we can control directly the velocity or the acceleration of the states. It is constructed an recursion algorithm "Variational Gradient Descent" (VGD), inspired in the classical Gradient Descent for Functions, by stablishing a differential equation using a generalized "gradient" for functional, based on the Euler-Lagrange Equation. We prove the algorithm attains a minimum using a multidimensional system, the "continuous" version of the recursion. Finally, we use a simulated example to explain the meaning of this version.**

Keywords* —*Variational Calculus, Optimal Control, Euler-Lagrange Equation, Gradient Descent, Feedback Linearization.

I. Introduction

There are many ways to solve Optimal Control Problems, but so many times, achieve the solution for nonlinear systems can be really hard This problem consists in finding a minimal area surface of revolution formed by a membrane stretched over two parallel disks of radii, respectively,R1 and R2. The application of the Euler-Lagrange principle leads to a second order differential equation and to a boundary problem, which does not have a solution for some values of the parameters R1, and R2. Although there are some Iterative Approaches, there are some issues to solve, for example, it's required to have initial conditions with the same basis as the real optimal solution have in order to achieve a good approximation.

This study is divided into six sections. The next part shows the required form of the dynamical system. In section 3, it is developed the Variational Gradient Descent when we can control completely the velocity, it is, the first derivative of the states. We call it 1st order VGD. Also using a multidimensional system, we prove that the algorithm converges to a local minimum. The same proceeding is applied in section 4 to a 2nd order VGD, this time to a dynamical system where we can fully control the acceleration. The simulated results will be shown in section five. Finally, we present conclusions in section 6

II. System Reduced Form

A. First Order Case

We have the following dynamical system:

$$\dot{x} = f(t, x, u); x(0) = x_0$$

With $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Suppose now, it is possible to perform Fully Feedback Linearization, i.e.

$$u = u(t, x, v)$$

With $v \in \mathbb{R}^n$ such that:

$$\dot{x} = v, x(0) = x_0$$

The optimal control problem consists in find an optimal input u in order to minimize a cost function:

$$J = \int_0^{t_f} g(t, x, u)dt \text{ s.t } \dot{x} = f(t, x, u), x(0) = x_0$$

With $g(t, x, u)$ an instantaneous cost dependent on the current time, position and input signal. Variable $J[x(t)]$ ***is called a functional*** because a set of functions M, a value J corresponds for every trajectory $x(t)$.

Considering $L(t, x, \dot{x}) = g(t, x, u(t, x, \dot{x}))$ We can reformulate the problem considering the change of variable of ():

$$J = \int_0^{t_f} L(t, x, \dot{x})dt, x(0) = x_0$$

Note that the restriction is included in the same problem, because after finding the optimal trajectory $x^*(t)$, we can define $v^* = \dot{x}^*$ and in consequence $u^* = u(t, x, v^*)$ as the optimal control input. The functional $J[x]$ has a minimum on an element $x^* \in M$ if for any $x \in M$ holds:

$$J[x^*] \leq J[x]$$

In order to minimize a functional, we can perform Euler-Lagrange Equation, necessary condition for optimality, at least local:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 0$$

Not always it's possible to solve in closed form this equation, and so many times it will be so tedious to solve analytically. In this paper, it is propose an numeric algorithm inspired on the classical Gradient descent algorithm.

B. Second Order Case

We have now the dynamical system:

$$\ddot{x} = f(t, x, \dot{x}, u); \begin{array}{l} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{array}$$

With $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^n$. Again, suppose it is possible to perform Fully Feedback Linearization:

$$u = u(t, x, \dot{x}, v)$$

*Research supported by UNI

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With $v \in \mathbb{R}^n$ such that:

$$\ddot{x} = v; \begin{matrix} x(0) = x_0 \\ \dot{x}(0) = \dot{x}_0 \end{matrix}$$

The optimal control problem now is:

$$J = \int_0^{t_f} g(t, x, \dot{x}, u)dt \text{ s.t } \ddot{x} = f(t, x, \dot{x}, u); \begin{smallmatrix} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{smallmatrix}$$

Considering $L(t, x, \dot{x}, \ddot{x}) = g(t, x, \dot{x}, u(t, x, \dot{x}, \ddot{x}))$ We can reformulate the problem considering the change of variable of ():

$$J = \int_0^{t_f} L(t, x, \dot{x}, \ddot{x})dt\, ; \begin{smallmatrix} x(0)=x_0 \\ \dot{x}(0)=\dot{x}_0 \end{smallmatrix}$$

So, once we have founded the optimal trajectory $x^*(t)$, we can define $v^* = \ddot{x}^*$ and then $u^* = u(t, x, \dot{x}, v^*)$ as the optimal control input. This time, Euler-Lagrange Equation takes a special form:

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) = 0$$

It is hard to solve this equation, so we propose again a numeric algorithm in chapter.

