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Solving the problem when saturating $\boldsymbol{f_{ground}}$ is really harder, we can still find some necessary conditions for the balance, although they aren't enough. Let's take the system:

$$\ddot{x} = \frac{x}{\sqrt{x^2 + z^2}} v$$

$$\ddot{z} = -g + \frac{z}{\sqrt{x^2 + z^2}} v$$

$$\ddot{x} = \sin\theta\, v$$

$$\ddot{z} = -g + \cos\theta\, v$$

$$0 \le v \le 1$$

Note that this is a normalized system for $\boldsymbol{f_{ground}} < \boldsymbol{f_{max}}$, u will apply now v instead f_{ground}, v is normalized in $[0,1]$. We have make a change of coordinates to get the last system, in fact there are two ways of doing it, one changing the time, other one scaling both, x and z (this additional to the change on the control signal). So we are going to use some polar coordinates to get conditions.

$$x = r\sin\theta$$

$$z = r\cos\theta$$

For now:

$$x < 0 \rightarrow \sin\theta < 0$$

$$z > 0 \rightarrow \cos\theta > 0$$

$$-\frac{\pi}{2} < \theta < 0, r > 0$$

Restrictions due unilateral contact:

$$T = -\frac{x}{\dot{x}} > 0$$

$$z_c = z + \dot{z}T - \frac{g}{2}T^2 = z - \dot{z}\frac{x}{\dot{x}} - \frac{g}{2}\left(\frac{x}{\dot{x}}\right)^2 > 0$$

Weaker condition:

$$\dot{\theta} > \frac{gx^2}{2\dot{x}r^2} = \frac{g\sin^2\theta}{2\dot{x}} \rightarrow \dot{\theta} > 0$$

Considering saturation

1. Using Extreme force to stop velocity in x before reaching $x = 0$:

$$\ddot{x} = \sin\theta\, v \geq \sin\theta$$

$$\dot{\theta} > 0 \rightarrow \theta(t) > \theta_0 \rightarrow 0 > \sin\theta > \sin\theta_0$$

$$\ddot{x} > \sin\theta_0$$

$$\dot{x} > \sin\theta_0\, t + \dot{x}_0$$

$$x(t) > \frac{\sin\theta_0}{2} t^2 + \dot{x}_0 t + x_0$$

In a stable trajectory we always have $0 \geq x$

$$0 \geq x(t) > \frac{\sin\theta_0}{2} t^2 + \dot{x}_0 t + x_0 \; \forall t > 0$$

Taking maximum of the function when:

$$\sin\theta_0\, t_m + \dot{x}_0 = 0 \rightarrow t_m = \frac{\dot{x}_0}{-\sin\theta_0}$$

$$0 > -\frac{\dot{x}_0^2}{2\sin\theta_0} + x_0 \rightarrow \frac{\dot{x}_0^2}{2\sin\theta_0} > x_0$$

This condition must hold for any condition on time, so:

$$\frac{\dot{x}^2}{2} < \frac{x^2}{r} \rightarrow T^2 > \frac{r}{2}$$

In order to get a stronger condition we recall the definition:

$$\frac{g\sin^2\theta}{2\dot{x}} < \dot{\theta}$$

$$0 \geq \ddot{x} = \sin\theta\, v$$

$$\frac{g\sin^2\theta}{2\dot{x}}\ddot{x} \geq \sin\theta\, v\dot{\theta}$$

$$\frac{g}{2\dot{x}}\ddot{x} \geq \frac{v\dot{\theta}}{\sin\theta} \geq \frac{\dot{\theta}}{\sin\theta}$$

$$\frac{g}{2}\ln\left|\frac{\dot{x}}{\dot{x}_0}\right| \geq -\ln\left|\frac{\cot\theta + \csc\theta}{\cot\theta_0 + \csc\theta_0}\right| = -\ln\left|\frac{\cot\frac{\theta}{2}}{\cot\frac{\theta_0}{2}}\right|$$

$$\left|\frac{\dot{x}}{\dot{x}_0}\right|^{\frac{g}{2}}\left|\frac{\cot\frac{\theta}{2}}{\cot\frac{\theta_0}{2}}\right| \geq 1$$

$$\left(\frac{\dot{x}}{\dot{x}_0}\right)^{\frac{g}{2}}\left(\frac{\cot\frac{\theta}{2}}{\cot\frac{\theta_0}{2}}\right) \geq 1$$

