Actividades día 2: Entrelazamiento y Protocolos Cuánticos

Ejercicio 1

1. a) $|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ $|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$

   b) $\langle +|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = 1$

   $\langle -|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1$

   $\langle +|-\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 = 0$

   $\therefore |+\rangle$ y $|-\rangle$ ortonormales

   c) $|+\rangle\langle +| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ $|-\rangle\langle -| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix}$

   $|+\rangle\langle +| + |-\rangle\langle -| = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

2. a) $|+i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix}$ $|-i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix}$

   b) $\langle -i|-i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{i}{\sqrt{2}}\right)^2 = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

   $\langle +i|+i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{i}{\sqrt{2}}\right)^2 = \frac{1}{2} - \left(-\frac{1}{2}\right) = 1$

   $\langle +i|-i\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{i}{\sqrt{2}}\right)^2 = \frac{1}{2} - \frac{1}{2} = 0$

   $\therefore |+i\rangle$ y $|-i\rangle$ ortonormales

   c) $|+\rangle = \langle +i|+\rangle|+i\rangle + \langle -i|+\rangle|-+i\rangle$

   $\alpha = \langle +i|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2 i = \boxed{\frac{1}{2} - \frac{1}{2}i = \alpha}$

   $\beta = \langle -i|+\rangle = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 i = \boxed{\frac{1}{2} + \frac{1}{2}i = \beta}$

Ejercicio 2

1. a) $|+\rangle\langle+| - |-\rangle\langle-| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{1}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbb{X}}$

   b) $|+i\rangle\langle+i| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$

   $|-i\rangle\langle-i| = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{bmatrix}$

   $|+i\rangle\langle+i| - |-i\rangle\langle-i| = \begin{bmatrix} \frac{1}{2} & \frac{-i}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \mathbb{Y}}$

   c) $|0\rangle\langle0| = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

   $|1\rangle\langle1| = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

   $|0\rangle\langle0| - |1\rangle\langle1| = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbb{Z}}$

2. a) $\mathbb{X}^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   $\mathbb{Y}^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   $\mathbb{Z}^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

   $\boxed{\text{En todos los casos se obtiene la identidad, las matrices de Pauli son sus propias inversas}}$

   b) $\mathbb{X}^\dagger = (\mathbb{X}^T)^* = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^T \right)^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \mathbb{X}$

   $\mathbb{Y}^\dagger = (\mathbb{Y}^T)^* = \left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}^T \right)^* = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}^* = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \mathbb{Y}$

   $\mathbb{Z}^\dagger = (\mathbb{Z}^T)^* = \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T \right)^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \mathbb{Z}$

   c) $Tr(\mathbb{X}) = Tr\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) = 0 + 0 = 0$

   $Tr(\mathbb{Y}) = Tr\left(\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \right) = 0 + 0 = 0$

   $Tr(\mathbb{Z}) = Tr\left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) = 1 - 1 = 0$

3. $\mathbb{XY} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \quad \mathbb{YX} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

   $\mathbb{XY} + \mathbb{YX} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} + \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

   $\therefore \mathbb{X}$ *e* $\mathbb{Y}$ *anticonmutan*

Ejercicio 3

1. a)

$$\cos\frac{\pi/2}{2}|0\rangle + \epsilon^{i0}\sin\frac{\pi/2}{2}|1\rangle = \cos\frac{\pi}{4}|0\rangle + \sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$\boxed{\text{Es el estado } |+\rangle}$

b)

$$\cos\frac{\pi/2}{2}|0\rangle + \epsilon^{i\pi/2}\sin\frac{\pi/2}{2}|1\rangle = \cos\frac{\pi}{4}|0\rangle + \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)\sin\frac{\pi}{4}|1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}i|1\rangle$$

$\boxed{\text{Es el estado } |+i\rangle}$

c)

$$\cos\frac{0}{2}|0\rangle + \epsilon^{i\phi}\sin\frac{0}{2}|1\rangle = 1|0\rangle + \epsilon^{i\phi}0|1\rangle = |0\rangle$$

$\boxed{\text{Es el estado } |0\rangle}$

2. a)

$$\widehat{\sigma}_n = \sin\theta.\cos\phi.\mathbb{X} + \sin\theta.\sin\phi.\mathbb{Y} + \cos\theta.\mathbb{Z} =$$
$$\sin\frac{\pi}{2}.\cos 0.\mathbb{X} + \sin\frac{\pi}{2}.\sin 0.\mathbb{Y} + \cos\frac{\pi}{2}.\mathbb{Z} =$$
$$\mathbb{X} + 0\mathbb{Y} + 0\mathbb{Z} = \mathbb{X}$$

b)

$$\widehat{\sigma}_n|S.\widehat{n},+\rangle = \mathbb{X}|+\rangle = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = \begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix} = |+\rangle = +1.|S.\widehat{n},+\rangle$$