III. The 1O-Variational Gradient Descent

A. The algorithm 1O-VGD

We use a functional, with an arbitrary differentiable $x(t) \in \mathbb{R}^n$ and $L(t, x, \dot{x})$ with existent second derivatives.

$$J = \int_0^{t_f} L(t, x, \dot{x})dt$$

Subject to:

$$x(0) = x_0$$

$$J(x) = \int_0^{t_f} L(t, x, \dot{x})dt$$

$$J(x + \varepsilon\eta) = \int_0^{t_f} L(t, x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta})dt$$

$$J(x + \varepsilon\eta) - J(x) = \int_0^{t_f} L(t, x + \varepsilon\eta, \dot{x} + \varepsilon\dot{\eta}) - L(t, x, \dot{x})dt$$

$$J(x + \varepsilon\eta) - J(x) = \int_0^{t_f} \frac{\partial L}{\partial x}\varepsilon\eta + \frac{\partial L}{\partial \dot{x}}\varepsilon\dot{\eta} + \mathcal{O}(\varepsilon^2\eta^2)dt$$

$$J(x + \varepsilon\eta) - J(x) = \varepsilon\int_0^{t_f} \frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta}dt + \mathcal{O}(\varepsilon^2\eta^2)$$

Now we'll define η as the solution of the following differential equation:

$$\ddot{\eta}(t) = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right); \begin{matrix} \eta(0) = 0 \\ \dot{\eta}(t_f) = -\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=t_f} \end{matrix}$$

With the values x and $\dot{x}$ evaluated in the current $x(t)$:

$$J(x + \varepsilon\eta) - J(x) = \varepsilon\int_0^{t_f} \frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta}dt + \mathcal{O}(\varepsilon^2\eta^2)$$

In order to define an iterative method we define the actual x as x_k and also we define from the differential equation:

$$x_0(0) = x_0$$
$$x_{k+1} = x_k + \varepsilon\eta_k$$

$$\ddot{\eta}_k(t) = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right); \begin{matrix} \eta_k(0) = 0 \\ \dot{\eta}_k(t_f) = -\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=t_f} \end{matrix}$$

We can see from the first two conditions that:

$$x_k(0) = x_0$$

So $\forall k, x_k$ holds the initial conditions of the dynamical system

$$J(x_k + \varepsilon\eta_k) - J(x_k) = \varepsilon\int_0^{t_f} \frac{\partial L}{\partial x}\eta_k + \frac{\partial L}{\partial \dot{x}}\dot{\eta}_k dt + \mathcal{O}(\varepsilon^2\eta_k^2)$$

$$\int_0^{t_f} \frac{\partial L}{\partial \dot{x}}\dot{\eta}_k dt = \eta_k(t_f)\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=t_f} - \eta_k(0)\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=0} - \int_0^{t_f} \eta_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)dt$$

$$J(x_k + \varepsilon\eta_k) - J(x_k) = \varepsilon\left(\eta_k(t_f)\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=t_f} + \int_0^{t_f} \eta_k(t)\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)dt\right) + \mathcal{O}(\varepsilon^2\eta_k^2)$$

$$\int_0^{t_f} \eta_k\ddot{\eta}_k dt = \eta_k(t_f)\dot{\eta}_k(t_f) - \int_0^{t_f} (\dot{\eta}_k)^2 dt$$

$$J(x_k + \varepsilon\eta_k) - J(x_k) = \varepsilon\left(\eta_k(t_f)\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=t_f} + \eta_k(t_f)\dot{\eta}_k(t_f) - \int_0^{t_f} (\dot{\eta}_k)^2 dt\right) + \mathcal{O}(\varepsilon^2\eta_k^2)$$

$$J(x_k + \varepsilon\eta_k) - J(x_k) = -\varepsilon\int_0^{t_f} (\dot{\eta}_k)^2 dt + \mathcal{O}(\varepsilon^2)$$