$$\dot{x}^{\frac{g}{2}}\cot\frac{\theta}{2} \leq \dot{x}_0{}^{\frac{g}{2}}\cot\frac{\theta_0}{2} = a_0 < 0$$

$$\cot\frac{\theta}{2} \leq a_0\dot{x}^{-\frac{g}{2}} < 0$$

$$\theta \geq 2\cot^{-1}\left(a_0\dot{x}^{-\frac{g}{2}}\right) = 2\tan^{-1}\left(a_0^{-1}\dot{x}^{\frac{g}{2}}\right)$$

$$\sin\theta \geq \sin 2\tan^{-1}\left(a_0^{-1}\dot{x}^{\frac{g}{2}}\right) = \left(\frac{1}{2}\left(a_0^{-1}\dot{x}^{\frac{g}{2}} + a_0\dot{x}^{-\frac{g}{2}}\right)\right)^{-1}$$

$$\ddot{x} = \sin\theta\, v \geq \sin\theta \geq \left(\frac{1}{2}\left(a_0^{-1}\dot{x}^{\frac{g}{2}} + a_0\dot{x}^{-\frac{g}{2}}\right)\right)^{-1}$$

$$0 \geq \dot{x}\frac{d\dot{x}}{dx} \geq \left(\frac{1}{2}\left(a_0^{-1}\dot{x}^{\frac{g}{2}} + a_0\dot{x}^{-\frac{g}{2}}\right)\right)^{-1}$$

$$\frac{1}{2}\left(a_0^{-1}\dot{x}^{\frac{g}{2}} + a_0\dot{x}^{-\frac{g}{2}}\right)\dot{x}\frac{d\dot{x}}{dx} \leq 1$$

$$\frac{1}{2}\left(a_0^{-1}\dot{x}^{1+\frac{g}{2}} + a_0\dot{x}^{1-\frac{g}{2}}\right)\frac{d\dot{x}}{dt} \leq \dot{x}$$

$$\frac{1}{2}\left(\frac{a_0^{-1}\dot{x}^{2+\frac{g}{2}}}{2+\frac{g}{2}} + \frac{a_0\dot{x}^{2-\frac{g}{2}}}{2-\frac{g}{2}}\right) - \frac{1}{2}\left(\frac{a_0^{-1}\dot{x}_0{}^{2+\frac{g}{2}}}{2+\frac{g}{2}} + \frac{a_0\dot{x}_0{}^{2-\frac{g}{2}}}{2-\frac{g}{2}}\right) \leq x - x_0$$

Considering $\dot{x} \to 0, x \to 0$ we have that the initial conditions holds:

$$-\frac{1}{2}\left(\frac{a_0^{-1}\dot{x}_0{}^{2+\frac{g}{2}}}{2+\frac{g}{2}} + \frac{a_0\dot{x}_0{}^{2-\frac{g}{2}}}{2-\frac{g}{2}}\right) \leq -x_0$$

$$x_0 \leq \frac{1}{2}\left(\frac{a_0^{-1}\dot{x}_0{}^{2+\frac{g}{2}}}{2+\frac{g}{2}} + \frac{a_0\dot{x}_0{}^{2-\frac{g}{2}}}{2-\frac{g}{2}}\right) = \frac{a_0^{-1}\dot{x}_0{}^{2+\frac{g}{2}}}{4+g} + \frac{a_0\dot{x}_0{}^{2-\frac{g}{2}}}{4-g}$$

With $a_0 = \dot{x}_0{}^{\frac{g}{2}}\cot\frac{\theta_0}{2}$ we have:

$$x_0 \leq \dot{x}_0{}^2\left(\frac{\tan\frac{\theta_0}{2}}{4+g} + \frac{\cot\frac{\theta_0}{2}}{4-g}\right) = \frac{\dot{x}_0{}^2}{\sin\theta_0}\left(\frac{1-\cos\theta_0}{4+g} + \frac{1+\cos\theta_0}{4-g}\right)$$

$$\frac{\dot{x}_0{}^2}{\sin\theta_0}\left(2\frac{4+g\cos\theta_0}{16-g^2}\right) \geq x_0$$

$$\frac{\dot{x}_0{}^2}{x_0^2}\left(2\frac{4+g\cos\theta_0}{16-g^2}\right) \leq \frac{1}{r}$$

$$T^2 \geq r\left(2\frac{4+g\cos\theta}{16-g^2}\right) > r\left(\frac{8}{16-g^2}\right) > \frac{r}{2}$$