So for small enough value of ε we have:

$$\mathcal{O}(\varepsilon^2) \le \varepsilon\int_0^{t_f} (\dot{\eta}_k)^2 dt$$

$$J(x_{k+1}) \le J(x_k)$$

B. Continuous system for the 1O-VGD

So let's consider the following multidimensional dynamical system, with τ the virtual temporal variable, $x(t, \tau)$ and $\eta(t, \tau)$ the solution of:

$$\frac{\partial x}{\partial \tau} = \eta(t, \tau)$$

$$\frac{\partial^2\eta}{\partial t^2} = F\left(x, \frac{\partial x}{\partial t}, \frac{\partial^2 x}{\partial t^2}\right); \begin{matrix} \eta(0, \tau) = 0 \\ \dot{\eta}(t_f, \tau) = -\left.\frac{\partial L}{\partial \dot{x}}\right|_{t=t_f} \end{matrix}$$

With

$$F(x,\dot{x},\ddot{x}) = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

Therefore, given that conditions, we will show that the value of J will decrease as the virtual time τ increases:

$$J(\tau) = \int_0^{t_f} L\left(t, x(t,\tau), \frac{\partial x}{\partial t}(t,\tau)\right) dt$$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial \tau} dt$$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \tau} dt$$

With $\dot{x} = \frac{\partial x}{\partial t}$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial x}\eta(t,\tau) + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \tau} dt$$

Assuming the second partial derivatives of x are continuos, and using Clairaut's theorem:

$$\frac{\partial \dot{x}}{\partial \tau} = \frac{\partial^2 x}{\partial \tau \partial t} = \frac{\partial^2 x}{\partial t \partial \tau} = \frac{\partial \eta(t,\tau)}{\partial t}$$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial x}\eta(t,\tau) + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta(t,\tau)}{\partial t} dt$$

$$\frac{\partial^2 \eta}{\partial t^2} = \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial^2 \eta}{\partial t^2}\eta(t,\tau) + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta(t,\tau) + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta(t,\tau)}{\partial t} dt$$

Considering that $\frac{\partial L}{\partial \dot{x}}$ is already replaced by the functions x and its derivatives, it will be a function of t and τ, so the term $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$ is equivalent to $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}(t,\tau)\right)$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial^2 \eta}{\partial t^2}\eta(t,\tau) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}(t,\tau)\right)\eta(t,\tau) + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta(t,\tau)}{\partial t} dt$$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \eta(t,\tau)\frac{\partial^2 \eta}{\partial t^2} dt + \int_0^{t_f} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}(t,\tau)\eta(t,\tau)\right) dt$$

$$\frac{dJ}{d\tau} = -\int_0^{t_f} \left(\frac{\partial \eta}{\partial t}\right)^2 dt + \frac{\partial \eta}{\partial t}(t,\tau)\eta(t,\tau)\Big|_{t=0}^{t=t_f} + \frac{\partial L}{\partial \dot{x}}(t,\tau)\eta(t,\tau)\Big|_{t=0}^{t=t_f}$$

Considering initial conditions on η, finally we have:

$$\frac{dJ}{d\tau} = -\int_0^{t_f} \left(\frac{\partial \eta}{\partial t}\right)^2 dt < 0$$

The function J is always non-increasing, and it will reach a stationary point when:

$$\frac{\partial \eta}{\partial t} \equiv 0$$

Taking into account the initial conditions:

$$\eta(0,\tau) = 0 \rightarrow \eta(t,\tau) = 0$$

$$\dot{\eta}(t_f,\tau) = -\frac{\partial L}{\partial \dot{x}}\Big|_{t=t_f}$$

We can see that necessarily $\frac{\partial L}{\partial \dot{x}}(t_f,\tau) = 0$, which according to [] is equivalent to the optimal moving end point condition.

In consequence, as the virtual time τ increases, the value of J will decrease. $J(\tau)$ depends on the value of $x(t,\tau)$, which, so:

$$J(\tau) = J[x(t,\tau)]$$

$$x^*(t) = \lim_{\tau\rightarrow\infty} x(t,\tau)$$

And so $x^*(t)$ is attains a minimum of the functional J. In the practice, high values of τ give us good approximations of x^*

IV. The 2O-Variational Gradient Descent

A. The algorithm 2O-VGD

We use a functional, with an arbitrary differentiable $x(t) \in \mathbb{R}^n$ and $L(t,x,\dot{x},\ddot{x})$ with existent second partial derivatives.

$$J = \int_0^{t_f} L(t,x,\dot{x},\ddot{x}) dt$$

Subject to:

$$x(0) = x_0, \dot{x}(0) = \dot{x}_0$$

$$J(x) = \int_0^{t_f} L(t,x,\dot{x},\ddot{x}) dt$$

Similarly to the previous section, we can see:

$$J(x+\varepsilon\eta) - J(x) = \varepsilon\int_0^{t_f} \frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\dot{\eta} + \frac{\partial L}{\partial \ddot{x}}\ddot{\eta}_k dt + O(\varepsilon^2)$$

Now we'll define η as the solution of the following differential equation:

$$\dddot{\eta}_k(t) = -\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)$$

$$\eta_k(0) = 0$$

$$\dot{\eta}_k(0) = 0$$

$$\ddot{\eta}_k(t_f) = -\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f}$$

$$\dddot{\eta}_k(t_f) = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\Big|_{t=t_f}$$

$$x_{k+1} = x_k + \varepsilon\eta_k$$

$$J(x_k+\varepsilon\eta_k) - J(x_k) = \varepsilon\int_0^{t_f} \frac{\partial L}{\partial x}\eta_k + \frac{\partial L}{\partial \dot{x}}\dot{\eta}_k + \frac{\partial L}{\partial \ddot{x}}\ddot{\eta}_k dt + O(\varepsilon^2)$$

$$\int_0^{t_f} \frac{\partial L}{\partial \ddot{x}}\ddot{\eta}_k dt = \dot{\eta}_k(t_f)\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f} - \int_0^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\dot{\eta}_k dt$$

$$\int_0^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\dot{\eta}_k dt$$
$$= \eta_k(t_f)\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\Big|_{t=t_f} - \int_0^{t_f} \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\eta_k dt$$

From () integrating by parts the second term of the Right-side:

$$\int_0^{t_f} \frac{\partial L}{\partial \dot{x}}\dot{\eta}_k dt = \eta_k(t_f)\frac{\partial L}{\partial \dot{x}}\Big|_{t=t_f} - \int_0^{t_f} \eta_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)dt$$

$$J(x_k + \varepsilon\eta_k) - J(x_k)$$
$$= \varepsilon\left(\dot{\eta}_k(t_f)\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f} - \eta_k(t_f)\frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\Big|_{t=t_f}\right.$$
$$\left. + \eta_k(t_f)\frac{\partial L}{\partial \dot{x}}\Big|_{t=t_f} + \int_0^{t_f}\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)\eta_k dt\right)$$
$$+ \mathcal{O}(\varepsilon^2)$$

$$J(x_k + \varepsilon\eta_k) - J(x_k)$$
$$= \varepsilon\left(\dot{\eta}_k(t_f)\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f} + \eta_k(t_f)\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\Big|_{t=t_f}\right.$$
$$\left. - \int_0^{t_f} \eta_k \ddddot{\eta}_k dt\right) + \mathcal{O}(\varepsilon^2)$$

$$\int_0^{t_f} \eta_k \ddddot{\eta}_k dt = \eta_k(t_f)\dddot{\eta}_k(t_f) - \int_0^{t_f} \dot{\eta}_k \dddot{\eta}_k dt$$

Using initial conditions, we finally get:

$$J(x_k + \varepsilon\eta_k) - J(x_k)$$
$$= \varepsilon\left(\dot{\eta}_k(t_f)\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f} + \int_0^{t_f} \dot{\eta}_k \dddot{\eta}_k dt\right) + \mathcal{O}(\varepsilon^2)$$

$$J(x_k + \varepsilon\eta_k) - J(x_k)$$
$$= \varepsilon\left(\dot{\eta}_k(t_f)\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f} + \dot{\eta}_k(t_f)\ddot{\eta}_k(t_f) - \int_0^{t_f} (\ddot{\eta}_k)^2 dt\right)$$
$$+ \mathcal{O}(\varepsilon^2)$$

$$J(x_{k+1}) - J(x_k) = -\varepsilon\int_0^{t_f} (\ddot{\eta}_k)^2 dt + \mathcal{O}(\varepsilon^2)$$

So for small enough value of ε we have:

$$\mathcal{O}(\varepsilon^2) \le \varepsilon\int_0^{t_f} (\ddot{\eta}_k)^2 dt$$
$$J(x_{k+1}) \le J(x_k)$$

B. Continuous system for the 2O-VGD

So let's consider the following multidimensional dynamical system, with $x(t,\tau)$ and $\eta(t,\tau)$ the solution of:

$$\frac{\partial x}{\partial \tau} = \eta(t,\tau)$$
$$\frac{\partial^4 \eta}{\partial t^4} = F\left(x, \frac{\partial x}{\partial t}, \frac{\partial^2 x}{\partial t^2}, \frac{\partial^3 x}{\partial t^3}, \frac{\partial^4 x}{\partial t^4}\right)$$

$$\eta(0,\tau) = 0$$
$$\dot{\eta}(0,\tau) = 0$$
$$\ddot{\eta}(t_f,\tau) = -\frac{\partial L}{\partial \ddot{x}}\Big|_{t=t_f}$$
$$\dddot{\eta}(t_f,\tau) = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\Big|_{t=t_f}$$

Where a dot represents a partial derivative respect t, and with:

$$F(x,\dot{x},\ddot{x},\dddot{x},\ddddot{x}) = -\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)$$

Therefore, given that conditions, we will show that the value of J will decrease as the virtual time τ increases:

$$J(\tau) = \int_0^{t_f} L\big(t, x(t,\tau), \dot{x}(t,\tau), \ddot{x}(t,\tau)\big)dt$$
$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial \tau} dt$$
$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \tau} + \frac{\partial L}{\partial \ddot{x}}\frac{\partial \ddot{x}}{\partial \tau} dt$$

With $\dot{x} = \frac{\partial x}{\partial t}$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial x}\eta(t,\tau) + \frac{\partial L}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \tau} + \frac{\partial L}{\partial \ddot{x}}\frac{\partial \ddot{x}}{\partial \tau} dt$$

Assuming again the second partial derivatives of x are continuos, and using Clairaut's theorem:

$$\frac{\partial \dot{x}}{\partial \tau} = \frac{\partial^2 x}{\partial \tau \partial t} = \frac{\partial^2 x}{\partial t \partial \tau} = \frac{\partial \eta(t,\tau)}{\partial t}$$
$$\frac{\partial \ddot{x}}{\partial \tau} = \frac{\partial^2 \dot{x}}{\partial \tau \partial t} = \frac{\partial^2 \dot{x}}{\partial t \partial \tau} = \frac{\partial^2 \eta(t,\tau)}{\partial t^2}$$

Considering η a function of (t,τ):

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial L}{\partial x}\eta + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta}{\partial t} + \frac{\partial L}{\partial \ddot{x}}\frac{\partial^2 \eta}{\partial t^2} dt$$
$$\frac{\partial^4 \eta}{\partial t^4} = -\left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)$$
$$\frac{dJ}{d\tau}$$
$$= \int_0^{t_f}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right) - \frac{\partial^4 \eta}{\partial t^4}\right)\eta + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta}{\partial t}$$
$$+ \frac{\partial L}{\partial \ddot{x}}\frac{\partial^2 \eta}{\partial t^2} dt$$

Considering that $\frac{\partial L}{\partial \dot{x}}$ and $\frac{\partial L}{\partial \ddot{x}}$ already replaced by the functions x and its derivatives, it will be a function of t and τ, so the term $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)$ is equivalent to $\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}(t,\tau)\right)$ and similarly $\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)$ is equivalent to $\frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial \ddot{x}}(t,\tau)\right)$

Also:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta - \frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\eta + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta}{\partial t} + \frac{\partial L}{\partial \ddot{x}}\frac{\partial^2 \eta}{\partial t^2}$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right)\eta + \frac{\partial L}{\partial \dot{x}}\frac{\partial \eta}{\partial t} + \frac{\partial L}{\partial \ddot{x}}\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2}{\partial t^2}\left(\frac{\partial L}{\partial \ddot{x}}\right)\eta$$
$$= \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\eta\right) + \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial t}\frac{\partial L}{\partial \ddot{x}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \ddot{x}}\right)\eta\right)$$

$$\frac{dJ}{d\tau} = \int_0^{t_f} \frac{\partial}{\partial t}\left(\frac{\partial \eta}{\partial t}\frac{\partial L}{\partial \ddot{x}} + \frac{\partial L}{\partial \dot{x}}\eta - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \ddot{x}}\right)\eta\right) - \frac{\partial^4 \eta}{\partial t^4}\eta dt$$
$$\frac{dJ}{d\tau} = \left(\frac{\partial \eta}{\partial t}\frac{\partial L}{\partial \ddot{x}} + \left(\frac{\partial L}{\partial \dot{x}} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \ddot{x}}\right)\right)\eta\right)\Bigg|_{t=0}^{t=t_f} - \int_0^{t_f} \frac{\partial^4 \eta}{\partial t^4}\eta dt$$

Using boundary conditions described in ()

$$\frac{dJ}{d\tau} = \frac{\partial \eta}{\partial t}\frac{\partial L}{\partial \ddot{x}}\bigg|_{t=0}^{t=t_f} + \dddot{\eta}(t_f,\tau)\eta(t_f,\tau) - \int_0^{t_f} \eta\frac{\partial^4 \eta}{\partial t^4}dt$$
$$\frac{dJ}{d\tau} = -\frac{\partial \eta}{\partial t}(t_f,\tau)\ddot{\eta}(t_f,\tau) + \int_0^{t_f} \frac{\partial \eta}{\partial t}\frac{\partial^3 \eta}{\partial t^3}dt$$

$$\frac{dJ}{d\tau} = -\int_0^{t_f}\left(\frac{\partial^2 \eta}{\partial t^2}\right)^2 dt$$

Considering initial conditions on η, finally we have:

$$\frac{dJ}{d\tau} = -\int_0^{t_f}\left(\frac{\partial^2 \eta}{\partial t^2}\right)^2 dt < 0$$

The function J is always non-increasing, and it will reach a stationary point when:

$$\frac{\partial^2 \eta}{\partial t^2} \equiv 0$$

Taking into account the initial conditions:

$$\eta(t,\tau) = 0$$
$$\dot{\eta}(t_f,\tau) = -\frac{\partial L}{\partial \dot{x}}\bigg|_{t=t_f}$$

$$\ddot{\eta}(t_f,\tau) = -\frac{\partial L}{\partial \ddot{x}}\bigg|_{t=t_f}$$
$$\dddot{\eta}(t_f,\tau) = \frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\bigg|_{t=t_f}$$

We can see that necessarily:

$$\frac{\partial L}{\partial \ddot{x}}(t_f,\tau) = 0$$
$$\frac{\partial L}{\partial \dot{x}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{x}}\right)\bigg|_{t=t_f} = 0$$

which are the conditions for a optimality with a moving end point.

In consequence, as the virtual time τ increases, the value of J will decrease. Defining again $x^*(t) = \lim_{\tau\to\infty} x(t,\tau)$, we know that $x^*(t)$ is a minimum of the functional J, and high values of τ give us good approximations of x^*

V. SIMULATION RESULTS

The algorithm was simulated numerically, and the results obtained for each controller are shown as follows. We took as an example an one dimensional LQR for a simple integrator problem:

$$g(t,x,u) = \tfrac{1}{2}(x^2 + u^2) \text{ s.t } \dot{x} = u$$
$$L(t,x,\dot{x}) = \frac{1}{2}(x^2 + \dot{x}^2)$$

$$\mathrm{F}(\mathrm{x},\dot{\mathrm{x}},\ddot{\mathrm{x}}) = \frac{\partial \mathrm{L}}{\partial \mathrm{x}} - \frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\partial \mathrm{L}}{\partial \dot{\mathrm{x}}}\right) = x - \ddot{x}$$

$$\frac{\partial x}{\partial \tau} = \eta(t,\tau); x(t,0) = x_0(t)$$
$$\frac{\partial^2 \eta}{\partial t^2} = x - \frac{\partial^2 x}{\partial t^2}; \begin{matrix}\eta(0,\tau) = 0 \\ \dot{\eta}(t_f,\tau) = -\dot{x}(t_f,\tau)\end{matrix}$$

The algorithm is a numeric method for solving that differential equation. The boundary condition of $x(t,0) = x_0(t)$ is arbitrary, so we choose $x_0(t) = 1 - \frac{t}{10}$, and the final time will be $t_f = 10$. We chose a step size of $\Delta\tau = \alpha = 0.041$, which in machine learning is called with "learning rate" and considering:

$$x_k(t) = x(t,k\Delta\tau), \eta_k(t) = \eta(t,k\Delta\tau)$$

We can solve it recursively over polynomials:

$$\ddot{\eta}_{k+1}(t) = x_k(t) - \ddot{x}_k(t); \begin{matrix}\eta_{k+1}(0) = 0 \\ \dot{\eta}_{k+1}(t_f) = -\dot{x}_k(t_f)\end{matrix}$$
$$x_{k+1}(t) = x_k(t) + \alpha\eta_k(t)$$

If we assume η and x are polynomial, we can see that on each iteration, the degree of x increases in two, because the first second order differential equation. So, solutions can be tedious. One way to solve it is storing the coefficients of the polynomials, and make recursion over them, or in another case solve numerically the differential equations over the time t. In fig. 1 we can see different curves for the iterations, and it's clear that they approaches to the optimal solution, showed as a black line.

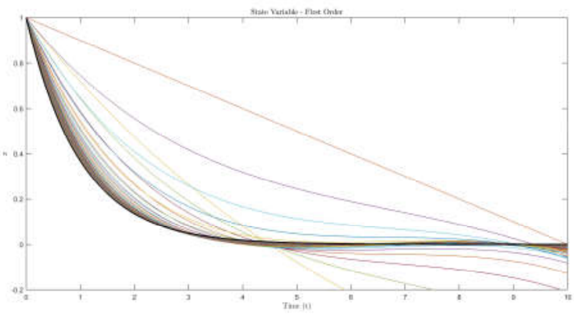

Figure 1: State variable vs time for the curves on different iterations. In black is the real optimal output

Also, in fig. 2 we can see the evolution of the cost function as the number of iterations increases. In some sense it is an approximation of the $J(\tau)$, because we can approximate $\tau \approx N\Delta\tau = N\alpha$, and in the graph we can see, it is always a decreasing.

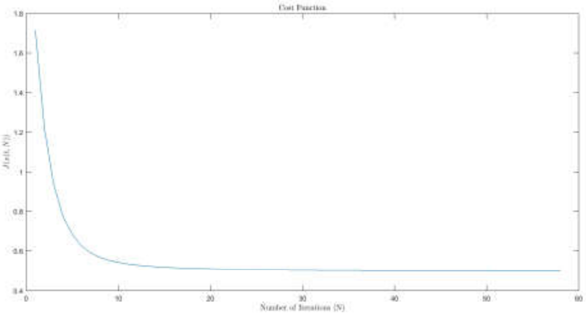

Figure 2: Cost Function vs number of iteration

VI. Conclusion

In this new approach, we design a generalization of the Gradient Descent Method, an algorithm that converges to a local minimum. We stablish the required form of the dynamical system, and we develop the algorithms, showing also the demonstration of convergence of the continuous version of it. Higher order approximations can be used to generate new and better algorithms, but it will increase the complexity and computational cost. Furthermore, results were obtained from simulation of a simple dynamical system. Finally, we explain the meaning of the continuous version using two simulated examples

References

[1] P. Seman, B. Rohal'-Ilkiv, M. Juh´as and M. Salaj, "Swinging up the Furuta Pendulum and its Stabilization Via Model Predictive Control", *Journal of Electrical Engineering*, vol. 64, no. 3, 2013.

[2] B. Cazzolato and Z. Prime, "On the Dynamics of the Furuta Pendulum", *Journal of Control Science and Engineering*, vol. 2011, pp. 1-8, 2011.

[3] P. La Hera, L. Freidovich, A. Shiriaev and U. Mettin, "New approach for swinging up the Furuta pendulum: Theory and experiments", *Mechatronics*, vol. 19, no. 8, pp. 1240-1250, 2009.

[4] I. Fantoni and R. Lozano, "Stabilization of the Furuta pendulum around its homoclinic orbit", *International Journal of Control*, vol. 75, no. 6, pp. 390-398, 2002.

[5] F. Gordillo, J. Acosta and J. Aracil, "A new swing-up law for the Furuta pendulum", *International Journal of Control*, vol. 76, no. 8, pp. 836-844, 2003.

[6] K. Ling, P. Falugi, J. Maciejowski and L. Chisci, "ROBUST PREDICTIVE CONTROL OF THE FURUTA PENDULUM", *IFAC Proceedings Volumes*, vol. 35, no. 1, pp. 37-42, 2002.

[7] J. Aracil, J. Acosta and F. Gordillo, "A nonlinear hybrid controller for swinging-up and stabilizing the Furuta pendulum", *Control Engineering Practice*, vol. 21, no. 8, pp. 989-993, 2013.

[8] S. Nair and N. Leonard, "A normal form for energy shaping: application to the Furuta pendulum", *Proceedings of the 41st IEEE Conference on Decision and Control, 2002.*

[9] B. Sharif and A. Ucar, "State feedback and LQR controllers for an inverted pendulum system", *2013 The International Conference on Technological Advances in Electrical, Electronics and Computer Engineering (TAEECE)*, 2013.

[10] M. Spong, "Partial feedback linearization of underactuated mechanical systems", *Proceedings of IEEE/RSJ International Conference on Intelligent Robots and Systems (IROS'94)*.

[11] J. Zhao and M. Spong, "Hybrid control for global stabilization of the cart–pendulum system", *Automatica*, vol. 37, no. 12, pp. 1941-1951, 2001.

[12] Hui Ye, A. Michel and Ling Hou, "Stability theory for hybrid dynamical systems", *IEEE Transactions on Automatic Control*, vol. 43, no. 4, pp. 461-474, 1998.

[13] I. Santos and G. Silva, "Some Results in Stability Analysis of Hybrid Dynamical Systems", *TEMA - Tendências em Matemática Aplicada e Computacional*, vol. 8, no. 3, 2007.gg

[14] A. Isidori, *Nonlinear Control Systems*, 3rd ed. London: Springer London, 1995, pp. 161-171.

[15] N. Četaev, *The stability of motion*, 1st ed. Oxf., 1961.

[16] S. Ulrich, J. Sasiadek and I. Barkana, "Modeling and Direct Adaptive Control of a Flexible- Joint Manipulator", *Journal of Guidance, Control, and Dynamics*, vol. 35, no. 1, pp. 25-39, 2012.

[17] E. Vinodh Kumar and J. Jerome, "Robust LQR Controller Design for Stabilizing and Trajectory Tracking of Inverted Pendulum", *Procedia Engineering*, vol. 64, pp. 169-178, 2013.

[18] S. Campbell, S. Crawford and K. Morris, "Friction and the Inverted Pendulum Stabilization Problem", *Journal of Dynamic Systems, Measurement, and Control*, vol. 130, no. 5, p. 054502, 2008.

[19] M. Spong, "The swing up control problem for the Acrobot", *IEEE Control Systems Magazine*, vol. 15, no. 1, pp. 49-55, 1995.

[20] K. Åström and K. Furuta, "Swinging up a pendulum by energy control", *Automatica*, vol. 36, no. 2, pp. 287-295, 2000.

[21] M. Ramírez-Neria, H. Sira-Ramírez, R. Garrido-Moctezuma and A. Luviano-Juárez, "Linear active disturbance rejection control of underactuated systems: The case of the Furuta pendulum", ISA Transactions, vol. 53, no. 4, pp. 920-928, 2014.

[22] L. Freidovich, A. Shiriaev, F. Gordillo, F. Gomez-Estern and J. Aracil, "Partial-Energy-Shaping Control for Orbital Stabilization of High-Frequency Oscillations of the Furuta Pendulum", *IEEE Transactions on Control Systems Technology*, vol. 17, no. 4, pp. 853-858, 2009.